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ON CONTACT METRIC IMMERSION

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Introduction. The theory of complex submanifolds in a complex manifold is one of the most fruitful aspects in the study of complex manifold. In fact, after Schouten and Yano [6] established the notion of so-called invariant submanifold of a complex manifold many beautiful theorems concerning this have been proved.

On the other hand Sasaki [4] established a differential geometric method to study a contact manifold and this permits us to study contact manifold by use of tensor calculus. Making use of Sasaki's method, Watanabe [7] and the present author [2, 3] studied some submanifolds of a contact manifold.

However, in their papers, they observed rather Riemannian structures of the submanifold than contact metric structures.

In this paper, the author tries to establish a theory of submanifold which is inherited a contact metric structure by the enveloping contact metric manifold.

In § 1 we give first of all the definition of contact metric manifold and in § 2 a summary of theory of submanifolds of codimension 2 in a Riemannian manifold. These two paragraphs are rather expository. After these preliminaries we give in § 3 some formulas in a submanifold of codimension 2 in a contact metric manifold.

In § 4 we define the notion of contact metric immersion of a manifold into a contact metric manifold of codimension 2 and show conditions for an immersion to be a contact metric one.

Further in this paragraph we study the relations between a contact metric immersion and an immersion which is called F-invariant one.

In § 5 we define the notion of normal contact immersion of a manifold into a normal contact manifold of codimension 2 and prove conditions for an immersion to be a normal contact one.

Finally in § 6 we show an example which is an umbilical submanifold in normal contact manifold but not a normal contact submanifold.

§ 1. Contact metric manifold.

A $(2n+1)$-dimensional differentiable manifold $\tilde{M}^{2n+1}$ is said to have a contact structure and called a contact manifold if there exists a 1-form $\tilde{\eta}$ on $\tilde{M}^{2n+1}$ such that

$$\tilde{\eta}\wedge(d\tilde{\eta})^n\neq 0 \tag{1.1}$$

everywhere on $\tilde{M}^{2n+1}$ where $d\tilde{\eta}$ is the exterior derivative of $\tilde{\eta}$ and the symbol $\wedge$

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means the exterior multiplication. $\tilde{\eta}$ is called a contact form on $\tilde{M}^{2n+1}$.

Since (1. 1) means that the two-form $d\tilde{\eta}$ is of rank $2n$ everywhere on $\tilde{M}^{2n+1}$ we can find a unique vector field $\tilde{E}$ on $\tilde{M}^{2n+1}$ satisfying

$$\tilde{\eta}(\tilde{E})=1, \qquad d\tilde{\eta}(\tilde{E}, \tilde{X})=0 \tag{1. 2}$$

for an arbitrary $\tilde{X}\in T(\tilde{M}^{2n+1})$.

It is known[1] that there exists a positive definite Riemannian metric $\tilde{g}$ such that if we define a linear transformation F on $\tilde{M}^{2n+1}$ by

$$2\tilde{g}(F\tilde{X}, \tilde{Y})=d\tilde{\eta}(\tilde{X}, \tilde{Y}), \tag{1. 3}$$

then $(F, \tilde{E}, \tilde{\eta}, \tilde{g})$ satisfies, for arbitrary $\tilde{X}\in T(\tilde{M}^{2n+1})$,

$$F^2\tilde{X}=-\tilde{X}+\tilde{\eta}(\tilde{X})\tilde{E}, \tag{1. 4}$$

$$\tilde{\eta}(F\tilde{X})=0, \tag{1. 5}$$

$$\tilde{g}(\tilde{E}, \tilde{X})=\tilde{\eta}(\tilde{X}), \tag{1. 6}$$

$$\tilde{g}(F\tilde{X}, F\tilde{Y})=\tilde{g}(\tilde{X}, \tilde{Y})-\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}). \tag{1. 7}$$

The set $(F, \tilde{E}, \tilde{\eta}, \tilde{g})$ which satisfies (1. 1), (1. 2), (1. 3), (1. 4) and (1. 5) is called a contact metric (or Riemannian) structure and the manifold with such a structure is called a contact metric (or Riemannian) manifold. If in a contact metric manifold the tensor defined by

$$N(\tilde{X}, \tilde{Y})=[\tilde{X}, \tilde{Y}]+F[F\tilde{X}, \tilde{Y}]+F[\tilde{X}, F\tilde{Y}]-[F\tilde{X}, F\tilde{Y}]+(\tilde{Y}\tilde{\eta}(\tilde{X})-\tilde{X}\tilde{\eta}(\tilde{Y}))\tilde{E} \tag{1. 8}$$

vanishes everywhere, the structure is said to be normal and the manifold is called a normal contact manifold.

In a normal contact manifold the following identities hold[2] for arbitrary vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $\tilde{M}^{2n+1}$:

$$\tilde{\nabla}_{\tilde{X}}\tilde{E}=F\tilde{X}, \tag{1. 9}$$

$$(\tilde{\nabla}_{\tilde{Z}}d\tilde{\eta})(\tilde{X}, \tilde{Y})=2\{\tilde{\eta}(\tilde{X})\tilde{g}(\tilde{Y}, \tilde{Z})-\tilde{\eta}(\tilde{Y})\tilde{g}(\tilde{X}, \tilde{Z})\}, \tag{1. 10}$$

where $\tilde{\nabla}_{\tilde{Z}}$ denotes the covariant differentiation with respect to the Riemannian metric $\tilde{g}$ in the direction of $\tilde{Z}$.

From (1. 9) it follows that

$$\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{E}, \tilde{Y})+\tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{E}, \tilde{X})=0,$$

which shows that the vector field $\tilde{E}$ is an infinitesimal isometry.

Now let $\{y^\lambda\}$ be a local coordinate of $\tilde{M}^{2n+1}$ and U the coordinate neighborhood. Then the set of vector fields

$$\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{2n+1}}\right)$$

is called the natural frame of $\tilde{M}^{2n+1}$ and it spans the tangent plane of $\tilde{M}^{2n+1}$ at

1) Sasaki [4], Hatakeyama [1].

2) Sasaki and Hatakeyama [5].

each point of U. The dual basis of the natural frame is given by the set of 1-forms $(dy^1, \cdots, dy^{2n+1})$.

If we represent N by

$$N=\frac{1}{2}N_{\mu\lambda}{}^{\kappa}dy^{\mu}\otimes dy^{\lambda}\otimes\frac{\partial}{\partial y^{\kappa}},$$

it follows that

$$\begin{aligned} N_{\mu\lambda}{}^{\kappa}\frac{\partial}{\partial y^{\kappa}} &= N\left(\frac{\partial}{\partial y^{\mu}}, \frac{\partial}{\partial y^{\lambda}}\right) \\ &= \{F_{\mu}{}^{\nu}(\partial_{\nu}F_{\lambda}{}^{\kappa}-\partial_{\lambda}F_{\nu}{}^{\kappa})-F_{\lambda}{}^{\nu}(\partial_{\nu}F_{\mu}{}^{\kappa}-\partial_{\mu}F_{\nu}{}^{\kappa})+\partial_{\lambda}\tilde{E}^{\kappa}\tilde{\eta}_{\mu}-\partial_{\mu}\tilde{E}^{\kappa}\tilde{\eta}_{\lambda}\}\frac{\partial}{\partial y^{\kappa}} \end{aligned}$$

or using the Riemannian connection $\tilde{\nabla}$,

$$N_{\mu\lambda}{}^{\kappa}\frac{\partial}{\partial y^{\kappa}}=\{F_{\mu}{}^{\nu}(\tilde{\nabla}_{\nu}F_{\lambda}{}^{\kappa}-\tilde{\nabla}_{\lambda}F_{\nu}{}^{\kappa})-F_{\lambda}{}^{\nu}(\tilde{\nabla}_{\nu}F_{\mu}{}^{\kappa}-\tilde{\nabla}_{\mu}F_{\nu}{}^{\kappa})+\tilde{\nabla}_{\lambda}\tilde{E}^{\kappa}\tilde{\eta}_{\mu}-\tilde{\nabla}_{\mu}\tilde{E}^{\kappa}\tilde{\eta}_{\lambda}\}\frac{\partial}{\partial y^{\kappa}}, \tag{1.11}$$

where we have put

$$F\frac{\partial}{\partial x^{\lambda}}=F_{\lambda}{}^{\kappa}\frac{\partial}{\partial y^{\kappa}}, \tag{1.12}$$

$$\tilde{E}=\tilde{E}^{\kappa}\frac{\partial}{\partial y^{\kappa}}, \qquad \tilde{\eta}\left(\frac{\partial}{\partial y^{\kappa}}\right)=\tilde{\eta}_{\kappa}. \tag{1.13}$$

§ 2. Submanifold of codimension 2 in a Riemannian manifold.

