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ON THE DEFICIENCIES OF MEROMORPHIC FUNCTIONS

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1. It was shown by R. Nevanlinna [3] that $k(\lambda)=\inf K(f)$ is positive, when f ranges over all meromorphic functions of positive non-integral order λ. Here

$$K(f)=\varlimsup_{r\to\infty}\frac{N(r;\ 0,f)+N(r;\ \infty,f)}{T(r,f)}.$$

He posed the problem of determining the exact value of $k(\lambda)$.

In this note we shall be concerned with an application of Nevanlinna's theorem to the sum of deficiencies. It seems to the present author that the results presented here are new and that they are worth while to be remarked.

THEOREM 1. *Let $f(z)$ be a meromorphic function of finite order λ. Assume that the sum of deficiencies of f' is equal to 2, that is,*

$$\sum\delta(a,f')=2.$$

Then λ is a positive integer.

THEOREM 2. *Let $f(z)$ be a meromorphic function of finite order λ. Then*

$$\sum\delta(a,f')\leqq 2-dk(\lambda)$$

with a positive constant d. Here for $p=[\lambda]$

$$\frac{1}{2}(5-\sqrt{21})\leqq\frac{1}{2}(4p+5-\sqrt{16p^2+36p+21})\leqq d\leqq 1.$$

It is famous that there are examples of meromorphic functions of order $p/2$ with an odd integer p and satisfying $\sum\delta(a,f)=2$ [2], [4], [5]. These examples offer a famous conjecture that $\sum\delta(a,f)=2$ would imply that λ is a half of an integer p ($\geqq 2$). Together with this hard conjecture theorems 1 and 2 show a significant difference between f and its derivative f'.

It is known that for an entire function of finite order

$$\sum_{a\neq\infty}\delta(a,f)\leqq 1-k(\lambda).$$

It would be a very interesting and significant problem to seek for the exact value of d.

2. LEMMAS. In order to prove our theorems we need some lemmas.

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LEMMA 1. *Assume that* $\delta(\infty, f)=1$ *and*

$$\sum_{a\neq\infty}\delta(a, f)=1.$$

Then $\delta(0, f')=1$ *and* $\delta(\infty, f')=1$.

This lemma is known [1], [8].

LEMMA 2. *Let* $f(z)$ *be a meromorphic function of finite order* λ. *Then*

$$\sum_{a\neq\infty}\delta(a, f^{(p)})+\frac{1}{p+1}\delta(\infty, f^{(p)})\leqq\frac{p+2}{p+1}.$$

Proof. By the second fundamental theorem for $f^{(p)}(z)$

$$m(r;\ \infty, f^{(p)})+\sum_1^q m(r;\ a_j, f^{(p)})+2N(r;\ \infty, f^{(p)})-N(r;\ \infty, f^{(p+1)})$$
$$\leqq 2T(r, f^{(p)})+O(\log r).$$

Since $2N(r;\ \infty, f^{(p)})-N(r;\ \infty, f^{(p+1)})=N(r;\ \infty, f^{(p-1)})$ and $pN(r;\ \infty, f^{(p)})\leqq(p+1)N(r;\ \infty, f^{(p-1)})$, we have

$$m(r;\ \infty, f^{(p)})+\sum_1^q m(r;\ a_j, f^{(p)})+\frac{p}{p+1}N(r;\ \infty, f^{(p)})\leqq 2T(r, f^{(p)})+O(\log r).$$

Hence we have

$$\delta(\infty, f^{(p)})+\sum_1^q\delta(a_j, f^{(p)})+\frac{p}{p+1}(1-\delta(\infty, f^{(p)}))\leqq 2,$$

which implies the desired result.

