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ON ANALYTIC MAPPINGS OF A CERTAIN RIEMANN SURFACE INTO ITSELF

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1. We shall be concerned with the study of analytic mappings of a Riemann surface into itself. Heins [5] showed that every non-constant analytic mapping of a Riemann surface of parabolic type with non-abelian fundamental group into itself is univalent. In the present paper we shall establish a similar result in a case of certain Riemann surfaces of hyperbolic type.

Let W be a Riemann surface of hyperbolic type, let $\mathfrak{G}_W(p, q)$ be the Green function with a pole at $q \in W$ and let π be a projection mapping of the universal covering surface W^∞ onto W. We take, as we may, W^∞ as $\{|z|<1\}$. Then $\mathfrak{G}_W(\pi(z), q)$ has the angular limit 0 a.e. on $\{|z|=1\}$. We denote by $\mathfrak{F}$ the set of all points of such kind on $\{|z|=1\}$. We say that two points z_1 and z_2 of $\mathfrak{F}$ are equivalent provided that there exists an element $T(z)$ of $\mathfrak{G}$ such that $z_2=T(z_1)$, where $\mathfrak{G}$ denotes the group of linear fractional transformations of $\{|z|<1\}$ onto itself which leave π invariant. This requirement defines an equivalence relation in $\mathfrak{F}$. We call an equivalence class of this relation an ideal boundary point of W and call the set of all points of $\mathfrak{F}$ belonging to an ideal boundary point its image. Each ideal boundary point belongs to a single ideal boundary component in the sense of Kerékjártó-Stoilow. Namely, let $e^{i\theta}$ be a point of the image of an ideal boundary point and let λ: $z=z(t)$ $(0\leqq t<1)$ be a curve in $\{|z|<1\}$ such that $\lim_{t\to 1} z(t)=e^{i\theta}$ and there exists a positive number ε satisfying

$$\left|\arg\frac{e^{i\theta}-z(t)}{e^{i\theta}}\right|<\frac{\pi}{2}-\varepsilon.$$

Then $\pi(z(t))$ tends to a single ideal boundary component α as $t\to 1$. This α is independent of a choice of $e^{i\theta}$ and λ. We denote by F the set of all ideal boundary points of W. If the image $\mathfrak{M}$ of a subset M of F is measurable on $\{|z|=1\}$, we say that M is measurable and call $\omega_M(p)=\omega_{\mathfrak{M}}(\pi^{-1}(p))$ the harmonic measure of M with respect to W, where $\omega_{\mathfrak{M}}(z)$ is the harmonic measure of $\mathfrak{M}$ with respect to $\{|z|<1\}$.

Let M be a subset of F of positive measure. According to Constantinescu-Cornea [1], we say that M is HB-indivisible if, for any bounded harmonic function $u(p)$ on W, $u(\pi(z))$ has the same angular limit a.e. on the image $\mathfrak{M}$ of M.

Let $\{\Omega_\nu\}_{\nu=1}^{\infty}$ be an exhaustion of W satisfying: for each ν, Ω_ν is relatively com-

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pact, $\bar{\Omega}_{\nu-1}\subset\Omega_\nu$, the relative boundary $\partial\Omega_\nu$ of Ω_ν consists of a finite number of regular analytic closed Jordan curves $\gamma_{\nu,1}, \cdots, \gamma_{\nu,N_\nu}$, and each component of $W-\bar{\Omega}_\nu$ is not relatively compact. We assume that each curve $\gamma_{\nu,i}$ divides W into two parts and denote by $G_{\nu,i}$ the part which does not contain Ω_ν. Denote by $\beta_{\nu,i}$ the part of the ideal boundary of W which belongs to $G_{\nu,i}$, and denote by $B_{\nu,i}$ the set of all ideal boundary points which belong to one of the ideal boundary components on $\beta_{\nu,i}$.

We shall consider a Riemann surface W having the following three properties: (1) W contains n $(1\leqq n<\infty)$ HB-indivisible sets on its ideal boundary, (2) for sufficiently large ν, each $B_{\nu,i}$ either consists of HB-indivisible sets or else contains no HB-indivisible sets, (3) for sufficiently large ν, there exist at least two $B_{\nu,i}$, $B_{\nu,j}$ of positive measure.

We shall call such a Riemann surface an admitted surface. For instance, a Riemann surface obtained by deleting a parametric disk from a Riemann surface, belonging to the class $O_{HB}-O_G$, and a Riemann surface belonging to the class $O_{HB_n}-O_{HB}$ with at least two ideal boundary components which contain HB-indivisible sets are admitted surfaces.

The purpose of this paper is to show that there exist only a finite number of non-constant analytic mappings of an admitted surface into itself and they are conformal automorphisms having finite periods.

2. Throughout this paper, we shall denote by W an admitted surface. Each HB-indivisible set is contained in a single ideal boundary component save for a set of measure zero [1]. Let $\alpha_1, \cdots, \alpha_m$ $(1\leqq m\leqq n)$ be ideal boundary components of W containing at least one HB-indivisible set. By the property (2), each A_i $(1\leqq i\leqq m)$, the set of all ideal boundary points belonging to α_i, consists of n_i HB-indivisible sets $M_1^i, \cdots, M_{n_i}^i$, $\sum_{i=1}^m n_i=n$. Let $\{\Omega_\nu\}_{\nu=1}^\infty$ be an exhaustion of W satisfying the requirements stated in §1. In the sequel, we may assume that $B_{1,i}$ $(1\leqq i\leqq m)$ consists of A_i and a set of measure zero, and $B_{1,i}$ $(m+1\leqq i\leqq N_1)$ contains no HB-indivisible sets. Further we may assume that there exist at least two $B_{1,i}$, $B_{1,j}$ of positive measure. For simplicity, we denote $\beta_{1,i}$, $B_{1,i}$, $G_{1,i}$ and $\gamma_{1,i}$ by β_i, B_i, G_i and γ_i, respectively.

