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A TURNING POINT PROBLEM OF AN n-TH ORDER DIFFERENTIAL EQUATION OF HYDRODYNAMIC TYPE

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§1. Introduction.

In this paper, we propose to study a linear ordinary differential equation of the n-th order of the form:

$$\varepsilon^{n-m}L_n(y)+L_m(y)=0, \tag{1. 1}$$

where $n-2\geqq m\geqq 0$ and

$$L_n(y)=-y^{(n)}+\sum_{\nu=m+1}^{n-1}R_{\nu+1}(x,\varepsilon)y^{(\nu)},$$

$$L_m(y)=\sum_{\nu=0}^{m}(P_{\nu+1}(x)+\varepsilon R_{\nu+1}(x,\varepsilon))y^{(\nu)}.$$

Here ε is a small positive parameter, x is a complex independent variable, y is an unknown function of x, $R_\nu(x,\varepsilon)$ are asymptotic power series of ε with coefficients holomorphic in x in the domain

$$0<\varepsilon\leqq\varepsilon_0,\qquad |x|\leqq c_0<1, \tag{1. 2}$$

and $P_\nu(x)$ are holomorphic functions in x, in particular, $P_{m+1}(x)$ has a zero of order q at the origin. Thus we can consider that the equation (1. 1) has a turning point of order q at the origin, and our purpose is to give complete informations about the asymptotic behavior of the solutions of (1. 1) in the neighborhood of the origin when ε tends to zero. Our method is based on the matching method which was used for the first time by Wasow [10] with the rigorous mathematical justification in the case of an almost diagonal second order system, and thereafter has been generalized by Wasow [11] and Nishimoto [5] to the n-th order equation with $m=0$. Introductory descriptions of this method are seen in Friedrichs [1] and Wasow [13].

When $n=4$, $m=2$ and $q=1$, the equation (1. 1) is equivalent to the well-known Orr-Sommerfeld equation which plays a fundamental role in the theory of stability of incompressible fluid dynamics. There are many investigations about this equation, for example, by Wasow [9], and by Lin and Labenstein [4]. They used the method of comparison equations to attempt to find a transformation which reduces the given equation to a simpler equation, and to attempt to solve the simplified equation

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by some explicit technique, for example, by Laplaces integral. In fact, the Orr-Sommerfeld equation was essentially solved by this procedure. To generalize this method to other cases, there are two approaches: one is to enlarge the class of differential equations which can be solved by some technique and are available for our purpose, and the other is to construct a nonsingular transformation which makes the given equation as simple as possible. About the second problem, Sibuya [6, 7] succeeded in obtaining a certain transformation for the equation (1. 1) with $n-2\geqq m$, and $q=1$, and some of the simplified equuations can be solved by the Laplace integral but there remain the equations which are unresolved, moreover when $q>1$ we can no longer construct such a transformation. On the first problem, our data of the equations whose behavior are already known can not almost be seen other than Sibuya [6], Wasow [11] and Nishimoto [5] in the general theory, and so this paper is devoted to this problem of the equation of the form (1. 1). Thus we may consider that the equation (1. 1) is already simplified by some transformation. Our method based on the matching method, in spite of rather complexity of the actual calculations of the solutions, enables us to understand the asymptotic natures of the solutions of (1. 1) in the full neighborhood of the origin under fairly reasonable assumptions. This method may also be applicable to the problem of the stability of boundary layers in a compressible gas (Lees and Lin [3]) which is not yet completely solved.

In § 2, we give notations, a preliminary transformation which makes further treatment simpler, and assumptions on the coefficients of (1. 1), one of which is so-called one segment condition and dominates all of the studies in this paper. In § 3, we construct the formal outer solution, and in § 4, § 5 obtain the outer domain D_1 where there exists the actual solution of (2. 1) whose asymptotic expansion coincides with the formal outer solution. The domain D_1 does not contain the turning point itself and then to understand the asymptotic behavior of the one outer solution at the turning point or beyond the boundary lines of D_1 is just the turning point problem. Therefore to solve this problem, it needs to construct an inner soluion in a direct neighborhood of the turning point itself. In § 6 and § 7, it is calculated the formal inner solution by introducing the stretching variable, and prove the existence of actual solutions in the inner domain D_2 in § 8. The domain D_2, in general, shrinks to the origin when ε tends to zero, but it is easily seen that D_1 and D_2 overlap with each other for an arbitrarily small ε. From this fact, we can match the two types of solution, and then in § 9 it is given an asymptotic expansion of the matching matrix between them, from which we can understand the asymptotic behavior of the one outer solution in the complete neighborhood of the turning point.

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§ 2. Notations and assumptions.

1. For the subsequent study, it is convenient to write the equation (1. 1) by the vector form, that is, by the usual transformation

$$y=y_1,\quad y^{(i)}=y_{i+1}\quad (i=0,1,\cdots,m),\qquad \varepsilon y^{(i)}=y_{i+1}\quad (i=m+1,\cdots,n-1),$$

the equation (1. 1) becomes

$$(2.\,1)'\qquad \varepsilon\begin{bmatrix} y_1 \\ y_m \\ y_{m+1} \\ y_n \end{bmatrix}' = \left[\begin{array}{ccc:ccc} 0 & \varepsilon & & & & \\ & \ddots & \varepsilon & & \mathbf{0} & \\ \mathbf{0} & & 0 & \varepsilon & & \\ \hdashline & & & 0 & 1 & \\ & \mathbf{0} & & & \ddots & 1 \\ P_1+\varepsilon R_1, & \cdots, & P_m+\varepsilon R_m & P_{m+1}+\varepsilon R_{m+1}, & \varepsilon^{n-m-1}R_{m+2}, \cdots, & \varepsilon R_n \end{array}\right] \begin{bmatrix} y_1 \\ y_m \\ y_{m+1} \\ y_n \end{bmatrix}.$$

To simplify the descriptions of further calculations we write the above equation as

$$(2.\,1)\qquad \begin{cases} U'=AU+BV, \\ \varepsilon V'=CU+DV, \end{cases}$$

where

$$U=\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},\quad V=\begin{bmatrix} y_{m+1} \\ \vdots \\ y_n \end{bmatrix},\quad A=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & 0 \end{bmatrix},\quad B=\begin{bmatrix} & \mathbf{0} & \\ 1, & 0, \cdots, & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} & \mathbf{0} & \\ P_1+\varepsilon R_1, & \cdots, & P_m+\varepsilon R_m \end{bmatrix},\quad D=\begin{bmatrix} 0 & 1 & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \\ & & \ddots & 1 \\ P_{m+1}+\varepsilon R_{m+1}, & \varepsilon^{n-m-1}R_{m+2}, & \cdots, & \varepsilon R_m \end{bmatrix}.$$

