

SOME INTEGRAL FORMULAS OF THE GAUSS-KRONECKER CURVATURE

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Let M^{2k} be a compact, oriented $2k$ -dimensional manifold and $f: M^{2k} \rightarrow E^{2k+1}$ be an immersion of M^{2k} into a $(2k+1)$ -dimensional Euclidean space, and let B_ν be the bundle of unit normal vectors of $f(M^{2k})$, and the mapping $\check{\nu}: B_\nu \rightarrow S_0^{2k}$ of B_ν into the unit sphere S_0^{2k} of E^{2k+1} , be defined by $\nu(p, e) = e$ for (p, e) in B_ν .

Let dV be the volume element of M^{2k} , $d\Sigma_{2k}$ be the volume element of S_0^{2k} . Then the integral:

$$T(f) = \int_{M^{2k}} |\check{\nu}^* d\Sigma_{2k}|$$

is called the *total curvature* of the immersion

$$(1) \quad f: M^{2k} \rightarrow E^{2k+1}.$$

Chern and Lashof [2] proves that $T(f)$ satisfies the following inequality:

$$(2) \quad T(f) \geq \left(\sum_{i=0}^{2k} \beta_i \right) c_{2k}$$

where c_{2k} denotes the volume of the unit sphere S_0^{2k} , and β_i is the i -th Betti number of M^{2k} . If the equality in (2) holds, then the immersion (1) is called a *minimal imbedding*.

Now, let N be the outer unit normal vector field on $f(M^{2k})$, and let

$$\eta: M^{2k} \rightarrow S_0^{2k}$$

be the sphere mapping defined by $\eta(p) = N(p)$, then the function $G(p)$ defined by

$$(3) \quad \eta^* d\Sigma_{2k} = G(p) dV,$$

where η^* is the dual mapping of η , is called the Gauss-Kronecker curvature of f .

The object of this note is to find some integral formulas for the Gauss-Kronecker curvature, and to prove that these integral formulas play a main role in the minimal imbedding of even-dimensional hypersurfaces in Euclidean spaces.

1. Some integral formulas for Gauss-Kronecker curvature.

THEOREM 1. *Let $f: M^{2k} \rightarrow E^{2k+1}$ be an immersion of an compact, oriented $2k$ -*

dimensional manifold in Euclidean space of dimension $(2k+1)$, and let $U=\{p \in M^{2k} : G(p) > 0\}$, and $V=\{p \in M^{2k} : G(p) < 0\}$, then

$$(4) \quad \int_U G(p) dV \cong \left(\sum_{i=0}^k \beta_{2i} \right) \frac{C_{2k}}{2},$$

and

$$(5) \quad \int_V G(p) dV \cong - \left(\sum_{i=1}^k \beta_{2i-1} \right) \frac{C_{2k}}{2}.$$

Proof. Let $\mathcal{G}(N)$ denote the index of the unit normal vector field N of M^{2k} then by the Hopf index theorem, we have

$$(6) \quad \mathcal{G}(N) = \frac{1}{C_{2k}} \int_{M^{2k}} \eta^* d\Sigma_{2k} = \frac{1}{2} \chi(M^{2k}),$$

where $\chi(M^{2k})$ denotes the Euler characteristic of M^{2k} . Therefore, we have

$$(7) \quad \int_{M^{2k}} G(p) dV = \frac{C_{2k}}{2} \chi(M^{2k}),$$

so that

$$(8) \quad \int_U G(p) dV + \int_V G(p) dV = \left(\sum_{i=0}^n (-1)^i \beta_i \right) \frac{C_{2k}}{2}.$$

On the other hand, by the definition of $G(p)$ and, we can easily prove that

$$|\tilde{\nu}^* d\Sigma_{2k}| = 2|G(p)| dV.$$

Therefore, by (2), we get

$$(9) \quad \int_{M^{2k}} |\tilde{\nu}^* d\Sigma_{2k}| = 2 \int_{M^{2k}} |G(p)| dV = \left(\sum_{i=0}^{2k} \beta_i \right) C_{2k},$$

hence, we have

$$(10) \quad \int_U G(p) dV - \int_V G(p) dV \cong \left(\sum_{i=0}^{2k} \beta_i \right) \frac{C_{2k}}{2}.$$

Combining (8) and (10), we can easily get the inequalities (4) and (5). This completes the proof of the theorem.