First we shall summarize certain properties of ω_{B_i} and $\omega_{M_k^i}$. Let $\omega_i^{(\nu)}$ be the harmonic function on Ω_ν whose boundary values are 1 on $\partial\Omega_\nu\cap G_i$ and 0 on $\partial\Omega_\nu\cap(W-\bar{G}_i)$, and put $\omega_i=\lim_{\nu\to\infty}\omega_i^{(\nu)}$. Let $\tilde{\omega}_i^{(\nu)}$ be the harmonic function on $\Omega_\nu\cap G_i$ whose boundary values are 1 on $\partial\Omega_\nu\cap G_i$ and 0 on γ_i, and put $\tilde{\omega}_i=\lim_{\nu\to\infty}\tilde{\omega}_i^{(\nu)}$. Then $\tilde{\omega}_i$ converges to 1 on every sequence of points on which ω_i converges to 1 and vice versa. This follows by the inequality

$$\tilde{\omega}_i\leqq\omega_i\leqq(1-\max_{\gamma_i}\omega_i)\tilde{\omega}_i+\max_{\gamma_i}\omega_i \qquad \text{on} \quad G_i.$$

Let $\tilde{\omega}'^{(\nu)}_i$ be the harmonic function on $\Omega_\nu\cap(W-\bar{G}_i)$ whose boundary values are 1 on $\partial\Omega_\nu\cap(W-\bar{G}_i)$ and 0 on γ_i, and put $\tilde{\omega}'_i=\lim_{\nu\to\infty}\tilde{\omega}'^{(\nu)}_i$. Similarly, $\tilde{\omega}'_i$ converges to 1 on every sequence of points on which ω_i converges to 0 and vice versa. Hence by the inequalities

$$\min_{\gamma_i} \mathfrak{G}_W(r, q)(1-\omega_i(p)) \leqq \mathfrak{G}_W(p, q) \leqq \max_{\gamma_i} \mathfrak{G}_W(r, q)(1-\tilde{\omega}_i(p)) \qquad \text{on } G_i,\ q \notin G_i,$$

$$\min_{\gamma_i} \mathfrak{G}_W(r, q)\omega_i(p) \leqq \mathfrak{G}_W(p, q) \leqq \max_{\gamma_i} \mathfrak{G}_W(r, q)(1-\tilde{\omega}_i'(p)) \qquad \text{on } W-\bar{G}_i,\ q \notin W-\bar{G}_i,$$

it follows that the image $\mathfrak{B}_i$ of B_i is the set of all points on $\{|z|=1\}$ at which $\omega_i(\pi(z))$ has the angular limit 1 and $\mathfrak{F}-\mathfrak{B}_i$ is the set of all points on $\{|z|=1\}$ at which $\omega_i(\pi(z))$ has the angular limit 0. Consequently, we have

$$\omega_{B_i} = \lim_{\nu\to\infty} \omega_i^{(\nu)}.$$

Let $\tilde{\pi}$ be a projection mapping of the universal covering surface G_i^∞ onto G_i. Take G_i^∞ as $\{|z|<1\}$. Denote by $\tilde{\mathfrak{F}}$ the set of all points on $\{|z|=1\}$ at which $\mathfrak{G}_{G_i}(\tilde{\pi}(z), q)$ has the angular limit 0 and denote by $\tilde{\mathfrak{B}}_i$ the set of all points $e^{i\theta} \in \tilde{\mathfrak{F}}$ such that $\tilde{\pi}(re^{i\theta})$ tends to β_i as $r \to 1$. Then in the same way as above we have that $\tilde{\mathfrak{B}}_i$ is the set of all points on $\{|z|=1\}$ at which $\tilde{\omega}_i(\tilde{\pi}(z))$ has the angular limit 1 and $\tilde{\mathfrak{F}}-\tilde{\mathfrak{B}}_i$ is the set of all points on $\{|z|=1\}$ at which $\tilde{\omega}_i(\tilde{\pi}(z))$ has the angular limit 0, and hence that $\tilde{\omega}_i(p) = \omega_{\tilde{B}_i}(p) = \omega_{\tilde{\mathfrak{B}}_i}(\tilde{\pi}^{-1}(p))$, where $\omega_{\tilde{\mathfrak{B}}_i}(z)$ is the harmonic measure of $\tilde{\mathfrak{B}}_i$ with respect to $\{|z|<1\}$.

It is essential in our study that every sequence of points on which $\mathfrak{G}_W(p, q)$ ($\mathfrak{G}_{G_i}(p, q)$) converges to 0 contains a subsequence on which ω_{B_i} ($\omega_{\tilde{B}_i}$) converges to either 1 or 0.