2. A following transformation of V and x into $\tilde{V}$ and t makes it possible to assume from the outset that $P_{m+1}(x)=x^q$ (for details, see Wasow [12]).

$$V=\operatorname{diag}(1,\omega(x),\cdots,\omega(x)^{n-m-1})\tilde{V},\qquad t=t(x),$$

where

$$\omega(x)=\frac{dt}{dx},\qquad t(x)=\left[\int_0^x P_{m+1}(x)^{1/(n-m)}dx\right]^{(n-m)/(n-m+q)}.$$

Clearly, the functions $t(x)$ and $\omega(x)$ are holomorphic in x, and can be written

$$t(x)=\alpha x+0(x^2)\ (\alpha\neq 0),\qquad \omega(x)=\alpha+0(x).$$

Since it is easily verified that this transformation does not make any essential change in the subsequent analyses, we assume that $p_{m+1}(x)=x^q$ already in the equation (2. 1).

3. Next, we write here the fundamental assumption which states that the characteristic polygon associated with (2. 1) consists of only one segment. Now let each of the elements of coefficient matrices C and D has the asymptotic expansion of the form

$$
(2.2)\qquad
\begin{aligned}
&P_j(x)+\varepsilon R_j(x,\varepsilon)\cong \sum_{\nu=0}^{\infty}\left(\sum_{\mu=0}^{\infty}p_{j\nu\mu}x^{\mu}\right)\varepsilon^{\nu} && (j=1,2,\cdots,m+1),\\
&\varepsilon^{n-j+1}R_j(x,\varepsilon)\cong \sum_{\nu=n-j+1}^{\infty}\left(\sum_{\mu=0}^{\infty}p_{j\nu\mu}x^{\mu}\right)\varepsilon^{\nu} && (j=m+2,\cdots,n).
\end{aligned}
$$

In the (X, Y) plane, we plot the points $\mathrm{P}_{j\nu\mu}$, for which the coefficients $p_{j\nu\mu}$ of the above expansions are not zeros, of the coordinates

$$
\mathrm{P}_{j\nu\mu}=\left(\frac{m-j+1+\nu}{n-j+1},\frac{\mu}{n-j+1}\right)\qquad (j=1,2,\cdots,m+1;\ \mu=0,1,\cdots),
$$

$$
\mathrm{P}_{j\nu\mu}=\left(\frac{\nu}{n-j+1},\frac{\mu}{n-j+1}\right)\qquad (j=m+2,\cdots,n;\ \mu=0,1,\cdots),
$$

and the point $\mathrm{R}=(+1,-1)$. The one segment condition means that all of the points $\mathrm{P}_{j\nu\mu}$ are on or above the segment L_0 which combines the point R and $\mathrm{P}_{m+1,0,q}=(0, q/n-m)$, or equivalently for nonzero coefficients $p_{j\nu\mu}$ the indices must satisfy the following inequality:

$$
(2.3)\qquad \mu+\frac{n-m+q}{n-m}\nu+m+1-(j+q)\geqq 0\qquad (j=1,2,\cdots,m+1).
$$

Here it is noticed that from the inequality (2. 3) we can easily see that when ε tends to zero in the equation (1. 1), the reduced equation:

$$
(2.4)\qquad \sum_{\nu=0}^{m}P_{\nu+1}(x)y^{(\nu)}=0
$$

has a regular singular point at the origin. About this equation, we make an assumption to avoid complexity that the difference of any two characteristic roots of (2. 4) is not an integer.

§ 3. Formal outer solution.

4. At first, if we transform the equation (2. 1) by the relations

$$
(3.1)\qquad U=\Omega_1(x)U_1,\qquad V=\Omega_2(x)V_1,\qquad t=\varepsilon x^{-a}\qquad (a=(n-m+q)/(n-m)),
$$

where

$$
\Omega_1(x)=\mathrm{diag}\,[x^m, x^{m-1},\cdots,x],\qquad \Omega_2(x)=\mathrm{diag}\,[1, x^{q/(n-m)},\cdots,x^{q(n-m-1)/(n-m)}],
$$

then after a short calulation we have

$$x\frac{dU_1}{dx}=A_1U_1+B_1V_1,$$

(3. 2)

$$tx\frac{dV_1}{dx}=C_1U_1+D_1V_1$$

with

$$A_1=\begin{bmatrix} -m & 1 & & 0 \\ & -m+1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & -1 \end{bmatrix},\qquad B_1=\begin{bmatrix} 0 \\ 1, 0, \cdots, 0 \end{bmatrix},$$

$$C_1=\begin{bmatrix} 0 \\ x^{-q+m}(P_1+\varepsilon R_1), \cdots, x^{-q+m+1-j}(P_j+\varepsilon R_j), \cdots, x^{-q+1}(P_m+\varepsilon R_m) \end{bmatrix},$$

$$D_1=\begin{bmatrix} 0 & 1 & & & 0 & \\ & 0 & \ddots & & \ddots & \\ & & & & & 1 \\ x^{-q}(x^q+R_{m+1}), & (tx)^{n-m-1}R_{m+2}, & \cdots, & (tx)^{n-j+1}R_j, & \cdots, & txR_m \end{bmatrix}-\frac{q}{n-m}t\begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & n-m-1 \end{bmatrix}.$$

Now we prove the following lemma.

LEMMA 3. 1. *Each element in the matrices C_1 and D_1 can be expanded asymptotically in power series of t whose coefficients are holomorphic functions of $x^{1/(n-m)}$ in the domain* (1. 2).

Proof. From (2. 2) we have for $j=1, 2, \cdots, m+1$,

$$x^{-q+m+1-j}(P_j+\varepsilon R_j)\cong \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty} p_{j\nu\mu}x^{\mu-q+m+1-j}\varepsilon^{\nu}$$

$$\cong \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} p_{j\nu\mu}x^{\mu-q+m+1-j+(n-m+q)\nu/(n-m)}t^{\nu}.$$

If we consider the assumption (2. 3), the above expression can be written

$$\cong \sum_{\nu=0}^{\infty} \tilde{p}_\nu(x)t^{\nu},$$

where $\tilde{p}_\nu(x)$ are power series of $x^{1/(n-m)}$. For the elements $(tx)^{n-j+1}R_j$, it is the same as above, and the lemma is proved.