THEOREM 2. *Let $f: M^{2k} \rightarrow E^{2k+1}$ be given as in Theorem 1, then the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding if and only if*

$$(11) \quad \int_U G(p) dV = \left(\sum_{i=0}^k \beta_{2i} \right) \frac{C_{2k}}{2},$$

and

$$(12) \quad \int_V G(p) dV = - \left(\sum_{i=1}^k \beta_{2i-1} \right) \frac{C_{2k}}{2}.$$

Proof. If equalities (11) and (12) hold, then by (9), we have

$$T(f) = \int_{M^{2k}} |\tilde{\nu}^* d\Sigma_{2k}| = 2 \int_U G(p) dV - 2 \int_V G(p) dV = \left(\sum_{i=0}^k \beta_i \right) c_{2k}.$$

Therefore, the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding.

Conversely, if the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding, then we have

$$T(f) = \int_{M^{2k}} |\tilde{\nu}^* d\Sigma_{2k}| = \left(\sum_{i=0}^{2k} \beta_i \right) c_{2k},$$

that is

$$(13) \quad \int_U G(p) dV - \int_V G(p) dV = \left(\sum_{i=0}^{2k} \beta_i \right) \frac{c_{2k}}{2}.$$

Therefore, by (8) and (13), we can easily get equalities (11) and (12). This completes the proof of the theorem.

2. Some Applications.

In this section, I want to use Theorem 1 and 2 to prove the following theorem on the minimal imbedding of a even dimensional manifold with the odd dimensional Betti numbers vanishing.

THEOREM 3. *Let $f: M^{2k} \rightarrow E^{2k+1}$ be an immersion of a compact, oriented $2k$ -dimensional manifold M^{2k} in E^{2k+1} with the Gauss-Kronecker curvature $G(p) \geq 0$, then the odd dimensional Betti number of M^{2k} are zero, and the immersion f is a minimal imbedding.*

Conversely, if the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding of a compact, oriented $2k$ -dimensional manifold in E^{2k+1} , with the odd dimensional Betti numbers vanishing, then the Gauss-Kronecker curvature $G(p) \geq 0$ for all p in M^{2k} .

Proof. By the hypothesis, if $G(p) \geq 0$ for all p in M^{2k} , then we have $V = \phi$, and therefore by inequality (5), we have

$$\sum_{i=1}^k \beta_{2i-1} = 0,$$

so that

$$\beta_{2i-1} = 0 \quad \text{for } i=1, 2, \dots, k.$$

Furthermore, by the inequality (7), we have

$$\begin{aligned} T(f) &= \int_{M^{2k}} |\tilde{\nu}^* d\Sigma_{2k}| = 2 \int_U G(p) dV = 2 \int_{M^{2k}} G(p) dV \\ &= \chi(M^{2k}) c_{2k} = \left(\sum_{i=0}^{2k} \beta_i \right) c_{2k}, \end{aligned}$$

so that the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding.

Conversely, if the immersion $f: M^{2k} \rightarrow E^{2k+1}$ is a minimal imbedding and $\beta_{2i-1} = 0$ for all $i=1, 2, \dots, k$. Now, let us suppose that there exist some points in M^{2k} with

the Gauss-Kronecker curvature $G(p)$ negative, then by the continuity of $G(p)$ on M^{2k} , the set $V = \{p \in M^{2k} : G(p) < 0\}$ is a positive measure set on M^{2k} , therefore the integral

$$\int_V G(p) dV$$

is negative, hence by (9) and (11), we have

$$\begin{aligned} T(f) &= \int_{M^{2k}} |\mathfrak{Y}^* d\Sigma_{2k}| = 2 \int_U G(p) dV - 2 \int_V G(p) dV \\ &> 2 \int_U G(p) dV = \left(\sum_{i=0}^k \beta_{2i} \right) c_{2k} = \left(\sum_{i=0}^k \beta_i \right) c_{2k}. \end{aligned}$$

This is a contradiction, hence $G(p) \geq 0$ for all p in M^{2k} . This completes the proof of the theorem.

In the special case, when $k=1$, then the Gauss-Kronecker curvature $G(p)$ is just the Gaussian curvature $K(p)$, hence by the fact of $c_2=4\pi$, we have the following corollary:

COROLLARY. *If $f: M^2 \rightarrow E^3$ is an immersion of a compact, oriented surface in the 3-dimensional Euclidean space, then we have*

$$(14) \quad \int_U K(p) dV \geq 4\pi,$$

and

$$(15) \quad \int_V K(p) dV \leq -4\pi,$$

where $U = \{p \in M^2 : K(p) > 0\}$ and $V = \{p \in M^2 : K(p) < 0\}$.

The immersion $f: M^2 \rightarrow E^3$ is a minimal imbedding if and only if the equalities in (14) and (15) hold. In particular if M^2 is a sphere, then the immersion f is a minimal imbedding if and only if the Gaussian curvature of f is non-negative everywhere.

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