Let M^m be an m-dimensional orientable differentiable manifold and ι be an immersion of M^m into an $m+2$-dimensional Riemannian manifold $\tilde{M}^{m+2}$. Then the Riemannian metric $\tilde{g}$ of $\tilde{M}^{m+2}$ induces naturally a Riemannian metric g on M^m by the immersion ι in such a way that

$$g(X, Y)=\tilde{g}(d\iota(X), d\iota(Y)),$$

where we denote by $d\iota$ the differential map of ι and by X, Y tangent vectors to M^m. In order to simplify the presentation we identify, for each point $p\in M^m$, the tangent space $T_p(M)$ with $d\iota(T_p(M))\subset T_{\iota(p)}(\tilde{M})$ by means of $d\iota$.

A vector in $T_{\iota(p)}(M)$ which is orthogonal, with respect to $\tilde{g}$, to the subspace $d\iota(T_p(M))$ is said to be normal to M^m at p. Since M^m is orientable, if we assume that $\tilde{M}^{m+2}$ is also orientable, in a certain neighborhood U of p we can choose two fields of mutually orthogonal unit normal vectors $\underset{1}{N}$ and $\underset{2}{N}$ to M^m at each point of U in such a way that, if $(B_1, \cdots, B_m)$ is a positively orientable frame of tangent vectors at p then the frame $(d\iota(B_1), \cdots, d\iota(B_m), \underset{1}{N}, \underset{2}{N})$ at $\iota(p)$ is positively oriented. Then we have

$$\begin{cases} \tilde{g}(X, \underset{1}{N})=\tilde{g}(X, \underset{2}{N})=\tilde{g}(\underset{1}{N}, \underset{2}{N})=0, \\ \tilde{g}(\underset{1}{N}, \underset{1}{N})=\tilde{g}(\underset{2}{N}, \underset{2}{N})=1. \end{cases} \tag{2.1}$$

If X and Y are tangent to M^m the covariant derivative of $d\iota(Y)$ in the direction of $d\iota(X)$ is expressed as

$$\tilde{\nabla}_{d\iota(X)}d\iota(Y)=\nabla_X Y+H(X, Y)\underset{1}{N}+K(X, Y)\underset{2}{N}. \tag{2.2}$$

Though $\nabla_X Y$ denotes the tangential components of $\tilde{\nabla}_{d\iota(X)}d\iota(Y)$, it is easily verified that $\nabla_X Y$ is identical with the covariant derivative of Y in the direction of X with respect to the induced Riemannian metric g. Thus we can write (2.2) as

$$\nabla_X Y=\tilde{\nabla}_X Y-H(X, Y)\underset{1}{N}-K(X, Y)\underset{2}{N}, \tag{2.3}$$

by means of the above identification.

The 2-forms H and K over M^m are called the second fundamental forms of M^m in $\tilde{M}^{m+2}$ with respect to the normal vectors $\underset{1}{N}$ and $\underset{2}{N}$ respectively.

Since $\underset{1}{N}$ and $\underset{2}{N}$ are both unit vectors we have from (2.1)

$$\tilde{g}(\underset{i}{N}, \tilde{\nabla}_X\underset{i}{N})=0, \qquad i=1, 2, \tag{2.4}$$

from which we can write

$$\tilde{\nabla}_X\underset{1}{N}=-A(X)+L(X)\underset{2}{N}, \tag{2.5}$$

$$\tilde{\nabla}_X\underset{2}{N}=-A'(X)-L(X)\underset{1}{N}, \tag{2.6}$$

where $A(X)$ and $A'(X)$ mean the tangential components of $\tilde{\nabla}_X\underset{1}{N}$ and $\tilde{\nabla}_X\underset{2}{N}$ to M^m respectively and L is the connection form of the normal bundle to M^m. Differentiating both members of the equations

$$\tilde{g}(\underset{i}{N}, Y)=0, \qquad i=1, 2,$$

covariantly in the direction of a tangent vector field X, we get

$$\tilde{g}(\tilde{\nabla}_X\underset{i}{N}, Y)+\tilde{g}(\underset{i}{N}, \tilde{\nabla}_X Y)=0, \qquad i=1, 2.$$

Substituting (2.3) into the above equation, we have the equations of Weingarten:

$$\tilde{g}(\tilde{\nabla}_X\underset{1}{N}, Y)=-H(X, Y), \qquad \tilde{g}(\tilde{\nabla}_X\underset{2}{N}, Y)=-K(X, Y). \tag{2.7}$$

Let $\{x^i\}$ $i=1, 2, \cdots, m$ be local coordinates in an open neighborhood V of $p\in M^m$. The set of vector fields

$$\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\right)$$

is called the natural frame of M^m and it spans the tangent plane of M^m at each point of V. We choose a positively oriented frame $(B_1, \cdots, B_m, \underset{1}{N}, \underset{2}{N})$, where

$$B_i=d\iota\left(\frac{\partial}{\partial x^i}\right), \qquad i=1, 2, \cdots, m,$$

at each point of the neighborhood $U\cap V$ of $\iota(p)\in\tilde{M}^{m+2}$. Then $A(X)$ and $A'(X)$ are represented as linear combinations of B_i, $i=1, 2, \cdots, m$ and consequently we have from (2. 5) and (2. 6)

$$\tilde{\nabla}_X \underset{1}{N} = -\sum_{i=1}^{m} H^i B_i + L(X)\underset{2}{N}, \tag{2. 8}$$

$$\tilde{\nabla}_X \underset{2}{N} = -\sum_{i=1}^{m} K^i B_i - L(X)\underset{1}{N}, \tag{2. 9}$$

or denoting $L(\partial/\partial x^j)$ by L_j

$$\tilde{\nabla}_{B_j} \underset{1}{N} = -\sum_{i=1}^{m} H_j{}^i B_i + L_j\underset{2}{N}, \tag{2. 10}$$

$$\tilde{\nabla}_{B_j} \underset{2}{N} = -\sum_{i=1}^{m} K_j{}^i B_i - L_j\underset{1}{N}. \tag{2. 11}$$

Hence, by virtue of (2. 7), we get

$$H\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = -\tilde{g}(\tilde{\nabla}_{B_j}\underset{1}{N}, B_i) = \sum_{i=1}^{m} H_j{}^k \tilde{g}(B_k, B_i) = H_j{}^k g_{ki}, \tag{2. 12}$$

$$K\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = -\tilde{g}(\tilde{\nabla}_{B_j}\underset{2}{N}, B_i) = \sum_{i=1}^{m} K_j{}^k \tilde{g}(B_k, B_i) = K_j{}^k g_{ki}, \tag{2. 13}$$

where g_{ji} means $g(\partial/\partial x^j, \partial/\partial x^i)$ and we use Einstein's summation convention for brevity. Further, in what follows, we use the standard identification by the induced Riemannian metric g and so we put

$$H\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = H_{ji} \quad \text{and} \quad K\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = K_{ji}.$$

Since the Riemannian connections $\tilde{\nabla}$ and ∇ are both torsionless we easily see that

$$H(X, Y) = H(Y, X), \qquad K(X, Y) = K(Y, X),$$

or equivalently that

$$H_{ji} = H_{ij}, \qquad K_{ji} = K_{ij}.$$

When at each point of M^m there exist differentiable functions h and k such that $H(X, Y) = hg(X, Y)$, $K(X, Y) = kg(X, Y)$ or equivalently

$$H_{ji} = hg_{ji}, \qquad K_{ji} = kg_{ji}, \tag{2. 14}$$

we call M^m a totally umbilical submanifold in $\tilde{M}^{m+2}$ and the immersion ι a totally umbilical immersion. Moreover when the proportional factors h and k vanish identically we call M^m a totally geodesic submanifold in $\tilde{M}^{m+2}$ and ι a totally geodesic immersion.