Let c be the union of a finite number of regular analytic closed Jordan curves in G_i which constitutes the relative boundary of a subregion G_c of G_i which has α_i as its ideal boundary component. In the sequel we shall call such a union an admitted union of curves associated with α_i. Denote by β_c the part of the ideal boundary of W belonging to G_c and by B_c the set of all ideal boundary points belonging to one of the ideal boundary components on β_c. Then we have

LEMMA 1. *Let $\omega_c^{(\nu)}$ be the harmonic function on Ω_ν whose boundary values are* 1 *on $\partial\Omega_\nu \cap G_c$ and* 0 *on $\partial\Omega_\nu \cap (W-\bar{G}_c)$, then*

$$\omega_{B_i} = \lim_{\nu\to\infty} \omega_c^{(\nu)} \qquad (1 \leqq i \leqq m).$$

Proof. Since $B_i \supseteqq B_c \supseteqq A_i$ and $\omega_{B_i} = \omega_{A_i}$, $\omega_{B_i} = \omega_{B_c}$. On the other hand, $\omega_{B_c} = \lim_{\nu\to\infty} \omega_c^{(\nu)}$. Hence we obtain

$$\omega_{B_i} = \lim_{\nu\to\infty} \omega_c^{(\nu)}.$$

As a corollary of this lemma, we have the following

LEMMA 2. *Every sequence of points on which $\omega_{M_k^i}$ (ω_{B_i}) converges to* 1 *tends* *to* α_i $(1 \leqq i \leqq m)$.

Proof. Let $\{p_\mu\}_{\mu=1}^\infty$ be a sequence of points such that $\lim_{\mu\to\infty} \omega_{M_k^i}(p_\mu)=1$. Since $\omega_{M_k^i} \leqq \omega_{B_i} \leqq 1$, $\lim_{\mu\to\infty} \omega_{B_i}(p_\mu)=1$. For an arbitrary positive integer ν_0, by Lemma 1 $\omega_{B_i}=\lim_{\nu\to\infty} \omega^{(\nu)}_{\gamma_{\nu_0},i}$, and further

$$\omega^{(\nu)}_{\gamma_{\nu_0},i}(p) \leqq \max_{\gamma_{\nu_0},i} \omega^{(\nu)}_{\gamma_{\nu_0},i}(r) \qquad \text{for} \quad p \in \Omega_\nu \cap (W-\bar{G}_{\nu_0,i}),$$

where α_i lies on $\beta_{\nu_0,i}$. Hence

$$\omega_{B_i}(p) \leqq \max_{\gamma_{\nu_0},i} \omega_{B_i}(r) \qquad \text{for} \quad p \in W-\bar{G}_{\nu_0,i}.$$

This implies that $p_\mu \in G_{\nu_0,i}$ for sufficiently large μ.

In the following sections we shall make free use of the notations introduced in this section.

3. Next we shall investigate some properties of φ_{G_i}, which is the restriction of an analytic mapping φ of W into itself to G_i $(1 \leqq i \leqq m)$.

Constantinescu-Cornea [2] showed: Let W_1 and W_2 be two Riemann surfaces, and let φ be a non-constant analytic mapping of W_1 into W_2. For a given non-negative superharmonic function u on W_1, we denote by $E_\varphi u$ the lower envelope of the set of non-negative superharmonic functions u' on W_2 satisfying $u' \circ \varphi \geqq u$. If u is a bounded minimal harmonic function on W_1, then $E_\varphi u$ is a bounded minimal harmonic function on W_2.

By this result we have the following

LEMMA 3. *If* φ *is a non-constant analytic mapping of* W *into itself, then for each* ω_{B_i} $(1 \leqq i \leqq m)$ *there exists a non-constant harmonic function* u *on* W *satisfying* $u \circ \varphi \geqq \omega_{B_i}$, $0<u<1$ *and* $u=\sum_{j=1}^r \omega_{M_j}$, *where* $M_1, \cdots, M_r$ *are HB-indivisible sets.*

Proof. Since $\omega_{M_k^i}$ is a bounded minimal harmonic function on W [1], $E_\varphi \omega_{M_k^i}$ is also a bounded minimal harmonic function on W. Obviously, $\omega_{M_k^i} \leqq (E_\varphi \omega_{M_k^i}) \circ \varphi \leqq 1$. It follows that $E_\varphi \omega_{M_k^i}$ is not a constant and $\sup E_\varphi \omega_{M_k^i}=1$. Hence there exists an *HB*-indivisible set M_{j_k} such that $E_\varphi \omega_{M_k^i}=\omega_{M_{j_k}}$. Put $u=\text{L.M.H.}\max_{1\leqq k\leqq n_i}\{E_\varphi \omega_{M_k^i}\}$. Then $u=\omega_{\cup_{k=1}^{n_i} M_{j_k}}=\sum_{j=1}^r \omega_{M_j}$, where $M_1, \cdots, M_r$ are *HB*-indivisible sets and $\cup_{k=1}^{n_i} M_{j_k} = \cup_{j=1}^r M_j$, $r \leqq n_i$. Since $\omega_{F-\cup_{j=1}^r M_j}>0$, u is not a constant and $0<u<1$. Moreover

$$\omega_{B_i}=\omega_{\bigcup_{k=1}^{n_i} M_k^i}=\text{L.H.M.}\max_{1\leqq k\leqq n_i}\{\omega_{M_k^i}\} \leqq \text{L.H.M.}\max_{1\leqq k\leqq n_i}\{E_\varphi \omega_{M_k^i}\}\circ\varphi=u\circ\varphi.$$

This is the desired result.

