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AN ELEMENTARY PROOF OF LOCAL MAXIMALITY FOR a_6

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1. In our previous paper [6] we proved the local maximality of $\Re a_6$ at the Koebe function. Our local maximality is that of Bombieri [1]. In the present paper we shall give another proof of local maximality for $\Re a_6$, which lies in an elementary level. Here we define that the proof is elementary if it depends only upon Golusin's and Grunsky's inequalities. In [6] we proved and made use of $y=O(x)$, $\eta=O(x)$, $\xi=O(x)$. These asymptotic equalities cannot be proved in any elementary method, we believe. We shall show that it is possible to avoid use of these asymptotic equalities. Another benefit of our method is that the expression becomes very small in contrast with that of [6]. Hence our method gives a possibility for the global consideration, that is, the Bieberbach conjecture for the sixth coefficient.

Our result may be formulated in the following manner:

Let $f(z)$ be a normalized regular function univalent in the unit circle

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Then

$$\Re a_6 \leqq 6-A(2-\Re a_2), \qquad A>0$$

holds for $0\leqq 2-\Re a_2<\varepsilon$. Equality occurs only for the Koebe function $z/(1-z)^2$.

We shall make use of the same notations as in [6].

2. Lemmas.

LEMMA 1. $\quad 7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)\leqq 4x-x^2-x'^2.$

By Lemma 1

$$x'=O(x^{1/2}), \qquad y'=O(x^{1/2}), \qquad \eta'=O(x^{1/2}), \qquad \xi'=O(x^{1/2}),$$
$$y=O(x^{1/2}), \qquad \eta=O(x^{1/2}).$$

In [6] we proved $y=O(x)$, $\eta=O(x)$ by a non-elementary method, so we cannot make use of this result.

LEMMA 2. $\quad \eta+(2\beta-1)y$

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$$\leqq(1+\beta^2)x+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}(y'+\beta x')^2$$

$$-\frac{3}{4}(\eta'+(\beta-1)y'+x')^2-\frac{5}{4}(\xi'+(\beta-1)\eta'+y')^2+O(x^{3/2}).$$

This was already proved in [6] and its proof depends only upon Golusin's inequality.

3. **Proof.** By Grunsky's inequality

$$\left|\sum_{\mu,\nu=1}^{m}\nu b_{\mu\nu}x_\mu x_\nu\right|\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2.$$

Here put $m=5$, $x_2=x_4=0$, $x_5=1/5$, $x_3=\beta/6$, $x_1=\delta$ with $\beta=2\Re a_2=2(2-x)$ and take the real part. Then we have

$$\Re a_6\leqq 6-(10-\delta^2)x+(11-4\delta)y+(5-2\delta)\eta$$

$$-12x'^2-\frac{29}{2}x'y'-\frac{7}{2}y'^2-7x'\eta'-3y'\eta'-2x'\xi'$$

$$+\frac{7}{2}y^2+3y\eta+O(x^{3/2}).$$

Here put $\delta=2+3y/2$. Then

$$\Re a_6\leqq 6-6x+\eta+3y-6y^2+\frac{7}{2}y^2$$

$$-12x'^2-\frac{29}{2}x'y'-\frac{7}{2}y'^2-7x'\eta'-3y'\eta'-2x'\xi'+O(x^{3/2}).$$

By Lemma 2 we have

$$\eta+3y\leqq 5x+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}(y'+2x')^2-\frac{3}{4}(\eta'+y'+x')^2$$

$$-\frac{5}{4}(\xi'+\eta'+y')^2+O(x^{3/2}).$$

Hence we have

$$\Re a_6\leqq 6-x-12x'^2-\frac{29}{2}x'y'-\frac{7}{2}y'^2-7x'\eta'-3y'\eta'-2x'\xi'$$

$$+\frac{1}{2}x'^2-\frac{1}{2}x'y'-\frac{1}{4}(y'+2x')^2-\frac{3}{4}(\eta'+y'+x')^2-\frac{5}{4}(\xi'+\eta'+y')^2$$

$$-\frac{5}{2}y^2+O(x^{3/2}).$$

By Lemma 1 we have

$$\Re a_6 \leqq 6 - F(x', y', \eta', \xi') - Q(y, \eta) + O(x^{3/2}),$$

$$4F(x', y', \eta', \xi') = 58x'^2 + 70x'y' + 26y'^2 + 34x'\eta' + 28y'\eta' + 13\eta'^2 + 8x'\xi' + 10y'\xi' + 10\eta'\xi' + 12\xi'^2,$$

$$Q(y, \eta) = \frac{13}{4} y^2 + \frac{5}{4} \eta^2.$$

It is easy to prove the positive definiteness of F and Q. By continuity we have the desired result.

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