Let $\tilde{R}$ and R be curvature tensors of $\tilde{M}^{m+2}$ and M^m respectively. Then the equation of Gauss, Mainardi-Codazzi and Ricci-Kühne are respectively given by

$$\tilde{g}(\tilde{R}(B_k, B_j)B_i, B_h) = R_{kjih} - H_{ji}H_{kh} + H_{ki}H_{jh} - K_{ji}K_{kh} + K_{ki}K_{jh}, \tag{2. 15}$$

$$\tag{2.16} \begin{cases} \tilde{g}(\tilde{R}(B_k, B_j)B_i, \underset{1}{N}) = \nabla_k H_{ji} - \nabla_j H_{ki} - L_k K_{ji} + L_j K_{ki}, \\ \tilde{g}(\tilde{R}(B_k, B_j)B_i, \underset{2}{N}) = \nabla_k K_{ji} - \nabla_j K_{ki} + L_k H_{ji} - L_j H_{ki}, \end{cases}$$

and

$$\tag{2.17} \tilde{g}(\tilde{R}(B_k, B_j)\underset{1}{N}, \underset{2}{N}) = \nabla_k L_j - \nabla_j L_k - K_{ki} H_j{}^i + K_{ji} H_k{}^i,$$

where

$$R_{kjih} = g\left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^h}\right)$$

and ∇_j denotes the operation of covariant differentiation in classical tensor calculus.

If the Riemannian manifold $\tilde{M}^{m+2}$ is of constant curvature, that is, if we have, for any $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ belonging to $T(\tilde{M}^{m+2})$,

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = c\{\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}\}, \qquad c = \text{const.},$$

then, from (2. 5), (2. 16) and (2. 17), we get

$$\tag{2.18} R_{kjih} = c(g_{ji}g_{kh} - g_{ki}g_{jh}) + H_{ji}H_{kh} - H_{ki}H_{jh} + K_{ji}K_{kh} - K_{ki}K_{jh},$$

$$\tag{2.19} \begin{cases} \nabla_k H_{ji} - \nabla_j H_{ki} = L_k K_{ji} - L_j K_{ki}, \\ \nabla_k K_{ji} - \nabla_j K_{ki} = -L_k H_{ji} + L_j H_{ki}, \end{cases}$$

and

$$\tag{2.20} \nabla_k L_j - \nabla_j L_k = K_{ki} H_j{}^i - K_{ji} H_k{}^i.$$

§ 3. Submanifold of codimension 2 in a contact Riemannian manifold.

Let $\tilde{M}^{2n+1}$ be a contact metric manifold and M^{2n-1} a submanifold of codimension 2 in $\tilde{M}^{2n+1}$. The transform FX of a tangent vector field $X \in T(M^{2n-1}) \subset T(\tilde{M}^{2n+1})$ can be expressed as a sum of its tangential part $(FX)^T$ to M^{2n-1} and its normal parts, that is,

$$FX = (FX)^T + \varphi(X)\underset{1}{N} + \psi(X)\underset{2}{N}.$$

The correspondence $X \in T(M^{2n-1})$ to $(FX)^T$ defines a linear transformation f: $T(M^{2n-1}) \to T(M^{2n-1})$ and the correspondence $X \in T(M^{2n-1})$ to $\varphi(X)$ and to $\psi(X)$ define respectively 1-forms φ and ψ on M^{2n-1}. So the equation above can be rewritten as

$$\tag{3.1} FX = fX + \varphi(X)\underset{1}{N} + \psi(X)\underset{2}{N},$$

from which we get

$$\tag{3.2} \varphi(X) = \tilde{g}(FX, \underset{1}{N}) = -\tilde{g}(X, F\underset{1}{N}),$$

$$\tag{3.3} \psi(X) = \tilde{g}(FX, \underset{2}{N}) = -\tilde{g}(X, F\underset{2}{N}).$$

By means of definitions of F and f we see immediately that

$$g(fX,\ Y)=-g(X, fY). \tag{3.4}$$

Let $\{x^i\}$ be a local coordinates in a neighborhood U of $p\in M^{2n-1}$. We choose a frame

$$(B_1, \cdots, B_{2n-1}, \underset{1}{N}, \underset{2}{N}),\quad B_i=d\iota\left(\frac{\partial}{\partial x^i}\right),\qquad i=1, 2, \cdots, 2n-1,$$

in $T(\tilde{M}^{2n+1})$, then, from (3.1), we have

$$FB_i=\sum_{h=1}^{2n-1} f_i{}^h B_h+f_i\underset{1}{N}+g_i\underset{2}{N}, \tag{3.5}$$

where $f_i{}^h$ are components of the matrix which defines the linear transformation f and f_i and g_i those of the 1-forms φ and ψ respectively.

On the other hand the transforms $F\underset{1}{N}$ and $F\underset{2}{N}$ are respectively expressed as

$$F\underset{1}{N}=\sum_{h=1}^{2n-1} h^h B_h+r\underset{2}{N}, \tag{3.6}$$

$$F\underset{2}{N}=\sum_{h=1}^{2n-1} k^h B_h+s\underset{1}{N}, \tag{3.7}$$

from which we have

$$\begin{cases} f_i=\tilde{g}(FB_i, \underset{1}{N})=-\tilde{g}(B_i, F\underset{1}{N})=-h^h\tilde{g}(B_h, B_i)=-h^h g_{ih}=-h_i, \\ g_i=\tilde{g}(FB_i, \underset{2}{N})=-\tilde{g}(B_i, F\underset{2}{N})=-k^j\tilde{g}(B_j, B_i)=-k^j g_{ji}=-k_i. \end{cases} \tag{3.8}$$

and

$$r=\tilde{g}(F\underset{1}{N}, \underset{2}{N})=-\tilde{g}(\underset{1}{N}, F\underset{2}{N})=-s. \tag{3.9}$$

The vector field $\tilde{E}$ being tangent to $\tilde{M}^{2n+1}$, it is represented as a linear combination of B_i, $\underset{1}{N}$ and $\underset{2}{N}$. Hence we put

$$\tilde{E}=\sum_{i=1}^{2n-1} u^i B_i+a\underset{1}{N}+b\underset{2}{N} \tag{3.10}$$

from which we get

$$\tilde{\eta}(B_j)=\tilde{g}(\tilde{E}, B_j)=u^i\tilde{g}(B_i, B_j)=u^i g_{ij}=u_j, \tag{3.11}$$

$$\tilde{\eta}(\underset{1}{N})=\tilde{g}(\tilde{E}, \underset{1}{N})=a,\quad \tilde{\eta}(\underset{2}{N})=\tilde{g}(\tilde{E}, \underset{2}{N})=b. \tag{3.12}$$

