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A REMARK ON COMPLEX HYPERSURFACES OF COMPLEX PROJECTIVE SPACE

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§ 1. Introduction.

In his dissertation, Smyth [2] has proved the following theorem:

Let M be a complete Einstein-Kaehler hypersurface of the complex projective space $P_{n+1}(\boldsymbol{C})$ for $n \geqq 2$. Then M is either a complex hyperplane $P_n(\boldsymbol{C})$ or a complex quadric in $P_{n+1}(\boldsymbol{C})$.

In the final step, his method depends on Cartan's classification of irreducible hermitian symmetric spaces.

The purpose of this note is to give an algebraic geometrical proof of the above theorem. For notations and terminologies, we refer to [1] and [2].

§ 2. Proof of Theorem.

Let M be a complete Einstein-Kaehler hypersurface of $P_{n+1}(\boldsymbol{C})$ with the induced metric $g=2\sum g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ and the fundamental 2-form $\Phi=(2/\sqrt{-1})\sum g_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$. Since M is complete with respect to the induced metric, it is compact. Hence, by the well known theorem of Chow, M is algebraic. The first Chern class $c_1(M)$ of M is represented by the closed 2-form

$$\gamma=\frac{1}{2\pi\sqrt{-1}}\sum R_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta,$$

where $R_{\alpha\bar{\beta}}$ denotes the component of the Ricci tensor of M. We denote by $[\Phi]$ and $[\gamma]$ the cohomology classes represented by Φ and γ, respectively, so that $c_1(M)=[\gamma]$.

Let ρ be the Ricci curvature of M so that $R_{\alpha\bar{\beta}}=\rho g_{\alpha\bar{\beta}}$. Then $\gamma=(\rho/4\pi)\Phi$ and hence

$$c_1(M)=\frac{\rho}{4\pi}[\Phi]. \tag{1}$$

Let h be the generator of $H^2(P_{n+1}(\boldsymbol{C}), \boldsymbol{Z})$ corresponding to the divisor class of a hyperplane $P_n(\boldsymbol{C})$. Then the first Chern class $c_1(P_{n+1}(\boldsymbol{C}))$ of $P_{n+1}(\boldsymbol{C})$ is given by

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$$c_1(P_{n+1}(\boldsymbol{C}))=(n+2)h. \tag{2}$$

Let d be the degree of the algebraic manifold M. Then the first Chern class $c_1(M)$ of M is given by

$$c_1(M)=(n-d+2)\tilde{h}, \tag{3}$$

where $\tilde{h}$ is the image of h under the homomorphism j^*: $H^2(P_{n+1}(\boldsymbol{C}), \boldsymbol{Z})\to H^2(M, \boldsymbol{Z})$ induced by the imbedding j: $M\to P_{n+1}(\boldsymbol{C})$.

Let Ψ be the fundamental 2-form of $P_{n+1}(\boldsymbol{C})$. Since $\dim H^2(P_{n+1}(\boldsymbol{C}), \boldsymbol{R})=1$, it follows that $[\Psi]=ah$, where a is a constant. Since $\Phi=j^*\Psi$, we have

$$[\Phi]=a\tilde{h}. \tag{4}$$

From (1), (3) and (4) we have

$$4\pi(n-d+2)=\rho a. \tag{5}$$

Let $\nu(M)$ be the normal bundle of $j(M)$ in $P_{n+1}(\boldsymbol{C})$. Then the first Chern class $c_1(\nu(M))$ of $\nu(M)$ is given by

$$c_1(\nu(M))=d\tilde{h}. \tag{6}$$

By Proposition 4 in [2], we have $2[ds]=4\pi c_1(M)-4\pi j^*c_1(P_{n+1}(\boldsymbol{C}))$. On the other hand, since $j^*T(P_{n+1}(\boldsymbol{C}))=T(M)\oplus\nu(M)$ (Whitney sum), we have $c_1(\nu(M))=j^*c_1(P_{n+1}(\boldsymbol{C}))-c_1(M)$. Hence we have

$$c_1(\nu(M))=-\frac{1}{2\pi}[ds]. \tag{7}$$

Let S and $\tilde{S}$ denote the Ricci tensors of M and $P_{n+1}(\boldsymbol{C})$ respectively. Since $S=\rho g$ and $j^*\tilde{S}=((n+2)/2)g$, we have, from Proposition 4 in [2],

$$2[ds]=\left(\rho-\frac{n+2}{2}\right)[\Phi]. \tag{8}$$

From (4), (6), (7) and (8) we have

$$4\pi d=\left(\frac{n+2}{2}-\rho\right)a. \tag{9}$$

The equations (5) and (9) imply

$$d=n+2-2\rho.$$

On the other hand, Takahashi [3] has proved that $\rho=(n+1)/2$ or $\rho=n/2$ according as M is totally geodesic or not. Hence we have $d=1$ or 2. This completes the proof.

Bibliography

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