5. From the above lemma, we can write the matrices C_1 and D_1 by

(3. 3)
$$C_1\cong \sum_{\nu=0}^{\infty} C_{1\nu}(x)t^{\nu},\qquad D_1\cong \sum_{\nu=0} D_{1\nu}(x)t^{\nu}$$

with

$$C_{10}(x)=\begin{bmatrix} & 0 & \\ c_{10l}(x), & \cdots, & c_{10m}(x)\end{bmatrix}, \qquad C_{11}(x)=\begin{bmatrix} & 0 & \\ c_{11l}(x), & \cdots, & c_{11m}(x)\end{bmatrix},$$

$$D_{10}=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ 1, & 0, & \cdots, & 0 \end{bmatrix}, \qquad D_{11}(x)=\begin{bmatrix} & 0 & \\ d_{11m+1}, & 0, \cdots, & 0\end{bmatrix}-\frac{q}{n-m}\begin{bmatrix} 0 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & n-m-1 \end{bmatrix}+O(x^{1/(n-m)}),$$

where the constants $c_{10j}(0)$ and $c_{11j}(0)$ are the quantities $p_{j0\mu_0}$ and $p_{j1\mu_1}$ respectively in the expression (2. 2) for which $\mu_\nu+m+1-(q+j)+a\nu=0$ $(\nu=0, 1)$, and d_{11m+1} is $p_{m+1,1,\mu}$ for which $\mu-q+(n-m+q)/(n-m)=0$.

To solve the equation (3. 2) by formal power series of t, it is conventient to make the principal parts of the coefficient matrices of (3. 2) diagonal and this is accomplished by the following lemma.

LEMMA 3. 2. *We can construct a nonsingular linear transformation of the form*

$$\text{(3. 4)}\qquad \begin{aligned} U_1&=Q_{11}(x)U_2+\{Q_{12}^{(1)}(x)t+Q_{12}^{(2)}(x)t^2\}V_2,\\ V_1&=\{Q_{21}^{(0)}(x)+Q_{21}^{(1)}(x)t\}U_2+\{Q_{22}^{(0)}(x)+Q_{22}^{(1)}(x)t\}V_2,\end{aligned}$$

where $Q_{ij}^{(k)}(x)$ are holomorphic functions of $x^{1/(n-m)}$, and the equation (3. 2) *is reduced by this transformation to*

$$\text{(3. 5)}\qquad x\frac{dU_2}{dx}=A_2U_2+B_2V_2, \qquad tx\frac{dV_2}{dx}=C_2U_2+D_2V_2,$$

where the coefficient matrices have asymptotic expansions in power series of t such that

$$\text{(3. 6)}\qquad \begin{aligned} A_2&\cong\sum_{\nu=0}^{\infty}A_{2\nu}(x)t^\nu, & B_2&\cong\sum_{\nu=1}^{\infty}B_{2\nu}(x)t^\nu,\\ C_2&\cong\sum_{\nu=2}^{\infty}C_{2\nu}(x)t^\nu, & D_2&\cong\sum_{\nu=0}^{\infty}D_{2\nu}(x)t^\nu\end{aligned}$$

in the domain

$$\text{(3. 7)}\qquad 0<\varepsilon\leqq\varepsilon_1, \qquad |x|\leqq c_1, \qquad |t|\leqq c_2$$

for sufficiently small positive number ε_1, c_1 and c_2. Here $A_{20}(x)$ is diagonal matrix, holomorphic in x, and the difference of any two diagonal elements at $x=0$ is not an integer, $D_{20}(x)$ is a constant diagonal matrix, and $D_{21}(x)$ is a diagonal holomorphic matrix function of $x^{1/(n-m)}$.

Proof. Firstly, if the equation (3. 2) is transformed by

$$\text{(3. 8)}\qquad U_1=\tilde{U}_2+tQ\tilde{V}_2, \qquad V_1=R\tilde{U}_2+\tilde{V}_2,$$

(here the symbol R is different from the one in the definition of equation (1. 1)) then we have

$$
\begin{aligned}
x\frac{d\tilde{U}_2}{dx} &= (I_m - tQR)^{-1}\left[\left\{A_1 + B_1R - Q\left(C_1 + D_1R - tx\frac{dR}{dx}\right)\right\}\tilde{U}_2\right. \\
&\quad \left. + \left\{tA_1Q + B_1 - x\frac{dtQ}{dx} - (C_1Q + D_1)\right\}\tilde{V}_2\right], \\
-tx\frac{d\tilde{V}_2}{dx} &= (I_{n-m} - tRQ)^{-1}\left[\left\{tR(A_1 + B_1R) - \left(C_1 + D_1R - tx\frac{dR}{dx}\right)\right\}\tilde{U}_2\right. \\
&\quad \left. + \left\{tR\left(tA_1Q + B_1 - x\frac{dtQ}{dx}\right) - (tC_1Q + D_1)\right\}\tilde{V}_2\right],
\end{aligned} \tag{3. 9}
$$

where I_r denotes the r-dim unit matrix. Here we choose the matrices Q and R by the form

$$Q = Q_0, \qquad R = R_0(x) + R_1(x)t$$

with

$$
\begin{aligned}
&Q_0D_{10} - B_1 = 0, \qquad C_{10}(x) + D_{10}R_0(x) = 0, \\
&D_{10}R_1(x) + D_{11}(x)R_0(x) + C_{11}(x) - R_0(x)(A_1 + B_1R_0(x)) - x\frac{dR_0(x)}{dx} = 0.
\end{aligned} \tag{3. 10}
$$

Then it is easily verified that

$$tA_1Q + B_1 - x\frac{dtQ}{dx} = Q(tC_1Q + D_1) + O(t),$$

$$C_1 + D_1R - tx\frac{dR}{dx} = tR(A_1 + B_1R) + O(t^2),$$

which imply that (3. 9) can be written

$$
\begin{aligned}
x\frac{d\tilde{U}_2}{dx} &= (A_1 + B_1R + O(t))\tilde{U}_2 + O(t)\tilde{V}_2, \\
tx\frac{d\tilde{V}_2}{dx} &= O(t^2)\tilde{U}_2 + (tC_1Q + D_1 + O(t^2))\tilde{V}_2.
\end{aligned} \tag{3. 11}
$$