Transforming again the both members of (3.5) by F and making use of (1.4), (3.5), (3.6), (3.7) and (3.10), we find

$$-B_i+u_i u^j B_j+au_i\underset{1}{N}+bu_i\underset{2}{N}=f_i{}^h f_h{}^j B_j+f_i{}^h f_h\underset{1}{N}+f_i{}^h g_h\underset{2}{N}-f_i f^j B_j+rf_i\underset{2}{N}-g_i g^j B_j-rg_i\underset{1}{N},$$

from which

(3. 13) $$f_i{}^h f_h{}^j = -\delta_i^j + u_i u^j + f_i f^j + g_i g^j,$$

(3. 14) $$f_i{}^h f_h = a u_i + r g_i, \qquad f_i{}^h g_h = b u_i - r f_i.$$

Transforming again the both members of (3. 6) by F and making use of (1. 4), (3. 5), (3. 7) and (3. 10), we find

$$-\underset{1}{N} + a u^i B_i + a^2 \underset{1}{N} + ab \underset{2}{N} = -f^j f_j{}^i B_i - f^j f_j \underset{1}{N} - f^j g_j \underset{2}{N} - r g^i B_i - r^2 \underset{1}{N},$$

from which

(3. 15) $$f_i f^i = 1 - a^2 - r^2,$$

(3. 16) $$g_i f^i = -ab.$$

In exactly the same way we get

(3. 17) $$g_i g^i = 1 - b^2 - r^2.$$

Since the second condition of (1. 2) is equivalent to $\tilde{g}(F\tilde{E}, \tilde{X}) = 0$ for any $\tilde{X} \in T(\tilde{M}^{2n+1})$, it follows that $F\tilde{E} = 0$ and consequently we have

$$F\tilde{E} = u^i F B_i + a F \underset{1}{N} + b F \underset{2}{N} = 0,$$

because of (3. 10). Substituting (3. 5), (3. 6) and (3. 7) into the equation above, we find

$$u^i (f_i{}^j B_j + f_i \underset{1}{N} + g_i \underset{2}{N}) + a(r \underset{1}{N} - f^j B_j) - b(g^j B_j + r \underset{2}{N}) = 0,$$

from which

(3. 18) $$u^i f_i{}^h = a f^h + b g^h,$$

(3. 19) $$u_i f^i = br, \qquad u_i g^i = -ar.$$

By virtue of (3. 10) the first condition of (1. 2) can be rewritten as

$$\tilde{\eta}(\tilde{E}) = \tilde{g}(\tilde{E}, \tilde{E}) = \tilde{g}(u^i B_i + a \underset{1}{N} + b \underset{2}{N},\ u^j B_j + a \underset{1}{N} + b \underset{2}{N}) = u_i u^i + a^2 + b^2 = 1,$$

that is,

(3. 20) $$u_i u^i = 1 - a^2 - b^2.$$

Let $\tilde{M}^{2n+1}$ is a normal contact manifold. Differentiating (1. 3) covariantly in the direction of $\tilde{Z}$ and taking account of (1. 9) and (1. 11), we find

(3. 21) $$\tilde{\nabla}_{\tilde{Z}}(d\tilde{\eta}(\tilde{X}, \tilde{Y})) - d\tilde{\eta}(\tilde{\nabla}_{\tilde{Z}} \tilde{X}, \tilde{Y}) - d\tilde{\eta}(\tilde{X}, \tilde{\nabla}_{\tilde{Z}} \tilde{Y}) = 2(\tilde{\eta}(\tilde{X}) \tilde{g}(\tilde{Y}, \tilde{Z}) - \tilde{\eta}(\tilde{Y}) \tilde{g}(\tilde{X}, \tilde{Z})).$$

Substituting B_i, B_j and B_k for $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ respectively in (3. 21) and observing the fact that

$$\frac{1}{2} d\tilde{\eta}(X, Y) = \tilde{g}(FX, Y) = g(fX, Y) \quad \text{for} \quad X, Y \in T(M^{2n+1}),$$

we obtain

(3. 22) $$\nabla_k f_{ji} = f_j H_{ki} - f_i H_{kj} + g_j K_{ki} - g_i K_{kj} + u_j g_{ki} - u_i g_{kj}.$$

Substituting $\underset{1}{N}$, B_i and B_j for $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ respectively in (3. 21), we find

$$\nabla_j f_i = -a g_{ji} - r K_{ji} - H_{jk} f_i{}^k + L_j g_i. \tag{3. 23}$$

In the same way, we get

$$\nabla_j g_i = -b g_{ji} + r H_{ji} - K_{jk} f_i{}^k - L_j f_i. \tag{3. 24}$$

On the other hand, in a normal contact manifold, we have (1. 9) and consequently

$$\tilde{\nabla}_{B_j} \tilde{E} = F B_j.$$

Substituting (3. 5) and (3. 10) into the equation above and making use of (2. 7), (2. 8) and (2. 9), we get

$$\nabla_j u^i = f_j{}^i + a H_j{}^i + b K_j{}^i, \tag{3. 25}$$

$$\nabla_j a = f_j - u^i H_{ji} + b L_j, \tag{3. 26}$$

$$\nabla_j b = g_j - u^i K_{ji} - a L_j. \tag{3. 27}$$

§ 4. Contact metric immersion.

Let $\tilde{M}^{2n+1}$ be a contact metric manifold, M^{2n-1} an orientable submanifold of $\tilde{M}^{2n+1}$ and ι the immersion of M^{2n-1} into $\tilde{M}^{2n+1}$. Making use of the contact form $\tilde{\eta}$ on $\tilde{M}^{2n+1}$ we define 1-form u on M^{2n-1} by

$$u(X) = \tilde{\eta}(d\iota(X))$$

or equivalently

$$u\left(\frac{\partial}{\partial x^i}\right) = \tilde{\eta}(B_i) = u_i. \tag{4. 1}$$

DEFINITION 4. 1. Let G be a Riemannian metric homothetic to the induced Riemannian metric g of M^{2n-1}. If there exists a pair of positive constants t and c such that $\eta = tu$ and $G = cg$ constitute a contact metric structure of M^{2n-1} we call the immersion ι a *contact metric immersion* and the submanifold M^{2n-1} a *contact metric submanifold.*

Since (η, G) constitutes a contact metric structure of M^{2n-1}, the linear mapping $\bar{f}$: $T(M^{2n-1}) \to T(M^{2n-1})$ and the vector field E, which are defined respectively by

$$G(\bar{f}X, Y) = \frac{1}{2} d\eta(X, Y), \qquad G(E, X) = \eta(X), \tag{4. 2}$$

satisfy

$$\eta(E) = G(E, E) = 1, \tag{4. 3}$$

$$\bar{f}(E) = 0, \qquad \eta(\bar{f}X) = 0, \tag{4. 4}$$

$$\bar{f}^2 X = -X + \eta(X) E. \tag{4. 5}$$

In the following, we put

$$\bar{f}\left(\frac{\partial}{\partial x^i}\right)=\bar{f}_i{}^j\frac{\partial}{\partial x^j}, \qquad E=\xi^j\frac{\partial}{\partial x^j}. \tag{4. 6}$$

DEFINITION 4. 2. Let $\tilde{M}^{2n+1}$ be a contact metric manifold and F a linear mapping $T(\tilde{M}^{2n+1})\to T(\tilde{M}^{2n+1})$ defined by (1. 3). If the immersion ι: $M^{2n-1}\to M^{2n+1}$ satisfies $F(T\circ\iota(M^{2n-1}))\subset T\circ\iota(M^{2n-1})$, the immersion ι is called an *F-invariant immersion* and the submanifold M^{2n-1} is called an *F-invariant submanifold.*

PROPOSITION 4. 3. *If an immersion of M^{2n-1} into a contact metric manifold $\tilde{M}^{2n+1}$ is F-invariant, the immersion is necessarily contact metric immersion.*

Proof. Since ι is F-invariant, (3. 5) shows that

$$FB_i=f_i{}^jB_j, \qquad F\underset{1}{N}=r\underset{2}{N}, \qquad F\underset{2}{N}=-r\underset{1}{N},$$

or equivalently $f_i=g_i=0$. Hence (3. 10) and $F\tilde{E}=0$ give

$$F\tilde{E}=u^jf_j{}^iB_i+ar\underset{2}{N}-br\underset{1}{N}=0.$$

Consequently we have

$$u^jf_j{}^i=0, \tag{4. 7}$$

$$ar=br=0. \tag{4. 8}$$

Now let P be a point of M^{2n-1} at which $r(\mathrm{P})=0$. Then at P we have

$$a^2(\mathrm{P})=b^2(\mathrm{P})=1,$$

because of (3. 15) and (3. 17). However, (3. 16) show that the absolute value of f_ig^i is equal to 1 at P. This contradicts to $f_i=g_i=0$. Thus, it follows that there is no zero point of r in M^{2n-1}. Consequently we have $a=b=0$ on M^{2n-1}. This shows that

$$f_i{}^hf_h{}^j=-\delta_i^j+u_iu^j \qquad \text{and} \qquad u_iu^i=1,$$

because of (3. 13) and (3. 20).