By using the Lindelöf principle [4] and Lemma 3, we obtain the following lemma.

LEMMA 4. *Let* φ *be a non-constant analytic mapping of* W *into itself and let* φ_{G_i} *denote the restriction of* φ *to* G_i $(1 \leqq i \leqq m)$. *Then every point of* W *is covered at most finitely often by* φ_{G_i}.

Proof. If a point $q \in W$ were covered infinitely often, we could find an infinite sequence of points $p_\mu \in G_i$ with $\varphi(p_\mu)=q$. This sequence must tend to β_i since φ is analytic on γ_i.

Suppose that $\liminf_{\mu\to\infty} \mathfrak{G}_{G_i}(p_\mu, r)=0$. For simplicity, we denote by $\{p_\mu\}_{\mu=1}^\infty$ a subsequence such that $\lim_{\mu\to\infty} \mathfrak{G}_{G_i}(p_\mu, r)=0$. Then $\lim_{\mu\to\infty} \omega_{B_i}(p_\mu)=1$. Hence by Lemma 3

$$u(q)=\lim_{\mu\to\infty} u\circ\varphi(p_\mu) \geqq \lim_{\mu\to\infty} \omega_{B_i}(p_\mu)=1.$$

This contradicts that u is not a constant and $0<u<1$.

Suppose that $\liminf_{\mu\to\infty} \mathfrak{G}_{G_i}(p_\mu, r)>0$. By the Lindelöf principle

$$\mathfrak{G}_W(\varphi_{G_i}(r), q) \geqq \sum_{\varphi_{G_i}(p)=q} n(p)\mathfrak{G}_{G_i}(r, p)=\infty,$$

where $n(p)$ denotes the multiplicity of φ_{G_i} at p. This is a contradiction. Thus we complete the proof of the lemma.

We shall now prove the following

LEMMA 5. *If* φ *is a non-constant analytic mapping of* W *into itself, then for each* i_0 $(1 \leqq i_0 \leqq m)$ *there exists an admitted union* c_{i_0} *of curves associated with* α_{i_0} *such that* $\varphi(c_{i_0})=k\cdot c_{j_0}$, *where* c_{j_0} *is an admitted union of curves associated with* α_{j_0} $(1 \leqq j_0 \leqq m)$, *and* k *is a positive integer.*

Proof. Let $\{p_\mu\}_{\mu=1}^\infty$ be a sequence of points such that $\lim_{\mu\to\infty} \omega_{M_1^{i_0}}(p_\mu)=1$. Then

$$\lim_{\mu\to\infty} \omega_{M_{j_0}}\circ\varphi(p_\mu)=\lim_{\mu\to\infty} (E_\varphi \omega_{M_1^{i_0}})\circ\varphi(p_\mu) \geqq \lim_{\mu\to\infty} \omega_{M_1^{i_0}}(p_\mu)=1,$$

where M_{j_0} is an HB-indivisible set. Denote by α_{j_0} the ideal boundary component of W containing M_{j_0}. We take a sufficiently large ν such that Ω_ν contains $\varphi(\gamma_{i_0})$, and let β_{ν, j_0} contain α_{j_0}. Since, by Lemma 2, $\{p_\mu\}_{\mu=1}^\infty$ and $\{\varphi(p_\mu)\}_{\mu=1}^\infty$ tend to α_{i_0} and α_{j_0} respectively, $\varphi^{-1}(G_{\nu, j_0}) \cap G_{i_0} \neq \phi$. Moreover $\varphi^{-1}(G_{\nu, j_0}) \cap \gamma_{i_0}=\phi$. Hence there exists a component Δ of $\varphi^{-1}(G_{\nu, j_0})$ whose closure is contained in G_{i_0}. The restriction φ_Δ of φ to Δ is an analytic mapping of Δ into G_{ν, j_0}. We shall see that φ_Δ is of type Bl. Let K be an arbitrary relatively compact subregion on G_{ν, j_0}, and let K' be a component of $\varphi_{\Delta}^{-1}(K)$. Assume that K' does not belong to the class SO_{HB}. There exists a positive harmonic function v on K' vanishing continuously on $\partial K'$ and $\sup_{K'} v=1$. Here we note that the closure of K' with respect to W is contained in Δ. Let $\{q_\mu\}_{\mu=1}^{\infty}$ be a sequence of points in K' such that $\lim_{\mu\to\infty} v(q_\mu)=1$ and $\{\varphi(q_\mu)\}_{\mu=1}^{\infty}$ converges to a point $q_0 \in K$. Since $\omega_{B_{i_0}} \geqq v$, by Lemma 3 $u(q_0)=\lim_{\mu\to\infty} u\circ\varphi(q_\mu) \geqq \lim_{\mu\to\infty} v(q_\mu)=1$. This is a contradition. Thus we conclude that φ_Δ is of type Bl [9]. Consequently, by Lemma 4, it follows that $\nu_{\varphi_\Delta}(p)=\nu_0$ save for a closed set of capacity zero, where ν_{φ_Δ} denotes the valence of φ_Δ and ν_0 is a positive integer [4]. Then we can take a regular analytic closed Jordan curve c_{j_0} in G_{ν,j_0} separating α_{j_0} from γ_{j_0} satisfying: $\nu_{\varphi_\Delta}(p)=\nu_0$ on c_{j_0}. Each component of $\varphi_\Delta^{-1}(G_{c_{j_0}})$ does not belong to the class SO_{HB} and its relative boundary is compact. Hence $\varphi_\Delta^{-1}(G_{c_{j_0}})$ must be connected and α_{i_0} lies on the part of the ideal boundary of W belonging to $\varphi_\Delta^{-1}(G_{c_{j_0}})$. Now we put $c_{i_0}=\varphi_\Delta^{-1}(c_{j_0})$, then c_{i_0} is a desired union of curves associated with α_{i_0}.