Here it is noted that from (3. 3) and (3. 10) we have

$$A_1 + B_1R = \begin{bmatrix} -m & 1 & & \\ & -(m-1) & \ddots & 0 \\ 0 & & \ddots & 1 \\ -c_{101}(x), & \cdots, & & -c_{10m}(x) - 1 \end{bmatrix} + O(t),$$

$$D_1 + tC_1Q = D_{10} + (D_{11}(x) + C_{10}(x)Q)t + O(t^2).$$

The assumption imposed on the coefficients of the reduced equation (2. 4) implies that the difference of any two characteristic roots of the matrix A_1+B_1R at $x=0$ is not an integer, and from this and from the form of D_{10}, we can easily verified by the usual method that there exists a non-singular transformation of the form

$$\tilde{U}_2=\tilde{Q}_{11}(x)U_2, \qquad \tilde{V}_2=\{\tilde{Q}_{22}^{(0)}(x)+\tilde{Q}_{22}^{(1)}(x)t\}\,V_2 \tag{3. 12}$$

which makes A_1+B_1R and $D_{10}+(D_{11}(x)+C_{10}(x)Q)t$ diagonal. If we combine the transformations (3. 8) and (3. 12), the lemma is proved.

6. Now we are ready to construct a fundamental system of formal solutions of the equation (3. 5) by the form

$$W=\left\{\sum_{\nu=0}^{\infty} W_\nu(x)t^\nu\right\}\exp\int^x E(x,t)dx \tag{3. 13}$$

with

$$E(x,t)=\begin{bmatrix} A_{20}(x)/x & 0 \\ 0 & [D_{20}+D_{21}(x)t]/tx \end{bmatrix}, \qquad W(x)=\begin{bmatrix} W^{11}(x) & W^{12}(x) \\ W^{21}(x) & W^{22}(x) \end{bmatrix},$$

where W^{11}, W^{12}, W^{21} and W^{22} are $m\times m$, $m\times(n-m)$, $(n-m)\times m$ and $(n-m)\times(n-m)$ matrices respectively, and the integral in (3. 13) is to be determined such that the constant term is zero. If we substitute (3. 13) into (3. 5), replace the matrices A_2, B_2, C_2 and D_2 by their asymptotic power series (3. 6) and compare the coefficients of t^μ $(\mu=0, 1, 2, \cdots)$ of the left and right hand sides, then we obtain the recursion formulas for $W_\mu(x)$;

(3. 14)$_\mu$

$$W_\mu\begin{bmatrix} 0 & 0 \\ 0 & D_{20} \end{bmatrix}+W_{\mu-1}\begin{bmatrix} A_{20} & 0 \\ 0 & D_{21} \end{bmatrix}+x\frac{dW_{\mu-1}}{dx}-a(\mu-1)W_{\mu-1}=\sum_{i+j=\mu}\begin{bmatrix} A_{2,i-1} & B_{2,i-1} \\ C_{2,i} & D_{2,i} \end{bmatrix}W_j,$$

where $W_{-1}\equiv 0$ and $W_0=I_n$ (n-dim unit matrix). From this equation, we can determine all of the matrices $W_\mu(x)$ by the following way. For $\mu=1$ (3. 14) becomes

$$W_1(x)\begin{bmatrix} 0, & 0 \\ 0, & D_{20} \end{bmatrix}=\begin{bmatrix} 0, & 0 \\ 0, & D_{20} \end{bmatrix}W_1(x), \quad \text{or} \quad \begin{bmatrix} 0, & W_1^{12}D_{20} \\ 0, & W_1^{22}D_{20} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ D_{20}W_1^{21}, & D_{20}W_1^{22} \end{bmatrix}, \tag{3. 14$_1$}$$

from which we can conclude by using the facts that the matrix D_{20} is nonsingular and any of the two diagonal elements does not coincide that $W_1^{12}=W_1^{21}=0$, and if each element of the matrix $W_\nu^{l,m}$ $(l, m=1, 2)$ is denoted by $w_{\nu jk}^{lm}$, then we have $w_{1jk}^{22}=0$ for $j\neq k$. Clearly we can not determine the matrix W_1^{11} and the elements w_{1jj}^{22} from (3. 14)$_1$, and these elements will be obtained from the equation (3. 14) with $\mu=2$;

$$
(3.14)_2 \qquad \begin{aligned}
&\begin{bmatrix} 0, & W_2^{12}D_{20} \\ 0, & W_2^{22}D_{20} \end{bmatrix} + \begin{bmatrix} W_1^{11}A_{20}, & W_1^{12}D_{21} \\ W_1^{21}A_{20}, & W_2^{22}D_{21} \end{bmatrix} + x\frac{d}{dx}\begin{bmatrix} W_1^{11}, & W_1^{12} \\ W_1^{21}, & W_1^{22} \end{bmatrix} - a\begin{bmatrix} W_1^{11}, & W_1^{12} \\ W_1^{21}, & W_1^{22} \end{bmatrix} \\
&= \begin{bmatrix} 0, & 0 \\ D_{20}W_2^{21}, & D_{20}W_2^{22} \end{bmatrix} + \begin{bmatrix} A_{20}W_1^{11}, & A_{20}W_1^{12} \\ D_{21}W_2^{21}, & D_{21}W_1^{22} \end{bmatrix} + \begin{bmatrix} A_{21}, & B_{21} \\ C_{22}, & D_{22} \end{bmatrix}.
\end{aligned}
$$

Firstly, since the elements W_1^{12} and W_1^{21} are known, W_2^{12} and W_2^{21} are determined uniquely, and also since w_{1jk}^{22} $(j \neq k)$ are known and the matrix D_{21} is diagonal, then we can obtain the elements w_{2jk}^{22} $(j \neq k)$. Next the elements w_{1jj}^{22} will be determined. From $(3.14)_2$ w_{1jj}^{22} satisfies the differential equation

$$
x\frac{dw_{1jj}^{22}}{dx} = aw_{1jj}^{22} + d_{22,jj},
$$

where d_{22jj} is the $j-j$ element of the matrix D_{22}. Since d_{22jj} is a holomorphic function of $x^{1/(n-n)}$, the above equation has a solution of the form

$$
w_{1jj}^{22} = f_1(x) + f_2(x) \log x,
$$

where $f_1(x)$ and $f_2(x)$ are holomorphic functions of $x^{1/(n-m)}$ and $f_2(0)=0$. At last, we will determine W_1^{11}. The equation for which w_{1jk}^{11} must satisfy are from $(3.14)_2$,

$$
x\frac{dw_{1jk}^{11}}{dx} = \{a + a_{20j}(x) - a_{20k}(x)\}w_{1jk}^{11} + a_{21jk}(x) \qquad (j \neq k),
$$

$$
x\frac{dw_{1jj}^{11}}{dx} = aw_{1jj}^{11} + a_{21jj}(x)
$$

where $a_{20j}(x)$ is the j-th diagonal element of $A_{20}(x)$ and $a_{21jk}(x)$ is the $j-k$ element of $A_{21}(x)$.