Now we put $\eta=u$, $G=g$, then it follows that

$$d\eta\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)=du\left(\frac{\partial}{\partial x^j}\ \frac{\partial}{\partial x^i}\right)=d\tilde{\eta}(B_j, B_i)=2\tilde{g}(FB_j, B_i)=2f_{ji}.$$

On the other hand, by the definition, we get

$$G\left(f\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)=g\left(f\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)=f_{ji}.$$

Combining the above two equations, we get $d\eta(X, Y)=2G(fX, Y)$ for any $X, Y\in T(M^{2n-1})$. This means that the pair (η, G) constitutes a contact metric structure on M^{2n-1}. Q.E.D.

PROPOSITION 4.4. *Let $\tilde{M}^{2n+1}$ be a contact metric manifold and ι an immersion of M^{2n-1} into $\tilde{M}^{2n+1}$. In order that ι is F-invariant it is necessary and sufficient that $r^2=1$.*

Proof. Necessity. If ι is F-invariant, we have $f_i=g_i=0$ and $a=b=0$. Hence, because of (3.15), $r^2=1$.

Sufficiency. Let ι be an immersion and $r^2=1$. Then it follows that

$$f_i f^i=-a^2, \qquad g_i g^i=-b^2$$

by virtue of (3.15) and (3.17). The Riemannian metric g being positive definite, we have $a=b=0$ and $f_i=g_i=0$. This, together with (3.5), implies that the immersion ι is an F-invariant immersion.

PROPOSITION 4.5. *Let $\tilde{M}^{2n+1}$ be a contact metric manifold and M^{2n-1} a contact metric submanifold of $\tilde{M}^{2n+1}$. If $n\geqq 2$ the linear mapping $\bar{f}$ is identical with f and we have $g(E, X)=u(X)$ or equivalently*

$$\xi^j=u^j. \tag{4.9}$$

Proof. First of all we have from (4.2)

$$\eta\left(\frac{\partial}{\partial x^i}\right)=tu\left(\frac{\partial}{\partial x^i}\right)=G\left(E, \frac{\partial}{\partial x^i}\right)=cg\left(E, \frac{\partial}{\partial x^i}\right),$$

which implies that

$$g(E, X)=\frac{t}{c}u(X), \quad \text{that is,} \quad \xi_j=\frac{t}{c}u_j. \tag{4.10}$$

By means of the definition of the exterior derivative, we get

$$\begin{aligned} d\eta\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)&=\frac{\partial}{\partial x^j}\eta\left(\frac{\partial}{\partial x^i}\right)-\frac{\partial}{\partial x^i}\eta\left(\frac{\partial}{\partial x^j}\right)-\eta\left(\left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right]\right)\\ &=t(\partial_j u_i-\partial_i u_j)=2\,cg\left(\bar{f}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right), \end{aligned} \tag{4.11}$$

and

$$2g\left(f\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)=2\tilde{g}(FB_j, B_i)=d\tilde{\eta}(B_j, B_i)=B_j u_i-B_i u_j-\tilde{\eta}([B_j, B_i]).$$

If we denote by $\{y^\lambda\}$ local coordinates of $\tilde{M}^{2n+1}$, the immersion ι is represented as $y^\lambda=y^\lambda(x^i)$ and this implies that

$$B_i f=\left(d\iota\left(\frac{\partial}{\partial x^i}\right)\right)f=\frac{\partial}{\partial x^i}(f\circ\iota)=\frac{\partial}{\partial x^i}f(y^\lambda(x^i))=\frac{\partial y^\lambda}{\partial x^i}\frac{\partial}{\partial y^\lambda}f.$$

Thus we have

$$d\tilde{\eta}(B_j, B_i)=2g\left(f\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)=\partial_j u_i-\partial_i u_j. \tag{4.12}$$

Combining (4.11) and (4.12), we get

$$f=\frac{c}{t}\bar{f}. \tag{4.13}$$

To show that the constant c/t is equal to 1, we first assume that there is a point $\mathrm{P}\in M^{2n-1}$ at which $(\underset{1}{FN})^T_{\mathrm{P}}$, $(\underset{2}{FN})^T_{\mathrm{P}}$ and E_{P} are linearly dependent. Then taking a vector

$$Y=p^i\frac{\partial}{\partial x^i}(\mathrm{P})$$

in such a way that at this point Y is orthogonal to the tangent plane which contains $(\underset{1}{FN})^T_{\mathrm{P}}$, $(\underset{2}{FN})^T_{\mathrm{P}}$ and E_{P}, we have

$$g(E,\ Y)_{\mathrm{P}}=\frac{1}{c}\eta(Y)_{\mathrm{P}}=\frac{t}{c}u(Y)_{\mathrm{P}}=\frac{t}{c}u_ip^i(\mathrm{P})=0,$$

$$g((\underset{1}{FN})^T,\ Y)_{\mathrm{P}}=-f_ip^i(\mathrm{P})=0, \qquad g((\underset{2}{FN})^T,\ Y)_{\mathrm{P}}=-g_ip^i(\mathrm{P})=0,$$

from which, together with (3.13),

$$f_i{}^hf_j{}^ip^j=-p^h.$$

On the other hand (4.6) and (4.13) show that

$$f_j{}^if_i{}^hp^j=\frac{c^2}{t^2}\bar{f}_j{}^i\bar{f}_i{}^hp^j=\frac{c^2}{t^2}\left(-\delta_j^h+\frac{t^2}{c}u_ju^h\right)p^j=-\frac{c^2}{t^2}p^h.$$

Comparing the last two equations, we have $c/t=1$. Thus in this case the mapping $\bar{f}$ is identical with f. Next suppose that $(\underset{1}{FN})^T$, $(\underset{2}{FN})^T$ and E are linearly independent at any point of M^{2n-1}. If there is a point of M^{2n-1} at which one of $(\underset{1}{FN})^T$ and $(\underset{2}{FN})^T$ vanishes, we can take an orthogonal vector to E and $(\underset{2}{FN})^T$ (or $(\underset{1}{FN})^T$). Using this vector we can prove the assertion quite similar way to the first case. If the vector fields do not vanish at any point of M^{2n-1} and linearly independent, (3.18) and (4.4) show that $a=b=0$. So by virtue of (3.13), (4.6) and (4.13) we get

$$f_j{}^if_i{}^hf^j=-r^2f^h=-\frac{c^2}{t^2}f^h$$

from which $r^2=c^2/t^2$. On the other hand (3.13) and (4.13) imply that

$$2(n-1)\frac{c^2}{t^2}=2(n-2+r^2).$$

Combining the above two relations, we have $r^2=1$. Thus, because of Proposition 4.4, $f_i=g_i=0$. This contradicts to our assumptions. This completes the proof.

PROPOSITION 4.6. *Under the same assumptions as those in Proposition* 4.5, *we have*

(4. 14) $$G=g(E, E)^{-1}g, \qquad g(E, E)=\text{const.}$$

The proof is quite easy.