REMARK. In Lemma 3 we saw that for each i_0 $(1\leqq i_0\leqq m)$ there exist HB-indivisible sets $M_1, \cdots, M_r$ such that $(\sum_{j=1}^{r} \omega_{M_j})\circ\varphi \geqq \omega_{B_{i_0}}$. Now we can infer that all $M_1, \cdots, M_r$ are contained in the same B_{j_0}, and hence $\omega_{B_{j_0}}\circ\varphi \geqq \omega_{B_{i_0}}$. In fact, for each M_j there exists an HB-indivisible set $M_k^{i_0}$ contained in B_{i_0} such that $\omega_{M_j}\circ\varphi \geqq \omega_{M_k^{i_0}}$. Let $\{p_\mu\}_{\mu=1}^{\infty}$ be a sequence of points such that $\lim_{\mu\to\infty} \omega_{M_k^{i_0}}(p_\mu)=1$. Then $\lim_{\mu\to\infty} \omega_{M_j}\circ\varphi(p_\mu)=1$. By Lemma 2 $\{p_\mu\}_{\mu=1}^{\infty}$ and $\{\varphi(p_\mu)\}_{\mu=1}^{\infty}$ tend to α_{i_0} and α_i respectively, where α_i is the ideal boundary component of W containing M_j. On the other hand as we saw in the proof of Lemma 5 φ maps a subregion of G_{i_0} having α_{i_0} as its ideal boundary component into G_{j_0}. Hence $\{\varphi(p_\mu)\}_{\mu=1}^{\infty}$ tends to α_{j_0} and hence $\alpha_i=\alpha_{j_0}$.

4. The harmonic length, the quantity assigned to cycles which was introduced by Landau-Osserman [8], is useful in our study.

Let W^* be a Riemann surface which does not belong to the class O_{HB}, and let c be a cycle on W^*. We define a quantity

$$h_W(c)=\sup_{u\in U}\int_c {}^*du,$$

where U denotes the set of all harmonic functions u on W^* satisfying $0<u<1$, and call $h_W(c)$ the harmonic length of c.

In this section we shall be concerned with the harmonic length of γ_i and an admitted union c_i of curves associated with α_i $(1\leqq i\leqq m)$, and

$$\int_{\gamma_i}\frac{\partial u}{\partial n}ds, \qquad \int_{c_i}\frac{\partial u}{\partial n}ds$$

for bounded harmonic functions u on W. Here γ_i and c_i are oriented positively with respect to $W-\bar{G}_i$ and $W-\bar{G}_{c_i}$ respectively, and $\partial/\partial n$ denotes the outer normal derivative. We shall prove two lemmas.

LEMMA 6. *Let c_i be an admitted union of curves associated with α_i ($1 \leqq i \leqq m$). Then*

$$\int_{\gamma_i} \frac{\partial u}{\partial n} ds = \int_{c_i} \frac{\partial u}{\partial n} ds$$

for all bounded harmonic functions u on W, and

$$h_W(c_i) = h_W(\gamma_i) = \int_{\gamma_i} \frac{\partial \omega_{B_i}}{\partial n} ds.$$

Proof. Since the region $G_i - \bar{G}_{c_i}$ belongs to the class SO_{HB},

$$\int_{\gamma_i} \frac{\partial u}{\partial n} ds = \int_{c_i} \frac{\partial u}{\partial n} ds$$

for all bounded harmonic functions u on W.

Let u be an arbitrary harmonic function on W satisfying $0 < u < 1$. Since

$$h_{\Omega_\nu}(\gamma_i) = \int_{\gamma_i} \frac{\partial \omega_i^{(\nu)}}{\partial n} ds$$

[8], we have

$$\int_{c_i} \frac{\partial u}{\partial n} ds = \int_{\gamma_i} \frac{\partial u}{\partial n} ds \leqq \int_{\gamma_i} \frac{\partial \omega_i^{(\nu)}}{\partial n} ds \qquad \text{for all } \nu.$$

Hence it follows that

$$\int_{c_i} \frac{\partial u}{\partial n} ds = \int_{\gamma_i} \frac{\partial u}{\partial n} ds \leqq \lim_{\nu \to \infty} \int_{\gamma_i} \frac{\partial \omega_i^{(\nu)}}{\partial n} ds = \int_{\gamma_i} \frac{\partial \omega_{B_i}}{\partial n} ds = \int_{c_i} \frac{\partial \omega_{B_i}}{\partial n} ds.$$

This implies that

$$h_W(c_i) = h_W(\gamma_i) = \int_{\gamma_i} \frac{\partial \omega_{B_i}}{\partial n} ds.$$