Here we assume for simplifications of further calculations of formal solution that

(3.15) *none of the quantities* $(n-m)\{a_{20j}(0) - a_{20k}(0)\}$ $(j, k = 1, 2, \cdots, m,\ j \neq k)$ *is integer.*

This assumption also simplifies descriptions of Proposition 6.1, Lemma 7.2 and the proof of Theorem 9.1 (see Remark of § 9).

Now since $a_{21jk}(x)$ is a holomorphic function of $x^{1/(n-m)}$, the above equations can be solved by the forms

$$
w_{1jk}^{11}(x) = f_1(x) \qquad (j \neq k),
$$

$$
w_{1jj}^{11}(x) = f_2(x) + f_3(x) \log x,
$$

where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are some holomorphic functions of $x^{1/(n-m)}$ and $f_3(0)=0$. Therefore from the equation $(3.14)_2$, we can obtain the elements W_1^{11}, w_{1jj}^{22}, W_2^{12}, W_2^{21} and w_{2jk}^{22} $(j \neq k)$ and undetermined elements are W_2^{11} and w_{2jj}^{22} which will be obtained from the equation $(3.14)_3$ by the same method as for W_1^{11} and w_{1jj}^{22}.

By repeating the above procedure, we can determine all of the coefficient matrices $W_\nu(x)$ and then the formal solution (3. 13). Here we summarize the results:

PROPOSITION. 3. 1. *The differential equation* (3. 5) *has a fundamental system of formal solutions of the form*

$$W \sim \left\{ \sum_{\nu=0}^{\infty} W_\nu(x) t^\nu \right\} F(t, x), \tag{3.16}$$

where the matrices $W_\nu(x)$ *are bounded in* $|x| \leqq c_1$ *and are polynomials of* $\log x$ *of degree at most* ν *with holomorphic coefficients of* $x^{1/(n-m)}$, *in particular* $W_0(x)=I$, *and* $F(t, x)$ *is diagonal and can be written*

$$F(t, x) = \begin{bmatrix} x^{A_{20}(0)}, & 0 \\ 0, & \exp\left\{ \dfrac{D_{20}}{at} \right\} \cdot x^{D_{21}(0)} \end{bmatrix}. \tag{3.17}$$

Henceforth we denote for convenience the diagonal elements of $A_{20}(0)$, D_{20} and $D_{21}(0)$ by the letters

$$\begin{aligned} &\operatorname{diag} A_{20}(0) = \{a_1, \cdots, a_m\}, \\ &\operatorname{diag} \frac{D_{20}}{a} = \{d_{0m+1}, \cdots, d_{0n}\}, \qquad d_{0k} = \frac{1}{a} \exp\left\{ \frac{2(k-m-1)_i}{n-m} + 2\pi r i \right\} \quad (i=\sqrt{-1}), \\ &\operatorname{diag} D_{21}(0) = \{d_{1m+1}, \cdots, d_{1n}\}. \end{aligned} \tag{3.18}$$

§ 4. Existence theorem of outer solution (1).

7. In this section we prove that for each formal solution there exists an actual solution whose asymptotic expansion coincides with it. The domain of existence D_1 is maximal in the sense of the angle of sector, and this fact is sometimes useful, for example, when we apply the results to the boundary value problems.

Our argument is given for the equation (3. 5), and rewrite this by the form

$$tx \frac{dW}{dx} = G(t, x) W, \tag{4.1}$$

where

$$W = \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}, \qquad G(t, x) = \begin{bmatrix} tA_2, & tB_2 \\ C_2, & D_2 \end{bmatrix}.$$

The equation (4. 1) has a fundamental system of formal solution (3. 15).

Let r be a positive integer, and define the matrix functions $W^{(r)}(t, x)$ and $G^{(r)}(t, x)$ by

$$W^{(r)}(t, x) = \left\{ \sum_{\nu=0}^{r+1} W_\nu(x) t^\nu \right\} F(t, x),$$

$$G^{(r)}(t, x)=tx\frac{dW^{(r)}(t, x)}{dx}\cdot W^{(r)}(t, x)^{-1}.$$

Clearly $W^{(r)}(t, x)$ is a fundamental solution of the differential equation

$$tx\frac{dW}{dx}=G^{(r)}(t, x)W$$

and $G^{(r)}(t, x)$ satisfies

$$G(t, x)-G^{(r)}(t, x)=O(t^{r+2}). \tag{4. 2}$$

We write (4. 1) in the form

$$tx\frac{dW}{dx}=\{G^{(r)}(t, x)+G(t, x)-G^{(r)}(t, x)\}W, \tag{4. 3}$$

then by the method of variation of constants, any solution of the integral equation

$$W(t, x)=W^{(r)}(t, x)+\int_{\Gamma(x)}(\tau\xi)^{-1}W^{(r)}(t, x)W^{(r)}(\tau, \xi)^{-1}\{G(\tau, \xi)-G^{(r)}(\tau, \xi)\}W(\tau, \xi)d\xi \tag{4. 4}$$

is a solution of (4. 1). Here $\Gamma(x)$ denotes a set of paths of integration $\lambda_{jk}(x)$ $(j, k=1, 2, \cdots, n)$ in the ξ plane which are chosen appropriately for each pair of (j, k), and $\tau=\varepsilon\xi^{a}$.