PROPOSITION 4. 7. *Let $\tilde{M}^{2n+1}$ be a contact metric manifold. In order that an immersion ι of M^{2n-1} into $\tilde{M}^{2n+1}$ is a contact metric immersion it is necessary and sufficient that the relations*

(4. 15) $$f^2X=-X+g(\tilde{E}^T, \tilde{E}^T)^{-1}u(X)\tilde{E}^T,$$

(4. 16) $$g(\tilde{E}^T, \tilde{E}^T)=\text{const.}$$

are both valid.

Proof. Let ι be a contact metric immersion of M^{2n-1} into $\tilde{M}^{2n+1}$. Then by Proposition 4. 6 and the proof of Proposition 4. 5 it follows that

(4. 17) $$\eta=g(\tilde{E}^T, \tilde{E}^T)^{-1}u, \quad E=\tilde{E}^T, \quad G=g(\tilde{E}^T, \tilde{E}^T)^{-1}g, \quad g(\tilde{E}^T, \tilde{E}^T)=\text{const.},$$

and consequently

$$f^2X=-X+\eta(X)E=-X+g(\tilde{E}^T, \tilde{E}^T)^{-1}u(X)\tilde{E}^T.$$

Conversely if (4. 15) and (4. 16) are both valid, we put

$$\eta=g(\tilde{E}^T, \tilde{E}^T)^{-1}u, \qquad G=g(\tilde{E}^T, \tilde{E}^T)^{-1}g.$$

Since $g(\tilde{E}^T, X)=u(X)$, we have

$$\eta(\tilde{E}^T)=g(\tilde{E}^T, \tilde{E}^T)^{-1}u(\tilde{E}^T)=g(\tilde{E}^T, \tilde{E}^T)^{-1}g(\tilde{E}^T, \tilde{E}^T)=1,$$

$$f^2X=-X+g(\tilde{E}^T, \tilde{E}^T)^{-1}u(X)\tilde{E}^T=-X+\eta(X)\tilde{E}^T.$$

As to relations of $d\eta$ and f we get, by (4. 16) and (1. 3),

$$\begin{aligned} d\eta\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) &= g(\tilde{E}^T, \tilde{E}^T)^{-1}du\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) \\ &= g(\tilde{E}^T, \tilde{E}^T)^{-1}d\tilde{\eta}(B_j, B_i)=2g(\tilde{E}^T, \tilde{E}^T)^{-1}\tilde{g}(FB_j, B_i) \\ &= 2g(\tilde{E}^T, \tilde{E}^T)^{-1}f_{ji}=2g(\tilde{E}^T, \tilde{E}^T)^{-1}g\left(f\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right). \end{aligned}$$

This shows that (η, G) thus defined is a contact metric structure of M^{2n-1}. Q.E.D.

PROPOSITION 4. 8. *Under the same assumptions as those in Proposition* 4. 5, *we have*

(4. 18) $$g(\tilde{E}^T, \tilde{E}^T)=g_{ji}u^ju^i=u_iu^i=r^2.$$

Proof. Since ι is a contact metric immersion, (3. 18), (4. 4) and (4. 9) show that $af^i+bg^i=0$, from which, together with (3. 15), (3. 16), we get

$$af_if^i+bg^if_i=a(1-r^2-a^2-b^2)=0.$$

In the same way, using (3. 17), we get

$$ag_i f^i + bg_i g^i = b(1-r^2-a^2-b^2) = 0.$$

The last two equations imply that

$$(a^2+b^2)(1-r^2-a^2-b^2)=0.$$

As we have, from (3. 20) and Proposition 4. 6, that $a^2+b^2=\text{const}$, it follows that

$$g(\tilde{E}^T, \tilde{E}^T) = g(E, E) = u_i u^i = 1-a^2-b^2 = r^2$$

or

$$a^2+b^2=0.$$

The first case is just our assertion. If the second case occurs, (3. 20) shows that $g(\tilde{E}^T, \tilde{E}^T)=1$. In this case, since we also have $a=b=0$ it follows that

$$f_i{}^h f_h{}^j f^i = -f^j + u_i f^i u^j + f_i f^i f^j + g_i f^i g^j = -r^2 f^j$$

because of (3. 13), (3. 15), (3. 16) and (3. 19).

On the other hand (3. 19) and (4. 15) show that

$$f_i{}^h f_h{}^j f^i = -f^j,$$

which implies that $r^2=1$. Thus, in this case we also have (4. 18). Q.E.D.

COROLLARY 4. 9. *r=constant.*

THEOREM 4. 10. *Let $\tilde{M}^{2n+1}$ be a contact metric manifold. In order that an immersion ι of M^{2n-1} into $\tilde{M}^{2n+1}$ is contact metric it is necessary and sufficient that the following relations are all valid.*

$$g(\tilde{E}^T, \tilde{E}^T) = u_i u^i = \text{const.}, \tag{4. 19}$$

$$(F\underset{1}{N})^T = -g(\tilde{E}^T, \tilde{E}^T)^{-1/2}\tilde{g}(\tilde{E}, \underset{2}{N})\tilde{E}^T = -b(u_r u^r)^{-1/2} u^i \frac{\partial}{\partial x^i}, \tag{4. 20}$$

$$(F\underset{2}{N})^T = g(\tilde{E}^T, \tilde{E}^T)^{-1/2}\tilde{g}(\tilde{E}, \underset{1}{N})\tilde{E}^T = a(u_r u^r)^{-1/2} u^i \frac{\partial}{\partial x^i}. \tag{4. 21}$$

Proof. Let ι be a contact metric immersion of M^{2n-1} into $\tilde{M}^{2n+1}$, then from Proposition 4. 6, (4. 19) is valid. Now, we put

$$(F\underset{1}{N})^T = A\tilde{E}^T + Z, \qquad (F\underset{2}{N})^T = B\tilde{E}^T + W, \tag{4. 22}$$

where Z and W are vectors orthogonal to $\tilde{E}^T$. Then Proposition 4. 8 gives

$$g(\tilde{E}^T, (F\underset{1}{N})^T) = Ag(\tilde{E}^T, \tilde{E}^T) = Ar^2, \qquad g(\tilde{E}^T, (F\underset{2}{N})^T) = Bg(\tilde{E}^T, \tilde{E}^T) = Br^2.$$

On the other hand (3. 19) implies that

$$g(\tilde{E}^T, (F\underset{1}{N})^T) = -\tilde{g}(\tilde{E}, \underset{2}{N})r = -br, \qquad g(\tilde{E}^T, (F\underset{2}{N})^T) = \tilde{g}(\tilde{E}, \underset{1}{N})r = ar.$$

Thus combining these relations, we get

$$A = -br^{-1}, \qquad B = ar^{-1}. \tag{4. 23}$$

Substituting (4. 23) into (4. 22), we have

$$(\underset{1}{FN})^T=-br^{-1}\tilde{E}^T+Z, \qquad (\underset{2}{FN})=ar^{-1}\tilde{E}^T+W,$$

from which

$$g((\underset{1}{FN})^T,(\underset{1}{FN})^T)=b^2r^{-2}g(\tilde{E}^T,\tilde{E}^T)+g(Z,Z)=b^2+g(Z,Z).$$

On the other hand (3. 15) and Proposition 4. 8 show that

$$g((\underset{1}{FN})^T,(\underset{1}{FN})^T)=1-a^2-r^2=1-a^2-(1-a^2-b^2)=b^2.$$

From the last two equations we have $g(Z,Z)=0$ and consequently $Z=0$. In exactly the same way we get $W=0$. Hence we have (4. 20) and (4. 21).

Conversely if an immersion ι satisfies the conditions of Theorem 4. 10, according to (3. 20) and Proposition 4. 8 we get,

$$\begin{aligned} f_j{}^i f_k{}^j &= -\delta_k^i+g(\tilde{E}^T,\tilde{E}^T)^{-1}(u_iu^i+b^2+a^2)u_ku^i \\ &= -\delta_k^i+g(\tilde{E}^T,\tilde{E}^T)^{-1}u^iu_k. \end{aligned}$$

Thus it satisfies the conditions of Proposition 4. 7. This completes the proof.