LEMMA 7. *If u is a positive harmonic function on W which converges to* 0 *on every sequence of points on which ω_{B_i} converges to* 1, *then*

$$\int_{\gamma_i} \frac{\partial u}{\partial n} ds < 0.$$

Proof. Let u_ν be the harmonic function on $\Omega_\nu \cap G_i$ whose boundary values are u on γ_i and 0 on $\partial\Omega_\nu \cap G_i$. Then we can verify that $u = \lim_{\nu\to\infty} u_\nu$. On the other hand, since $u_\nu - \min_{\gamma_i} u(1-\tilde{\omega}_i^{(\nu)})$ is positive on $\Omega_\nu \cap G_i$ and vanishes on $\partial\Omega_\nu \cap G_i$,

$$\int_c \frac{\partial u_\nu}{\partial n} ds + \min_{\gamma_i} u \int_c \frac{\partial \tilde{\omega}_i^{(\nu)}}{\partial n} ds = \int_{\partial\Omega_\nu \cap G_i} \frac{\partial}{\partial n} \{u_\nu - \min_{\gamma_i} u(1-\tilde{\omega}_i^{(\nu)})\} ds < 0$$

for all ν, where c is a regular analytic closed Jordan curve in G_i being homologous to γ_i. Hence we have

$$\int_{\gamma_i} \frac{\partial u}{\partial n} ds = \int_c \frac{\partial u}{\partial n} ds = \lim_{\nu\to\infty} \int_c \frac{\partial u_\nu}{\partial n} ds \leqq -\min_{\gamma_i} u \cdot \lim_{\nu\to\infty} \int_c \frac{\partial \tilde{\omega}_i^{(\nu)}}{\partial n} ds$$

$$= -\min_{\gamma_i} u \cdot \int_c \frac{\partial \omega_{\tilde{B}_i}}{\partial n} ds = -\min_{\gamma_i} u \int_{\gamma_i} \frac{\partial \omega_{\tilde{B}_i}}{\partial n} ds < 0.$$

This is the desired result.

5. Now we turn to the study of global properties of analytic mappings of W into itself.

We shall first prove the following

LEMMA 8. *Let φ be a non-constant analytic mapping of W into itself and let $h_W(\gamma_{i_0}) = \min_{1\leqq i\leqq m} \{h_W(\gamma_i)\}$. Then $\omega_{B_{j_0}} \circ \varphi(p) = \omega_{B_{i_0}}(p)$ for an integer j_0 $(1 \leqq j_0 \leqq m)$.*

Proof. By Lemma 5 there exists an admitted union c_{i_0} of curves associated with α_{i_0} such that $\varphi(c_{i_0}) = kc_{j_0}$, where c_{j_0} is an admitted union of curves associated with α_{j_0} $(1 \leqq j_0 \leqq m)$. Since $h_W(c_{i_0}) \geqq h_W(\varphi(c_{i_0}))$ [8], we have

$$h_W(c_{i_0}) \geqq h_W(kc_{j_0}) = kh_W(c_{j_0})$$

and hence by Lemma 6

$$h_W(\gamma_{i_0}) = h_W(c_{i_0}) \geqq kh_W(c_{j_0}) = kh_W(\gamma_{j_0}) \geqq kh_W(\gamma_{i_0}).$$

On the other hand by Lemmas 6 and 7

$$h_W(\gamma_{i_0}) = \int_{\gamma_{i_0}} \frac{\partial \omega_{B_{i_0}}}{\partial n} ds > 0.$$

Then it follows that $\varphi(c_{i_0})=c_{j_0}$ and

$$\int_{c_{i_0}}\frac{\partial\omega_{B_{i_0}}}{\partial n}ds=\int_{c_{j_0}}\frac{\partial\omega_{B_{j_0}}}{\partial n}ds.$$

Hence we obtain

$$\int_{\gamma_{i_0}}\frac{\partial}{\partial n}(\omega_{B_{j_0}}\circ\varphi-\omega_{B_{i_0}})ds=\int_{c_{i_0}}\frac{\partial}{\partial n}(\omega_{B_{j_0}}\circ\varphi-\omega_{B_{i_0}})ds$$

$$=\int_{c_{j_0}}\frac{\partial\omega_{B_{j_0}}}{\partial n}ds-\int_{c_{i_0}}\frac{\partial\omega_{B_{i_0}}}{\partial n}ds=0.$$

Moreover, since $\omega_{B_{j_0}}\circ\varphi\geqq\omega_{B_{i_0}}$ it follows that $\omega_{B_{j_0}}\circ\varphi-\omega_{B_{i_0}}$ is not negative and converges to 0 on every sequence of points on which $\omega_{B_{i_0}}$ converges to 1. Consequently, by Lemma 7, we conclude that $\omega_{B_{j_0}}\circ\varphi=\omega_{B_{i_0}}$.