If we put

$$\begin{aligned} W(t, x)&=\hat{W}(t, x)F(t, x),\\ W^{(r)}(t, x)&=\hat{W}^{(r)}(t, x)F(t, x), \end{aligned} \tag{4. 5}$$

then (4. 4) becomes

$$\begin{aligned} \hat{W}(t, x)=\hat{W}^{(r)}(t, x)+\int_{\Gamma(x)}&(\tau\xi)^{-1}F(t, x)F^{-1}(\tau, \xi)\hat{W}^{(r)}(t, x)\hat{W}^{(r)}(\tau, \xi)^{-1}\\ &\times\{G-G^{(r)}\}\hat{W}(\tau, \xi)F(\tau, \xi)F(t, x)^{-1}d\xi. \end{aligned} \tag{4. 6}$$

From (3. 16) and (4. 2), the integral term of the above equation can be written for each j, k,

$$\begin{aligned} &\varepsilon^{r+1}\int_{\lambda_{jk}}\xi^{-a(r+1)-1}(x/\xi)^{a_j-a_k}L_{jk}[\hat{W}(\tau, \xi)]d\xi \qquad (j, k=1, \cdots, m),\\ &\varepsilon^{r+1}\int_{\lambda_{jk}}\xi^{-a(r+1)-1}(x/\xi)^{a_j-d_{1k}}\{\exp\varepsilon^{-1}d_{0k}(\xi^{a}-x^{a})\}L_{jk}[\hat{W}(\tau, \xi)]d\xi\\ &\qquad\qquad\qquad\qquad (j=1, \cdots, m, k=m+1, \cdots, n),\\ &\varepsilon^{r+1}\int_{\lambda_{jk}}\xi^{-a(r+1)-1}(x/\xi)^{d_{1j}-a_k}\{\exp\varepsilon^{-1}d_{0j}(x^{a}-\xi^{a})\}L_{jk}[\hat{W}(\tau, \xi)]d\xi\\ &\qquad\qquad\qquad\qquad (j=m+1, \cdots, n, k=1, \cdots, m),\\ &\varepsilon^{r+1}\int_{\lambda_{jk}}\xi^{-a(r+1)-1}(x/\xi)^{d_{1j}-d_{1k}}\{\exp\varepsilon^{-1}(d_{0j}-d_{0k})(x^{a}-\xi^{a})\}L_{jk}[\hat{W}(\tau, \xi)]d\xi\\ &\qquad\qquad\qquad\qquad (j, k=m+1, \cdots, n), \end{aligned} \tag{4. 7}$$

where $L_{jk}[\hat{W}]$ is a linear form of the n-components in the k-th column of $\hat{W}$. From (4. 2) and the form of $\hat{W}(t, x)$, the coefficients of this linear form are bounded if t, x, τ and ξ are bounded.

Here we define a sector $S_{1\gamma}^{(k)}$. Let $k=1, 2, \cdots, m$ and chose arbitrarily one of the arguments of d_{0j} $(j=m+1, \cdots, n)$, denote it by $d_{0\mu_1}$ and reorder the remainder arguments of d_{0j} such that

$$\arg d_{0\mu_1} < \arg d_{0\mu_2} < \cdots < \arg d_{0\mu_\beta} < \arg d_{0\mu_1} + 2\pi (\beta \leqq n-m).$$

Then the sector $S_{1\gamma}^{(k)}$ is of the central angle less than $(n-m+2)\pi/(n-m+q)$ of the form

$$S_{1\gamma}^{(k)}: \frac{n-m}{n-m+q}\left\{-\frac{3}{2}\pi - \arg d_{0\mu_1} + \gamma\right\} \leqq \arg x \leqq \frac{n-m}{n-m+q}\left\{\frac{3}{2}\pi - \arg d_{0\mu_\beta} - \gamma\right\},$$

where γ is a sufficiently small positive constant.

Next let $k=m+1, \cdots, n$, and order the arguments of $-d_{0k}$, $d_{0j}-d_{0k}$ $(j=m+1, \cdots, n, j \neq k)$ by

$$0 \leqq \arg d_{0\mu_1} < \arg d_{0\mu_2} < \cdots < \arg d_{0\mu_\beta} < \arg d_{0\mu_1} + 2\pi (\beta \leqq n-m+1),$$

and define the sector $S_{1\gamma}^{(k)}$

$$S_{1\gamma}^{(k)}: \frac{n-m}{n-m+q}\left\{-\frac{3}{2}\pi - \arg d_{0\mu_1} + \gamma\right\} \leqq \arg x \leqq \frac{n-m}{n-m+q}\left\{\frac{3}{2}\pi - \arg d_{0\mu_\beta} - \gamma\right\}.$$

The central angle of this $S_{1\gamma}^{(k)}$ is less than either $(n-m+1)\pi/(n-m+q)$ or $2(n-m+1)\pi/(n-m+q)$ according to the selection of $d_{0\mu_1}$.

In the next section we will prove a following proposition.

PROPOSITION 4. 1. *For each k $(k=1, 2, \cdots, n)$, there exists a region $\tilde{D}_1^{(k)}$ which contains a domain $D_1^{(k)}$ defined by*

$$D_1^{(k)}: \arg x \in S_1^{(k)}, \qquad 0 < \varepsilon \leqq \varepsilon_1, \qquad \tilde{c}_2 \varepsilon^{1/a} \leqq x \leqq c_1$$

such that for all $x \in \tilde{D}_1^{(k)}$, we can construct paths of integration $\lambda_{jk}(x)$ which are contained in $\tilde{D}_1^{(k)}$ except of its end point and for ξ on $\lambda_{jk}(x)$, we have

$$\int_{\lambda_{jk}(x)} |\xi|^{-ar-1} |d\xi| \leqq K |x|^{-ar}, \tag{4. 8}$$

(4. 9) *the exponential factors in the integrands of* (4. 7) *are bounded for arbitrarily small ε.*

Here ε_1, c_1 and $\tilde{c}_2$ are sufficiently small positive constants, and K is some positive number independent of x.

If it is assumed that the above proposition is true, we can estimate that the every integral in (4. 7) is of the order $O(t^{r+1})$ and so we can show that there exists a solution of the integral equation (4. 6) by a standard method of successive approximation or a fixed point theorem, and therefore exists a corresponding solution of the differential equation (4. 1) in the domain $D_1^{(k)}$. Furthermore we can prove that the actual solution thus obtained does not depend on r and has an asymptotic expansion which coincides with the formal solution. The details of this procedure are here omitted and are rendered to the previous paper [5]. From the above descriptions we obtain immediately an existence theorem of fundamental system of solutions.

