KŌDAI MATH. SEM. RFP.
20 (1968), 448–453

ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

By H. P. Dikshit

1. Definitions and notations. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write $P_n=p_0+p_1+\cdots+p_n$, $P_{-1}=p_{-1}=0$. The sequence-to-sequence transformation:

$$t_n=\sum_{\nu=0}^{n} p_{n-\nu}s_\nu/P_n, \qquad (P_n\neq 0) \tag{1. 1}$$

defines the sequence $\{t_n\}$ of Nörlund means[1] of $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. If $\{t_n\}\in BV$, i.e., $\sum_n|t_n-t_{n-1}|<\infty$,[2] we say that $\sum a_n$ or $\{s_n\}$ is summable $|N, p_n|$.[3]

In the special case in which $p_n=\binom{n+\alpha-1}{\alpha-1}$, $\alpha>-1$, the (N, p_n) mean reduces to familiar (C, α) mean.

Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense over $(-\pi, \pi)$ and let the Fourier series of $f(t)$ be

$$\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt)=\sum_{n=1}^{\infty}A_n(t). \tag{1. 2}$$

Then the conjugate series of (1. 2) is

$$\sum_{n=1}^{\infty}(b_n \cos nt-a_n \sin nt)=\sum_{n=1}^{\infty}B_n(t). \tag{1. 3}$$

We write throughout:

$$\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}, \qquad \psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\};$$

$$R_n=(n+1)p_n/P_n; \qquad S_n=\sum_{\nu=0}^{n}(\nu+1)^{-1}P_\nu/P_n;$$

$$P_n^*=\sum_{\nu=0}^{n}|p_\nu|; \qquad S_n^*=\sum_{\nu=0}^{n}(\nu+1)^{-1}|P_\nu|/|P_n|;$$

Received April 1, 1968.

1) Nörlund [6]. See also Woronoi [14].

2) Similarly by '$F(x)\in BV(a, b)$', we mean that $F(x)$ is a function of bounded variation in the interval (a, b) and '$\{\mu_n\}\in B$' means that $\{\mu_n\}$ is a bounded sequence.

3) Mears [5].

$$V_n=|P_n^{-1}|\sum_{\nu=1}^{n}\nu|p_\nu-p_{\nu-1}|; \qquad \Delta f_n=f_n-f_{n+1};$$

$\tau=[1/t]$, i.e., the greatest integer contained in $1/t$. K denotes a positive constant, not necessarily the same at each occurrence.

2. Introduction. Generalising the classical results of Bosanquet [1] and Bosanquet and Hyslop [2] on $|C|$ summability of Fourier series and its conjugate series respectively, Pati has proved the following results.

THEOREM A.[4] *Let $\{p_n\}$ be a positive monotonic sequence such that $P_n\to\infty$ as $n\to\infty$, and $\{R_n\}\in BV$, $\{S_n\}\in BV$.*

i) *If $\varphi(t)\in BV(0,\pi)$, then the Fourier series of $f(t)$, at $t=x$, is summable $|N,p_n|$.*

ii) *If*

$$\psi(t)\in BV(0,\pi) \quad \text{and} \quad \int_0^\pi t^{-1}|\psi(t)|dt\leqq K, \tag{2.1}$$

then the conjugate series of the Fourier series of $f(t)$, at $t=x$, is summable $|N,p_n|$.

Generalisations of the results (i) and (ii) of Theorem A by dropping the monotonicity of $\{p_n\}$ and replacing the hypothesis: $\{S_n\}\in BV$ by the equivalent hypothesis: $P_n\sum_{\nu=n}^{\infty}\{(\nu+1)P_\nu\}^{-1}\leqq K$, has been effected recently, in the form of the following.

THEOREM B.[5] *Let $\{p_n\}$ be a positive sequence such that $\{R_n\}\in BV$ and $P_n\sum_{\nu=n}^{\infty}\{(\nu+1)P_\nu\}^{-1}\leqq K$.*

(i) *If $\varphi(t)\in BV(0,\pi)$ and $P_n\to\infty$ as $n\to\infty$, then the Fourier series of $f(t)$, at $t=x$, is summable $|N,p_n|$.*

(ii) *If (2.1) holds, then the conjugate series of the Fourier series of $f(t)$, at $t=x$, is summable $|N,p_n|$.*

In [10], Pati has shown that it is possible to modify his original proof of Theorem A to do away with the monotonicity of $\{p_n\}$, and thus obtained a theorem equivalent to Theorem B. He has also pointed out in [10] that one need not use $P_n\to\infty$ as $n\to\infty$ in Theorem B. Shorter proofs of these theorems are due to Pati and Dikshit [11] and Dikshit [3].