§ 5. Normal contact immersion.

In this paragraph we define the notion of normal contact immersion of M^{2n-1} into a normal contact manifold $\tilde{M}^{2n+1}$ and give conditions of a contact metric immersion to be a normal contact immersion.

DEFINITION 5. 1. Let $\tilde{M}^{2n+1}$ be a normal contact manifold and ι be a contact metric immersion of M^{2n-1} into $\tilde{M}^{2n+1}$. We say that ι is a *normal contact immersion* if the contact metric structure induced on M^{2n-1} by ι is normal.

THEOREM 5. 2. *An F-invariant immersion ι of M^{2n-1} into $\tilde{M}^{2n+1}$ is a normal contact immersion.*

Proof. By the definition of normality we have only to examine $N_{ji}{}^h=0$. As is seen in the proof of Proposition 4. 3, we have for an F-invariant immersion, $f_i=g_i=0$ and $a=b=0$.

Using this facts and substituting (3. 22), (3. 25) and (4. 2) into the similar equation to (1. 11) we have

$$\begin{aligned} N_{ji}{}^h &= f_j{}^r(\nabla_r f_i{}^h-\nabla_i f_r{}^h)-f_i{}^r(\nabla_r f_j{}^h-\nabla_j f_r{}^h)+\nabla_i\xi^h\eta_j-\nabla_j\xi^h\eta_i \\ &= \left(1-\frac{1}{r^2}\right)(f_j{}^hu_i-f_i{}^hu_j). \end{aligned}$$

However, for an F-invariant immersion, $r^2=1$ by virtue of (3. 15). This shows that $N_{ji}{}^h=0$. Q.E.D.

THEOREM 5. 3. *In order that a contact metric immersion ι of M^{2n-1} into a* *normal contact manifold* $\tilde{M}^{2n+1}$ *be normal contact immersion it is necessary and sufficient that one of the following holds*:

1) ι *is an F-invariant immersion,*

2) *the second fundamental tensors* H_{ji} *and* K_{ji} *have the following forms*:

$$H_{ji}=Hg_{ji}+hu_ju_i, \tag{5.1}$$

$$K_{ji}=Kg_{ji}+ku_ju_i. \tag{5.2}$$

REMARK. As is easily checked the forms of the second fundamental tensors (5. 1) and (5. 2) are independent of the choice of unit normal vectors to $T(M^{2n-1})$.

Proof of Theorem 5. 3. Let ι be a normal contact immersion of M^{2n-1} into $\tilde{M}^{2n+1}$. Then by the definition of normality we get

$$N_{ji}{}^h=f_j{}^r(\nabla_r f_i{}^h-\nabla_i f_r{}^h)-f_i{}^r(\nabla_r f_j{}^h-\nabla_j f_r{}^h)+\nabla_i\xi^h\eta_j-\nabla_j\xi^h\eta_i=0.$$

Substituting (3. 22), (3. 25), (4. 2) into the equation above and making use of (3. 18), (4. 9) and Theorem 4. 10 we find

$$\begin{aligned}N_{ji}{}^h=&\frac{1}{r}f_j{}^r(bH_r{}^hu_i-aK_r{}^hu_i+r\delta_r^hu_i)-\frac{1}{r}f_i{}^r(bH_r{}^hu_j-aK_r{}^hu_j+r\delta_r^hu_j)\\&+\frac{1}{r^2}(f_i{}^h+aH_i{}^h+bK_i{}^h)u_j-\frac{1}{r^2}(f_j{}^h+aH_j{}^h+bK_j{}^h)u_i=0.\end{aligned} \tag{5.3}$$

On the other hand we know that if a contact metric structure is normal the vector field ξ^i is a Killing vector field. By Proposition 4. 5, ξ^i being equal to u^i, we have

$$aH_j{}^i+bK_j{}^i=0 \tag{5.4}$$

because of (3. 25).

Thus (5. 3) takes the form

$$rf_j{}^r(bH_r{}^hu_i-aK_r{}^hu_i+r\delta_r^hu_i)-rf_i{}^r(bH_r{}^hu_j-aK_r{}^hu_j+r\delta_r^hu_j)+f_i{}^hu_j-f_j{}^hu_i=0.$$

Substituting (5. 4) into the last equation, we find

$$f_j{}^r\{(a^2+b^2)rH_r{}^hu_i+b(r^2-1)\delta_r^hu_i\}-f_i{}^r\{(a^2+b^2)rH_r{}^hu_j+b(r^2-1)\delta_r^hu_j\}=0.$$

Transvecting this with u^i and making use of (3. 20), Proposition 4. 8 and Theorem 4. 10, we get

$$(a^2+b^2)f_j{}^r\left(H_r{}^h-\frac{b}{r}\delta_r^h\right)=0.$$

Since $a^2+b^2=\text{const}$, we have $a^2+b^2=0$ or

$$f_j{}^r\left(H_r{}^h-\frac{b}{r}\delta_r^h\right)=0. \tag{5.5}$$

If $a^2+b^2=0$, it follows that $r^2=1$. Thus in this case the immersion is F-invariant. If $a^2+b^2\neq0$, transforming the both sides of (5. 5) by $f_i{}^j$, we have

$$H_i{}^h=\frac{b}{r}\delta_i^h+\frac{1}{r^2}\left(H_r{}^h u^r u_i-\frac{b}{r}u^h u_i\right).$$

H_{ji} being symmetric with respect to j and i, it follows that

$$H_r{}^h u^r=pu^h, \qquad p=\frac{1}{r^2}H_{ji}u^j u^i.$$

Hence

$$H_i{}^h=\frac{b}{r}\delta_i^h+\frac{1}{r^2}\left(\frac{1}{r^2}H_{jk}u^j u^k-\frac{b}{r}\right)u_i u^h.$$

Substituting this into (5. 4), we have

$$K_i{}^h=-\frac{a}{r}\delta_i^h+\frac{1}{r^2}\left(\frac{1}{r^2}K_{jk}u^j u^k+\frac{a}{r}\right)u_i u^h.$$

This proves the necessity of our assertions.

Conversely if, in a contact metric submanifold M^{2n-1}, the second fundamental tensors H_{ji} and K_{ji} have the forms of (5. 1) and (5. 2).

Differentiating (4. 20) covariantly, we have

$$\nabla_j f_h=\frac{1}{r}(u_h\nabla_j b+b\nabla_j u_h). \tag{5. 6}$$

Substituting (3. 23), (3. 25) and (3. 27) into (5. 6), we have

$$-arg_{jh}-r^2K_{jh}-rf_h{}^iH_{ji}=-\frac{a}{r}u_h u_j-u_hK_{ji}u^i+b(f_{jh}+aH_{jh}+bK_{jh}).$$

Substituting (5. 1) (5. 2) into the above, we get

$$-r(ag_{jh}+rKg_{jh}+Hf_{hj})=-\left(\frac{a}{r}+K\right)u_h u_j+bf_{jh}+(abH+b^2K)g_{jh}+(abh+b^2k)u_j u_h.$$

Transvecting this with f^{hj}, we have

$$H=\frac{b}{r}.$$

Next differentiating (4. 21) covariantly we have also

$$K=-\frac{a}{r}.$$

Thus we get

$$H_i{}^h=\frac{b}{r}\delta_i^h+hu_i u^h, \tag{5. 7}$$

$$K_i{}^h=-\frac{a}{r}\delta_i^h+ku_i u^h. \tag{5. 8}$$

Substituting (5. 7) and (5. 8) into the left hand members of (5. 3), we have

$$N_{ji}{}^h=\frac{1}{r}f_j{}^h\left(\frac{1}{r}(a^2+b^2)+r\right)u_i-\frac{1}{r}f_i{}^h\left(\frac{1}{r}(a^2+b^2)+r\right)u_j+\frac{1}{r^2}(f_i{}^h u_j-f_j{}^h u_i).$$

Thus, by virtue of (4.18), we have $N_{ji}{}^h=0$. This, together with Theorem 5.2, completes the proof.