This lemma allows us to infer the following

LEMMA 9. *If φ is a non-constant analytic mapping of W into itself, then φ is univalent and $W-\varphi(W)$ is a closed set of capacity zero.*

Proof. First we shall see that φ is of type Bl. Let K be an arbitrary relatively compact subregion of W and let K' be a component of $\varphi^{-1}(K)$. Assume that K' does not belong to the class SO_{HB}. There exists a positive harmonic function v on K' vanishing continuously on $\partial K'$ and satisfying $\sup_{K'}v=1$. By the same argument as in the proof of Lemma 5, we can find admitted unions c_{i_0} and c_{j_0} $(1\leqq i_0, j_0\leqq m)$ of curves associated with α_{i_0} and α_{j_0} respectively satisfying: (1) φ maps $G_{c_{i_0}}$ into $G_{c_{j_0}}$, (2) $K\cap G_{c_{j_0}}=\phi$, where $h_W(\gamma_{i_0})=\min_{1\leqq i\leqq m}\{h_W(\gamma_i)\}$. Since $K'\cap G_{c_{i_0}}=\phi$, $1-v \geqq\lim_{\nu\to\infty}\omega^{(\nu)}_{c i_0}=\omega_{B_{i_0}}$ on K'. Hence we have $\inf_{K'}\omega_{B_{i_0}}=0$. Let $\{q_\mu\}_{\mu=1}^{\infty}$ be a sequence of points in K' such that $\lim_{\mu\to\infty}\omega_{B_{i_0}}(q_\mu)=0$ and the sequence $\{\varphi(q_\mu)\}_{\mu=1}^{\infty}$ converges to a point $q_0\in K$. By Lemma 8

$$\omega_{B_{j_0}}(q_0)=\lim_{\mu\to\infty}\omega_{B_{j_0}}\circ\varphi(q_\mu)=\lim_{\mu\to\infty}\omega_{B_{i_0}}(q_\mu)=0.$$

This is a contradiction, whence follows that φ is of type Bl.

Using the Lindelöf principle and Lemma 8 we can prove, in the same way as in the proof of Lemma 4, that every point of W is covered at most finitely often by φ.

Hence it follows that $\nu_\varphi(p)=\nu_0$ save for a closed set of capacity zero, where ν_φ denotes the valence of φ and ν_0 is a positive integer.

Now we shall see that $\nu_0=1$. Let $h_W(\gamma_{i_0})=\min_{1\leqq i\leqq m}\{h_W(\gamma_i)\}$ and let c_{i_0} and c_{j_0} be admitted unions of curves associated with α_{i_0} and α_{j_0} respectively obtained in

Lemma 5, so that, the restriction of φ to $G_{c_{i_0}}$ is an analytic mapping of $G_{c_{i_0}}$ into $G_{c_{j_0}}$ of type Bl. Let Δ be an arbitrary component of $\varphi^{-1}(G_{c_{j_0}})$, then the restriction φ_Δ of φ to Δ is an analytic mapping of Δ into $G_{c_{j_0}}$ of type Bl and $\nu_{\varphi_\Delta}\leqq\nu_0$. Hence it follows that Δ contains n'_{j_0} $(\geqq n_{j_0})$ HB-indivisible sets on its ideal boundary [3]. By the same reasoning each component of $\varphi^{-1}(G_i)$ $(1\leqq i\leqq m,\ i\neq j_0)$ contains n'_i $(\geqq n_i)$ HB-indivisible sets on its ideal boundary. Moreover the sets $\varphi^{-1}(G_{c_{j_0}})$ and $\varphi^{-1}(G_i)$ $(1\leqq i\leqq m,\ i\neq j_0)$ are mutually disjoint. This implies that W contains at least $\sum_{i=1}^m l_i n_i$ $(\geqq\sum_{i=1}^m n_i=n)$ HB-indivisible sets on its ideal boundary, where l_{j_0} and l_i are the numbers of the components of $\varphi^{-1}(G_{c_{j_0}})$ and $\varphi^{-1}(G_i)$ respectively. Hence each l_i $(1\leqq i\leqq m)$ must be equal to 1, whence $\varphi^{-1}(G_{c_{j_0}})=G_{c_{i_0}}$. Consequently we obtain $\varphi(c_{i_0})=\nu_0 c_{j_0}$. By the inequality

$$h_W(\gamma_{i_0})=h_W(c_{i_0})\geqq h_W(\varphi(c_{i_0}))=h_W(\nu_0 c_{j_0})=\nu_0 h_W(c_{j_0})=\nu_0 h_W(\gamma_{j_0})\geqq\nu_0 h_W(\gamma_{i_0})$$

we obtain $\nu_0=1$. This completes the proof of the lemma.

Finally we shall prove two theorems which lead us to our main result.

THEOREM 1. *The number of conformal automorphisms of W onto itself is finite.*

Proof. Assume that there exist infinitely many distinct conformal automorphisms of W onto itself, $\{\varphi^{(k)}\}_{k=1}^\infty$. Let $h_W(\gamma_{i_0})=\min_{1\leqq i\leqq m}\{h_W(\gamma_i)\}$. By Lemma 8 we may assume that

$$\omega_{B_{j_0}}\circ\varphi^{(k)}=\omega_{B_{i_0}} \qquad \text{for all } k.$$

Put $\psi^{(k)}=\varphi_{-1}^{(1)}\circ\varphi^{(k)}$, where $\varphi_{-1}^{(1)}$ denotes the inverse mapping of $\varphi^{(1)}$, then

