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ON CONSTANTS IN EXTREMAL PROBLEMS OF ANALYTIC FUNCTIONS

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1. In the forthcoming book of Oikawa and Sario [3] the following question is given as an open problem. When a plane region has a boundary consisting of a finite number of analytic Jordan curves and its connectivity is more than one, does the strict inequality $C_D<C_B$ hold? Here C_D and C_B are the domain constants depending on a reference point which will be defined below.

This paper gives a necessary and sufficient condition for $C_D=C_B$ which contains an affirmative answer of the question above.

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2. Let W be an open Riemann surface and ζ a point of W with a fixed local parameter t around it. Let $\mathcal{A}_\zeta=\mathcal{A}_\zeta(W)$ be the family of analytic functions f on W satisfying the following normalization condition:

(1) $$f(\zeta)=0, \qquad \left.\frac{df}{dt}\right|_{t=t(\zeta)}=1.$$

Constants C_D and C_B are defined as follows:

$$C_D=C_D(\zeta,\ W)=1\Big/\min_{f\in\mathcal{A}_\zeta}\sqrt{\frac{D[f]}{\pi}}, \qquad D[f]=\iint_W df\cdot\overline{df}^*,$$

and

$$C_B=C_B(\zeta,\ W)=1\Big/\min_{f\in\mathcal{A}_\zeta}M[f], \qquad M[f]=\sup_{z\in W}|f(z)|.$$

LEMMA. *C_D and C_B are nonnegative and finite and satisfy the inequality*

$$C_D\leqq C_B.$$

Proof. When $W\in O_{AD}$, the inequality holds clearly. Hence we assume that $W\notin O_{AD}$. Let $w=f_D(z)$ be the extremal function satisfying

(2) $$C_D(\zeta,\ W)=1\Big/\sqrt{\frac{D[f_D]}{\pi}}.$$

Let G be the image $f_D(W)$ and S the area of G. Then we have

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(3) $$S \leqq D[f_D].$$

Let w be the fixed local parameter around 0. Then we have

(4) $$1\Big/\sqrt{\frac{S}{\pi}} \leqq C_D(0, G).$$

From Ahlfors-Beurling [1] we have

(5) $$C_D(0, G) \leqq C_B(0, G).$$

If $f \in \mathcal{A}_0(G)$, then $f \circ f_D \in \mathcal{A}_\zeta(W)$. Therefore we have

(6) $$C_B(0, G) \leqq C_B(\zeta, W).$$

From (2)~(6) we have completed the proof of the lemma.

By virtue of (3) we have

THEOREM 1. *Let W be a Riemann surface not belonging to O_{AD} and f_D an extremal function satisfying* (2). *If f_D is not univalent, then we have*

$$C_D(\zeta, W) < C_B(\zeta, W).$$

If W does not belong to O_{AD} and not planar, theorem 1 implies $C_D < C_B$. In order to derive a condition for the equality $C_D = C_B$, we may assume that W is a region on z-plane. Let $\mathcal{S}_\zeta = \mathcal{S}_\zeta(W)$ be the family of univalent analytic functions f on W satisfying the normalization condition (1). Constants C_{SD} and C_{SB} are defined similarly as C_D and C_B. Then we have

COROLLARY. *A necessary and sufficient condition for $C_D = C_B$ is given by*

$$C_B = C_{SB}.$$

Proof. $C_D = C_B$ implies $C_D = C_{SD}$ by means of theorem 1. Hence we have $C_B = C_{SB}$. Conversely, if $C_B = C_{SB}$, then $C_D = C_B$ follows by the well-known inequality [1]

$$C_{SB} = C_{SD} \leqq C_D \leqq C_B.$$

THEOREM 2. *A necessary and sufficient condition for $C_D = C_B$ is either*

i) *W belongs to O_{AB}, or*

ii) *W is conformally equivalent to the unit disc $\{|z| < 1\}$ less a relatively closed set which is expressed as the union of at most a countable number of compact sets of class N_B.*

Proof. It is clear that the condition is sufficient. We suppose that $C_D = C_B$. If W belongs to O_{AD}, then W belongs necessarily to O_{AB}. If W does not belong to O_{AD}, then from theorem 1 and corollary, W is planar and satisfies $C_B = C_{SB}$. It is known that the extremal function f_B satisfying

$$C_B=1/M[f_B]$$

is unique and that the function $w=C_B\cdot f_B(z)$ takes all the values in the unit disc $\{|w|<1\}$ except for the union of at most a countable number of compact sets of class N_B (Havinson [2]). By the condition $C_B=C_{SB}$, an extremal function for the class $\mathcal{S}_\zeta(W)$ with respect to the constant C_{SB} coincides with the function f_B. Hence $C_B\cdot f_B$ maps W conformally onto a region which is stated in ii).

REMARK. If $C_D(\zeta, W)=C_B(\zeta, W)$ at a point ζ of W, then by theorem 2 we have $C_D(z, W)=C_B(z, W)$ for all the points z of W.

REFERENCES

[1] AHLFORS, L. V., AND A. BEURLING, Conformal invariants and function-theoretic null-sets. Acta Math. **83** (1950), 101–129.

[2] HAVINSON, S. JA., Analytic capacity of sets, joint nontriviality of various classes of analytic functions and the Schwarz lemma in arbitrary domains. Amer. Math. Soc. Translations (2) **43** (1964), 215–266.

[3] OIKAWA, K., AND L. SARIO, Capacity functions. To appear.

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