Let us draw $(n-m)(n-m+1)$ vectors d_{0j}, $-d_{0j}$ and $d_{0j}-d_{0k}$ $(j, k=m+1,\cdots, n, j\neq k)$ from the origin in the complex plane, select arbitrarily one of them and denote it d_1 and then order counterclockwise the remainder vectors such that

$$\arg d_1<\arg d_2<\cdots<\arg d_\beta<\arg d_1+2\pi(\beta\leqq(n-m)(n-m+1)),$$

and we define the sector S_1 in the x-plane by

$$S_1\colon \frac{n-m}{n-m+q}\left\{-\frac{3}{2}\pi-\arg d_1\right\}<\arg x<\frac{n-m}{n-m+q}\left\{\frac{3}{2}\pi-\arg d_\beta\right\}$$

Now we have a following theorem:

THEOREM 4. 1. *Let*

$$W\sim\left\{\sum_{\nu=0}^{\infty} W_\nu(x)t^\nu\right\}F(t, x)$$

be a formal solution of (4. 1) *defined in Proposition* 3. 1. *Then there exists a fundamental system of actual solutions of* (4. 1) *of the form*

$$W(t, x)=\hat{W}(t, x)F(t, x),$$

and for every positive integer r, there exists a domain D_1 of x, ε plane defined by

$$D_1\colon \arg x\in S_1, \qquad 0<\varepsilon\leqq\varepsilon_1, \qquad |x|\leqq c_1, \qquad |t|\leqq c_2,$$

(ε_1, c_1 *and* c_2 *are certain constants independent of* ε) *in which it holds*

$$\hat{W}(t, x)-\sum_{\nu=0}^{r} W_\nu(x)t^\nu=E_r(t, x)t^{r+1}$$

where $E_r(t, x)$ is a matrix function bounded in the domain D_1.

The k-th colnmn vector of the fundamental system of the solutions is called the solution of the k-th asymptotic type, and in particular the balanced solution if $k=1, 2, \cdots, m$, and the dominant-recessive solution if $k=m+1, m+2, \cdots, n$ respectively.

§ 5. Proof of Proposition 4. 1.

8. In this section we prove the Proposition 4. 1, that is, we show the existence of the domain $D_1^{(k)}$ and the paths of integration $\lambda_{jk}(x)$ satisfying the condition (4. 8) and (4. 9), by using the method in Iwano [2] without any essential modifications.

Note at first that

$$S_{1\gamma}^{(k)}=\bigcap_{j=1}^{\beta}\frac{n-m}{n-m+q}\left\{-\frac{3}{2}\pi-\arg d_{0\mu_j}+\gamma\right\}\leqq\arg x\leqq\frac{n-m}{n-m+q}\left\{\frac{3}{2}\pi-\arg d_{0\mu_j}-\gamma\right\},$$

and since the central angle of $S_{1\gamma}^{(k)}$ is larger than $(n-m)\pi/(n-m+q)$ for sufficiently small γ, it contains at least one singular direction: $\mathrm{Re}\, d_{0\mu_j}x^a=0$ ($\mathrm{Re}\, z$ denotes the real part of z), or more precisely

$$\text{(5. 1)}\qquad \begin{aligned} &l_j^+;\ \arg x=\theta_j^+\equiv\frac{n-m}{n-m+q}\left\{\frac{1}{2}\pi-\arg d_{0\mu_j}\right\},\\ &l_j^-;\ \arg x=\theta_j^-\equiv\frac{n-m}{n-m+q}\left\{-\frac{1}{2}\pi-\arg d_{0\mu_j}\right\}\end{aligned}$$

for each $j=1, 2, \cdots, \beta$), but no more than two singular directions. It is apparent that in the region $\theta_j^-<\arg x<\theta_j^+$, we have $\mathrm{Re}\, d_{0\mu_j}x^a>0$. Here we denote for simplicity the angles of boundary lines of $S_{1\gamma}^{(k)}$ by

$$\text{(5. 2)}\quad \Theta^-=\frac{n-m}{n-m+q}\left\{-\frac{3}{2}\pi-\arg d_{0\mu_1}+\gamma\right\},\qquad \Theta^+=\frac{n-m}{n-m+q}\left\{+\frac{3}{2}\pi-\arg d_{0\mu_\beta}-\gamma\right\}.$$

Now we divide the integrals in (4. 7) into four classes of the indices j for each fixed k such that

J_1: the integral whose integrand does not carry the exponential factor,

J_2: the sector $S_{1\gamma}^{(k)}$ contains only the singular direction l_j^+,

J_3: the sector $S_{1\gamma}^{(k)}$ contains only the singular direction l_j^-,

J_4: the sector $S_{1\gamma}^{(k)}$ contains both the singular directions l_j^+ and l_j^-.

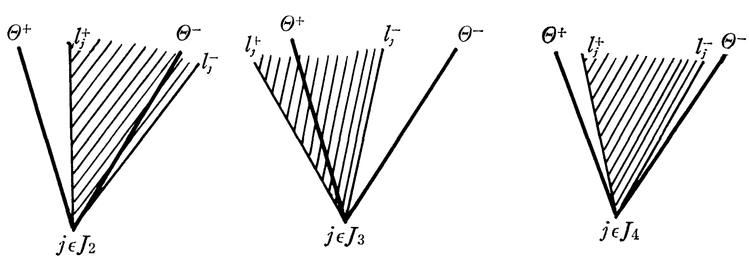

Fig. 1

The shadow regions in the above figure mean that $\mathrm{Re}\, d_{0\mu_j}x^a \geqq 0$, and note that $\theta_j^+ - \theta_j^- = \pi/a$.

Denote by $|x|e^{i\theta}$ and by $|\xi|e^{i\varphi}$ the polar coordinates of the points x and ξ, and define the angles $\Theta_i^\pm$ $(i=2, 3, 4)$, θ_{j0} $(j=1, 2, \cdots, n)$ and the initial point x_{j0} of the integral path $\lambda_{\mu_j}(x)$ $(j=1, 2, \cdots, n)$ by

$$\Theta_i^+ = \min_{j\in J_i} \theta_j^+, \qquad \Theta_i^- = \max_{j\in J_i} \theta_j^- \quad (i=2, 3, 4), \tag{5.3}$$

$$\theta_{j0} = \begin{cases} \Theta^- & \text{for } j\in J_1, \\ \Theta^- & \text{for } j\in J_2, \\ \Theta^+ & \text{for } j\in J_3, \\ \dfrac{\Theta_4^+ + \Theta_4^-}{2} & \text{for } j\in J_4, \end{cases} \tag{5.4}$$

$$|x_{j0}| = c_1' \exp\int_{\theta_0}^{\theta_{j0}} \cot \Phi(\varphi)d\varphi,$$

where c_1' is a certain constants, θ_0 is an arbitrary constant angle in $[\Theta^-, \Theta^+]$ and $\Phi(\varphi)$ is to be determined as a piecewise continuous function in the interval $[\Theta^-, \Theta^+]$ satisfying the inequality