COROLLARY 5.4. *Let ι be a contact metric immersion of M^{2n-1} into M^{2n+1}. If ι is a totally umbilical or totally geodesic immersion, ι is a normal contact immersion.*

THEOREM 5.5. *Let $\tilde{M}^{2n-1}$ be a normal contact metric manifold of constant curvature. In order that a contact metric immersion ι of M^{2n-1} into $\tilde{M}^{2n+1}$ is a normal contact immersion it is necessary and sufficient that one of the following is satisfied:*

1) *ι is an F-invariant immersion,*
2) *ι is a totally umbilical immersion.*

Proof. The sufficiency of the conditions are clear by Theorem 5.3. So, we have only to examine the necessity of the conditions. The proof of the sufficiency of Theorem 5.3 show that if ι is normal contact

$$H=\frac{b}{r},\qquad K=-\frac{a}{r}.$$

Thus we express the second fundamental tensors as (5.7) and (5.8). On the other hand the fact that $\tilde{M}^{2n+1}$ is a manifold of constant curvature implies that

$$u^i f^{kj}\nabla_k H_{ji}=u^i f^{kj}L_k K_{ji},\tag{5.9}$$

$$u^i f^{kj}\nabla_k K_{ji}=-u^i f^{kj}L_k H_{ji},\tag{5.10}$$

because of (2.19). Substituting (5.7), (5.8) into (5.9), (5.10), we get

$$u^i f^{kj}\nabla_k\left(\frac{b}{r}g_{ji}+hu_ju_i\right)=hu^iu_if^{kj}\nabla_k u_j=2(n-1)r^2h=0,$$

$$u^i f^{kj}\nabla_k\left(-\frac{a}{r}g_{ji}+ku_ju_i\right)=ku^iu_if^{kj}\nabla_k u_j=2(n-1)r^2k=0.$$

Thus we have

$$H_{ji}=\frac{b}{r}g_{ji},\qquad K_{ji}=-\frac{a}{r}g_{ji},\tag{5.11}$$

This completes the proof.

In concluding this paragraph we state a theorem on an F-invariant immersion.

DEFINITION 5.6. An immersion ι of M^{2n-1} into $\tilde{M}^{2n+1}$ is said to be *minimal* if it satisfies at arbitrary point of M^{2n-1}

$$\text{trace } H=\text{trace } K=0.$$

THEOREM 5.7. *An F-invariant immersion of M^{2n-1} into a normal contact manifold $\tilde{M}^{2n+1}$ is a minimal immersion.*

Proof. Since ι is F-invariant it follows that $f_i=g_i=0$ and $a=b=0$ on M^{2n-1}, which implies that

$$rH_{ji}=K_{jk}f_i{}^k, \qquad rK_{ji}=H_{jk}f_i{}^k$$

because of (3.23) and (3.24). As H_{jk}, K_{jk} are both symmetric with respect to their indices and f_{ji} is skew symmetric, we have

$$rH_i{}^i=K_{jk}f^{jk}=0, \qquad rK_i{}^i=H_{jk}f^{jk}=0.$$

This completes the proof.

§ 6. An example.

It is well known[3] that an odd dimensional sphere is one of the most typical example of a normal contact manifold.

In this paragraph we show an example of submanifolds in an odd dimensional sphere.

In Theorem 5.3 and 5.5 we have assumed that the immersion ι is a contact metric immersion and shown a condition for ι to be normal contact. This assumption, however, can not be omitted. To show this we give an example.

Let S^{2n+1} be an odd dimensional sphere which is represented by the equation

$$\sum_{A=1}^{2n+2}(y^A)^2=1 \tag{6.1}$$

in a $(2n+2)$-dimensional Euclidean space E^{2n+2} with rectangular coordinates y^A $(A=1,2,\cdots,2n+2)$. We put

$$\tilde{\eta}=\frac{1}{2}\sum_{\alpha=1}^{n+1}(y^{n+1+\alpha}dy^\alpha-y^\alpha dy^{n+1+\alpha}), \tag{6.2}$$

then the 1-form $\tilde{\eta}$ defines a contact form on S^{2n+1}.

The Riemannian metric g on S^{2n+1} is naturally induced from the Euclidean space E^{2n+2} in such a way that

$$\tilde{g}_{\lambda\kappa}=\delta_{\lambda\kappa}+\frac{y^\lambda y^\kappa}{(y^{2n+2})^2}, \qquad \tilde{g}^{\lambda\kappa}=\delta^{\lambda\kappa}-y^\lambda y^\kappa. \tag{6.3}$$

The exterior derivative of $\tilde{\eta}$ given by (6.2) becomes

$$d\tilde{\eta}=\frac{1}{2}\sum_{\alpha=1}^{n+1}(dy^{n+1+\alpha}\wedge dy^\alpha-dy^\alpha\wedge dy^{n+1+\alpha}). \tag{6.4}$$

Since S^{2n+1} is defined by (6.1), we have

3) Sasaki and Hatakeyama [5].

$$y^{2n+2}dy^{2n+2} = -\sum_{A=1}^{2n+1} y^A dy^A.$$

From these two relations we have

$$(2F_{\lambda\kappa}) = 2(d\eta)_{\lambda\kappa} = \begin{pmatrix} 0 & \cdots & 0 & -\dfrac{y^1}{y^{2n+2}} & -1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & -\dfrac{y^n}{y^{2n+2}} & 0 & \cdots & -1 \\ \dfrac{y^1}{y^{2n+2}} & \cdots & \dfrac{y^n}{y^{2n+2}} & 0 & \dfrac{y^{n+2}}{y^{2n+2}} & \cdots & \dfrac{y^{2n+1}}{y^{2n+2}} \\ 1 & \cdots & 0 & -\dfrac{y^{n+2}}{y^{2n+2}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & -\dfrac{y^{2n+1}}{y^{2n+2}} & 0 & \cdots & 0 \end{pmatrix}.$$

Now we consider a submanifold of S^{2n+1} whose local representation is given by

(6. 5)
$$\begin{cases} y^A = x^A \ (A=1,2,\cdots,2n-1), \\ (y^{2n})^2 = t - \sum_{\alpha=1}^{2n-1} (x^\alpha)^2, \quad 0<t<1, \quad y^{2n+1}=0, \\ y^{2n+2} = \sqrt{1-t}. \end{cases}$$

Then we have

(6. 6)
$$B_i^k = \delta_i^\kappa \ (\kappa=1,2,\cdots,2n-1), \qquad B_i^{2n} = -\frac{y^\imath}{y^{2n}}, \qquad B_i^{2n+1}=0.$$

We put

(6. 7)
$$\underset{1}{N} = (C^\kappa) = (0,\cdots,0,1), \qquad \underset{2}{N} = (D^\kappa) = (y^1,\cdots,y^{2n},0).$$

Then $\underset{1}{N}$ and $\underset{2}{N}$ are mutually orthogonal vectors to the submanifold defined by (6. 5). The submanifold is, as is easily seen, totally umbilical submanifold of codimension 3 in E^{2n+2}. Since S^{2n+1} is a totally umbilical submanifold of E^{2n+2} the submanifold defined by (6. 5) is a totally umbilical submanifold of S^{2n+1}. Now we calculate $\tilde{g}(F\underset{1}{N}, \underset{2}{N})$ and find

(6. 8)
$$2r = 2F_{\lambda\kappa}C^\lambda D^\kappa = -y^n + \frac{y^{2n+1}}{y^{2n+2}} \cdot y^{n+1} = -y^n = -x^n.$$

Thus the function r is not constant. Hence the submanifold (6. 5) cannot be a normal contact submanifold.

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