3. Proof of theorem 1. Theorem 1 is a simple consequence of theorem 2, However we shall give here a different proof. Since

$$2=\sum\delta(a, f')=\sum_{a\neq\infty}\delta(a, f')+\delta(\infty, f')$$
$$\leqq\sum_{a\neq\infty}\delta(a, f')+\Theta(\infty, f')\leqq\sum\Theta(a, f')\leqq 2,$$

we have $\delta(\infty, f')=\Theta(\infty, f')$. Further by $N(r;\ \infty, f')\geqq 2\bar{N}(r;\ \infty, f)=2\bar{N}(r;\ \infty, f')$ we have

$$2\delta(\infty, f')\geqq 1+\Theta(\infty, f').$$

Hence we have

$$\delta(\infty, f')=\Theta(\infty, f')=1.$$

Now by Lemma 1 we have

$$\delta(0, f'')=\delta(\infty, f'')=1.$$

Hence

$$k(\lambda)\leqq 2-\delta(0, f'')-\delta(\infty, f'')=0.$$

Thus λ must be a positive integer by Nevanlinna's theorem.

4. **Proof of theorem 2.** Assume that

$$2-d_0k(\lambda)<\sum_{a\neq\infty}\delta(a, f')+\delta(\infty, f').$$

This and Lemma 2 imply that $1-\delta(\infty, f')<2d_0k(\lambda)$. For simplicity's sake we denote k, δ' instead of $k(\lambda)$, $\delta(\infty, f')$, respectively. Then by Ullrich's fundamental inequalities [6] we have

$$2-\delta'-d_0k<\sum_{a\neq\infty}\delta(a, f')\leqq \varliminf_{r\to\infty}\frac{T(r, f'')}{T(r, f')}$$

$$\leqq \varlimsup_{r\to\infty}\frac{T(r, f'')}{T(r, f')}\leqq 2-\delta'.$$

Further we have

$$2-\delta'-d_0k<\sum_{a\neq\infty}\delta(a, f')\leqq(2-\delta')\delta(0, f'').$$

Hence

$$1-\delta(0, f'')<\frac{d_0k}{2-\delta'}.$$

By $2N(r;\ \infty, f'')\leqq 3N(r;\ \infty, f')$

$$1-\delta(\infty, f'')\leqq\frac{3}{2}(1-\delta')\varlimsup_{r\to\infty}\frac{T(r, f')}{T(r, f'')}$$

$$\leqq\frac{3}{2}(1-\delta')\frac{1}{2-\delta'-d_0k}.$$

Hence

$$2-\delta(0, f'')-\delta(\infty, f'')<\frac{d_0k}{2-\delta'}+\frac{3}{2}\frac{1-\delta'}{2-\delta'-d_0k}.$$

In order to estimate the right hand side from above we make use of $0\leqq 1-\delta'<2d_0k$ and then we have

$$\frac{d_0k}{2-\delta'}+\frac{3}{2}\frac{1-\delta'}{2-\delta'-d_0k}<d_0k+\frac{3d_0k}{1-d_0k}.$$

Let $\varphi(k)$ be

$$\frac{1}{2k}(4+k-\sqrt{16+4k+k^2}),$$

which is the smaller root of $kd_0^2-(4+k)d_0+1=0$. $\varphi(k)$ is monotone decreasing for $k>0$. It is known that

$$k(\lambda) \leqq \frac{|\sin(\pi\lambda)|}{p+|\sin(\pi\lambda)|} \leqq \frac{1}{p+1}$$

for $p=[\lambda]\leqq\lambda<p+1$. Hence

$$\varphi(k) \geqq \varphi\left(\frac{1}{p+1}\right) = \frac{1}{2}(4p+5-\sqrt{16p^2+36p+21})$$

for $0\leqq k\leqq 1/(p+1)$. If d_0 is not greater than $\varphi(1/(p+1))$, we have

$$2-\delta(0, f'')-\delta(\infty, f'')<k,$$

which contradicts Nevanlinna's theorem. Thus the existence of d in question has been proved and d must satisfy

$$d \geqq \varphi\left(\frac{1}{p+1}\right).$$

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