Very recently Si-Lei has obtained the following results by replacing the hypothesis: '$\{p_n\}$ is a positive monotonic sequence' of Theorem A by the lighter hypothesis: '$\{p_n\}$ is any sequence such that $P_n^*=O(|P_n|)$ and $V_n=O(1)$'.

THEOREM C.[6] *Let $\{p_n\}$ be any sequence such that $P_n^*=O(|P_n|)$, $V_n=O(1)$, $\{R_n\}\in BV$ and $\{S_n\}\in BV$.*

(i) *If $\varphi(t)\in BV(0,\pi)$, then the Fourier series of $f(t)$, at $t=x$, is summable*

4) Pati [7], [8] and [9].

5) The result (i) of Theorem B is due to Varshney [13] while the result (ii) is due to Dikshit [4]. The equivalence of the two hypotheses has been demonstrated in [10].

6) Si-Lei [12].

$|N, p_n|$.

(ii) *If* (2.1) *holds, then the conjugate series of the Fourier series of* $f(t)$, *at* $t=x$, *is summable* $|N, p_n|$.

Theorem A is a special case of Theorem C, since the additional hypotheses of Theorem C are automatically satisfied whenever $\{p_n\}$ is a positive monotonic sequence.

The object of the present paper is to show that it is, indeed, possible to drop the hypothesis: $V_n=O(1)$ in Theorem C, by following a shorter and more direct method of proof. We in fact prove the following theorem, which contains as special cases Theorem A and Theorem B.

THEOREM 1. *Let* $\{p_n\}$ *be any sequence such that* $P_n^*=O(|P_n|)$, $\{R_n\}\in BV$ *and* $\{S_n\}\in BV$.

(i) *If* $\varphi(t)\in BV(0, \pi)$, *then the Fourier series of* $f(t)$, *at* $t=x$, *is summable* $|N, p_n|$.

(ii) *If* (2.1) *holds, then the conjugate series of the Fourier series of* $f(t)$, *at* $t=x$, *is summable* $|N, p_n|$.

We require the following lemmas for the proof of our Theorem 1.

LEMMA 1. *If* $0\leqq\nu\leqq n$ *and* $\{R_n\}\in BV$, *then*

$$(n+1)\left|\sum_{k=0}^{\nu} p_k c_{n,k}(t)\right|\leqq KP_\nu^*,$$

where $c_{n,k}(t)=\{\sin (n-k)t\}/(n-k)$.

Proof. We have

$$(n+1)\left|\sum_{k=0}^{\nu} p_k c_{n,k}(t)\right|=\left|\sum_{k=0}^{\nu} p_k \frac{n-k+k+1}{n-k}\sin (n-k)t\right|$$

$$\leqq \sum_{k=0}^{\nu}|p_k|+\left|\sum_{k=0}^{\nu} R_k P_k c_{n,k}(t)\right|.$$

Now by Abel's transformation

$$\sum_{k=0}^{\nu} R_k P_k c_{n,k}(t)=\sum_{k=0}^{\nu-1}\{P_k\Delta R_k-p_{k+1}R_{k+1}\}\sum_{\mu=0}^{k} c_{n,\mu}(t)+R_\nu P_\nu\sum_{\mu=0}^{\nu} c_{n,\mu}(t).$$

Therefore, since $\sum_{\nu=1}^{n}(\sin \nu t)/\nu=O(1)$ and $\{R_n\}\in B$,

$$\left|\sum_{k=0}^{\nu} R_k P_k c_{n,k}(t)\right|\leqq KP_\nu^*|\Delta R_k|+K\sum_{k=0}^{\nu-1}|p_{k+1}|+K|P_\nu|$$

$$\leqq KP_\nu^*,$$

since $\{R_n\}\in BV$ by the hypothesis. This completes the proof of our Lemma 1.

LEMMA 2. *For any sequence* $\{p_n\}$ *such that* $P_n^*=O(|P_n|)$, $\{S_n\}\in BV$ *implies* $\{S_n^*\}\in B$.

Proof. We write

$$S_n^*=\frac{1}{|P_n|}\sum_{\nu=1}^{n}|(\nu+1)^{-1}P_\nu|+|P_0|/|P_n|$$

$$\leqq\frac{1}{|P_n|}\sum_{\nu=1}^{n}|P_\nu S_\nu-P_{\nu-1}S_{\nu-1}|+K$$

$$\leqq\frac{1}{|P_n|}\sum_{\nu=1}^{n}|P_{\nu-1}||\Delta S_{\nu-1}|+\frac{1}{|P_n|}\sum_{\nu=1}^{n}|p_\nu||S_\nu|+K$$