$$\omega_{B_{i_0}}\circ\psi^{(k)}=\omega_{B_{i_0}} \qquad \text{for all } k.$$

Since $\{\psi^{(k)}\}_{k=1}^\infty$ are also infinitely many distinct conformal automorphisms of W onto inself, there exists an integer k such that $\psi^{(k)}(\gamma_{i_0})\cap\gamma_{i_0}=\phi$ [6]. Let $\{p_\mu\}_{\mu=1}^\infty$ be a sequence of points such that $\lim_{\mu\to\infty}\omega_{B_{i_0}}(p_\mu)=1$, then $\lim_{\mu\to\infty}\omega_{B_{i_0}}\circ\psi^{(k)}(p_\mu)=1$ and hence $\{p_\mu\}_{\mu=1}^\infty$ and $\{\psi^{(k)}(p_\mu)\}_{\mu=1}^\infty$ tend to α_{i_0} by Lemma 2. This implies that either $\psi^{(k)}(G_{i_0})\subsetneqq G_{i_0}$ or $G_{i_0}\subsetneqq\psi^{(k)}(G_{i_0})$. If $G_{i_0}\subsetneqq\psi^{(k)}(G_{i_0})$, $\psi_{-1}^{(k)}(G_{i_0})\subsetneqq G_{i_0}$. Consequently there exists a conformal automorphism φ of W onto itself having the following properties: (i) $\omega_{B_{i_0}}\circ\varphi=\omega_{B_{i_0}}$, (ii) $\varphi(G_{i_0})\subsetneqq G_{i_0}$ and (iii) if $\lim_{\mu\to\infty}\omega_{B_{i_0}}(p_\mu)=1$, then $\{\varphi_n(p_\mu)\}_{\mu=1}^\infty$ tends to α_{i_0} for all n (this is a consequence of (i)). Here φ_n denotes the n-th iterate of φ. By the properties (ii) and (iii), $\{\varphi_n(G_{i_0})\}_{n=1}^\infty$ is a defining sequence of α_{i_0}. We note that $\{\varphi_n(\gamma_{i_0})\}_{n=1}^\infty$ tends to the ideal boundary of W. Since the region $G_{i_0}-\overline{\varphi_n(G_{i_0})}$ belongs to the class SO_{HB},

$$\omega_{B_{i_0}}\leqq\max_{\gamma_{i_0}\cup\varphi_n(\gamma_{i_0})}\omega_{B_{i_0}} \qquad \text{on } G_{i_0}-\overline{\varphi_n(G_{i_0})} \quad \text{for all } n.$$

On the other hand by the property (i)

$$\max_{r_{i_0}} \omega_{B_{i_0}} = \max_{\varphi_n(r_{i_0})} \omega_{B_{i_0}}.$$

Hence we have

$$\omega_{B_{i_0}} \leqq \max_{r_{i_0}} \omega_{B_{i_0}} \quad \text{on} \quad G_{i_0} - \overline{\varphi_n(G_{i_0})} \quad \text{for all } n,$$

whence follows

$$\omega_{B_{i_0}} \leqq \max_{r_{i_0}} \omega_{B_{i_0}} \quad \text{on} \quad G_{i_0}.$$

This is a contradiction.

THEOREM 2. *If φ is a non-constant analytic mapping of W into itself, then φ is a conformal automorphism of W onto itself having a finite period.*

Proof. Assume that the set $W-\varphi(W)$ is not empty. By Lemma 9 φ is univalent and $W-\varphi(W)$ is a closed set of capacity zero. Hence there exists a Riemann surface W_1 such that $W_1 \supsetneqq W$, and φ is extended to an analytic mapping $\varphi^{(1)}$ of W_1 into itself which maps W_1, W_1-W topologically onto $W, W-\varphi(W)$ respectively. Again, since $\varphi^{(1)}$ is univalent and $W_1-W=W_1-\varphi^{(1)}(W_1)$ is a non-empty closed set of capacity zero, there exists a Riemann surface W_2 such that $W_2 \supsetneqq W_1$, and $\varphi^{(1)}$ is extended to an analytic mapping $\varphi^{(2)}$ of W_2 into itself which maps W_2, W_2-W_1 topologically onto W_1, W_1-W respectively. Repeating this argument we obtain a sequence $\{W_k\}_{k=1}^{\infty}$ of Riemann surfaces and a sequence $\{\varphi^{(k)}\}_{k=1}^{\infty}$ of analytic mappings satisfying: (i) $W_{k-1} \subsetneqq W_k$ for all k, (ii) $\varphi^{(k)}$ is an analytic mapping of W_k into itself which maps W_k, W_k-W_{k-1} topologically onto $W_{k-1}, W_{k-1}-W_{k-2}$ respectively, where $W_0=W$. We put $W^*=\cup_{k=1}^{\infty} W_k$, and $\varphi^*(p)=\varphi(p)$ for $p \in W$, $=\varphi^{(k)}(p)$ for $p \in W_k-W_{k-1}$. Then φ^* is a conformal automorphism of W^* onto itself and $\{\varphi_n^*\}$ are distinct, where φ_n^* denotes the n-th iterate of φ^*. On the other hand, since W^*-W is a closed set of capacity zero, every bounded minimal harmonic function on W is extended to a bounded minimal harmonic function on W^*. Conversely, the restriction of a bounded minimal harmonic function on W^* to W is a bounded minimal harmonic function on W. Further, let G_i^* be the component of $W^*-\Omega_1$ containing G_i, and let B_i^* be the set of all ideal boundary points of W^* which belong to one of the ideal boundary components on the part of the ideal boundary of W^* belonging to G_i^*. Then we can verify that B_i is of positive measure if and only if B_i^* is of positive measure, and that the harmonic measure of B_i^* is equal to the extension of the harmonic measure of B_i. Hence it follows that W^* is also an admitted surface. This contradicts Theorem 1. Consequently φ must be a conformal automorphism of W onto itself. Moreover, by Theorem 1 φ has a finite period.

Summing up these two theorems we have our main result.

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