$$\leqq\frac{P_n^*}{|P_n|}\sum_{\nu=1}^{n}|\Delta S_{\nu-1}|+K\leqq K,$$

since $\{S_n\}\in BV$ and $P_n^*=O(|P_n|)$, by the hypotheses of the Lemma

3. Proof of Theorem 1. In his proof of Theorem C, Si-Lei has used the hypothesis $V_n=O(1)$, in showing that,

$$\Sigma_{21}\equiv\sum_{n=1}^{\tau}\left|\frac{P_n}{P_nP_{n-1}}\sum_{k=0}^{n-1}(k+1)^{-1}P_k\sin(n-k)t\right|\leqq K, \tag{3.1}$$

and

$$t\sum_{n=1}^{\tau+1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-1}(k+1)^{-1}P_k(n-k)\sin(n-k)t\right|\leqq K, \tag{3.2}$$

and also in the proof of a Lemma [12, Lemma 2], which is required for his results $\Sigma_{11}\leqq K$, $\Sigma_{12}\leqq K$ and

$$\Sigma_1\equiv\sum_{n=1}^{\infty}\left|\frac{(n+1)}{P_nP_{n-1}}\sum_{k=0}^{n-1}\left(\frac{P_n}{n+1}p_k-p_n\frac{P_k}{k+1}\right)\int_0^{\pi}\phi(t)\sin(n-k)t\,dt\right|\leqq K,$$

([12], pp. 284–286 and pp. 291–292, p. 287 (3.9), p. 288 (3.10) and p. 290 respectively).

Thus, in order to prove our theorem, it is sufficient to prove (3.1), (3.2) and the Lemma 2 of Si-Lei [12], without using the hypothesis $V_n=O(1)$.

Now, since $|\sin(n-k)t|\leqq nt$ and $\{R_n\}\in B$.

$$\Sigma_{21}\leqq t\sum_{n=1}^{\tau}\left|\frac{np_n}{P_n}\right|S_{n-1}^*\leqq Kt\sum_{n=1}^{\tau}1\leqq K,$$

by Lemma 2.

Similarly,

$$t\sum_{n=1}^{\tau+1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-1}(k+1)^{-1}P_k\sin(n-k)t(n-k)\right|$$

$$\leqq t^2\sum_{n=1}^{\tau+1}n|R_n|S_{n-1}^*\leqq Kt^2\sum_{n=1}^{\tau+1}n\leqq K,$$

Since $\{R_n\}\in B$ and $\{S_n^*\}\in B$, by virtue of our Lemma 2.

In the form of our Lemma 1, we have already shown that it is possible to replace the hypothesis: $V_n=O(1)$ by the hypothesis: $\{R_n\}\in BV$ of Theorem C, in Si-Lei's Lemma 2 [12], in order to get precisely the same order estimate.

This completes the proof of our Theorem 1.

4 **Remarks**: It may be observed that the result of our Theorem 1 is essentially the same as that of the Theorem C of Si-Lei. This is due to the fact that the hypothesis: $V_n=O(1)$, which we have dropped from Si-Lei's theorem is ensured by the other hypotheses which are common to our Theorem 1 and Theorem C. This fact is brought out in the following theorem.

THEOREM 2. *If any sequence such that* $\{R_n\}\in BV$, $\{S_n\}\in BV$ *and* $P_n^*=O(|P_n|)$, *then* $V_n=O(1)$.

5. Proof of Theorem 2. We have

$$p_{\nu-1}-p_\nu=\Delta(p_{\nu-1})=\Delta(P_{\nu-1}R_{\nu-1}/\nu)$$

$$=\frac{P_{\nu-1}R_{\nu-1}}{\nu(\nu+1)}-\frac{R_{\nu-1}p_\nu}{(\nu+1)}+\frac{P_\nu}{\nu+1}\Delta(R_{\nu-1}).$$

Therefore

$$\sum_{\tau=1}^{n}\nu|p_{\nu-1}-p_\nu|\leqq\sum_{\tau=1}^{n}(\nu+1)^{-1}|P_{\nu-1}||R_{\nu-1}|+\sum_{\tau=1}^{n}|p_\nu||R_{\nu-1}|$$

$$+\sum_{\tau=1}^{n}|P_\nu||\Delta R_{\nu-1}|$$

$$\leqq K|P_{n-1}|S_{n-1}^*+KP_n^*+KP_n^*\sum_{\tau=1}^{n}|\Delta R_{\nu-1}|$$

$$\leqq KP_n^*\leqq K|P_n|,$$

since $\{S_n^*\}\in B$, by Lemma 2, $\{R_n\}\in BV$ and $P_n^*=O(|P_n|)$ by the hypotheses.

This completes the proof of our Theorem 2.

My grateful thanks are due to Dr. T. Pati of the University of Jabalpur, for his kind interest in the preparation of this paper.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ALLAHABAD, INDIA, AND
DEPARTMENT OF POST-GRADUATE STUDIES AND RESEARCH IN MATHEMATICS,
UNIVERSITY OF JABALPUR, INDIA.