$$a\delta \leqq \Phi(\varphi) \leqq \pi - a\delta \qquad \left(a = \frac{n-m+q}{n-m}\right) \tag{5.5}$$

for sufficiently small positive constant δ. Then, the path of integration $\lambda_{\mu_j}(x)$ combining the initial point x_{j0} to x consists in general of a curvilinear part $\lambda'_{\mu_j}(x)$:

$$\begin{aligned} |\xi| &= |x| \exp\left(\int_\theta^\varphi \cot \Phi(\varphi)d\varphi\right) && \text{for } \theta_{j0} \leqq \varphi \leqq \theta \quad \text{if } j\in J_1, J_2, J_4, \\ |\xi| &= |x| \exp\left(\int_\theta^\varphi \cot \Phi(\varphi)d\varphi\right) && \text{for } \theta \leqq \varphi \leqq \theta_{j0} \quad \text{if } j\in J_3, J_4, \end{aligned} \tag{5.6}$$

and of a rectilinear part $\lambda''_{\mu_j}(x)$:

$$|x| \exp\left(\int_\theta^{\theta_{j0}} \cot \Phi(\varphi)d\varphi\right) \leqq |\xi| \leqq c_1' \exp\left(\int_{\theta_0}^{\theta_{j0}} \cot \Phi(\varphi)d\varphi\right), \qquad \varphi = \theta_{j0}. \tag{5.7}$$

If we define the region $\tilde{D}_1^{(k)}$ as a set of points $x = |x|e^{i\theta}$ satisfying the inequalities

$$c_2'\varepsilon^{1/a} \exp\left(\int_{\theta_0}^\theta \cot \Phi(\varphi)d\varphi\right) \leqq |x| \leqq c_1' \exp\left(\int_{\theta_0}^\theta \cot \Phi(\varphi)d\varphi\right), \quad \Theta^- \leqq \theta \leqq \Theta^+, \quad 0 < \varepsilon \leqq \varepsilon_1' \tag{5.8}$$

for suitably chosen positive constant c_2', then every point x in $\tilde{D}_1^{(k)}$ can be reached from the initial point x_{j0} along $\lambda_{\mu j}(x)$ contained in $\tilde{D}_1^{(k)}$ (Fig. 2).

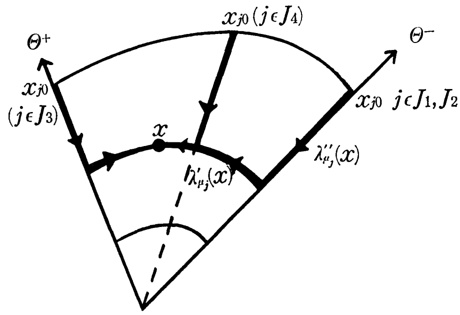

Fig. 2

Now we will show that the condition (4. 8) and (4. 9) are satisfied on the integral path $\lambda_{\mu j}(x)$ defined as above if we choose the function $\Phi(\varphi)$ appropriately. Suppose at first that $\Phi(\varphi)$ was determined so that it satisfies (5. 5), and if we notice that the line element ds is expressed by

$$ds=-d|\xi| \quad \text{on} \quad \lambda''_{\mu j}(x),$$

(5. 9) $$ds=\frac{|\xi|}{\sin\Phi(\varphi)}d\varphi \quad \text{on} \quad \lambda'_{\mu j}(x) \quad \text{for} \quad \theta_{j0}\leqq\varphi\leqq\theta \qquad (j\in J_1, J_2 \text{ or } J_4),$$

$$ds=-\frac{|\xi|}{\sin\Phi(\varphi)}d\varphi \quad \text{on} \quad \lambda'_{\mu j}(x) \quad \text{for} \quad \theta\leqq\varphi\leqq\theta_{j0} \qquad (j\in J_3 \text{ or } J_4),$$

then we have

$$\int_{\lambda_{\mu j}(x)}|\xi|^{-ra-1}|d\xi|\leqq\int_{\lambda'_{\mu j}(x)}|\xi|^{-ra-1}ds+\int_{\lambda''_{\mu j}(x)}|\xi|^{-ra-1}ds$$

$$\leqq\frac{1}{ra}|x|^{-ra}\exp\left\{-ra\int_\theta^{\theta_{j0}}\cot\Phi(\varphi)d\varphi\right\}+|x|^{-ra}\left|\int_\theta^{\theta_{j0}}\frac{1}{\sin\Phi(\varphi)}\left\{\exp\int_\theta^\varphi-ar\cot\Phi(\phi)d\phi\right\}d\varphi\right|$$

and this proves the condition (4. 8).

In order to prove the condition (4. 9) it is sufficient to show that the quantity $-\mathrm{Re}\,d_{0\mu}\xi^a$ is monotonically increasing along the integral path $\lambda_{\mu j}(x)$, because then we have

$$\mathrm{Re}\,d_{0\mu}x^a-\mathrm{Re}\,d_{0\mu}\xi^a\leqq 0,$$

and apparently this is valid on the rectilinear part $\lambda''_{\mu j}(x)$. Therefore we want only to show that there exists a piecewise continuous function $\Phi(\varphi)$ on the interval $[\Theta^-, \Theta^+]$ satisfying (5. 5) and at the same time $-\mathrm{Re}\, d_{0\mu}\xi^a$ is monotonically increasing along the curvilinear part $\lambda'_{\mu j}(x)$, that is,

$$\frac{-d\,\mathrm{Re}\, d_{0\mu j}\xi^a}{ds} \geqq 0 \qquad \text{on} \quad \lambda'_{\mu j}(x). \tag{5. 10}$$

After a short calculation we have from (5. 9)

$$\frac{d\xi}{ds}=\frac{d}{d\varphi}|\xi|e^{i\varphi}\frac{d\varphi}{ds}=\pm\frac{\xi}{|\xi|}\{\cot\Phi(\varphi)+i\}\sin\Phi(\varphi)=\pm\frac{\xi}{|\xi|}e^{i\Phi(\varphi)}$$

according as $\theta_{j0}\leqq\varphi\leqq\theta$ or $\theta\leqq\varphi\leqq\theta_{j0}$, and hence

$$-\frac{d}{ds}\mathrm{Re}\, d_{0\mu j}\xi^a=-\mathrm{Re}\frac{d}{d\xi}d_{0\mu j}\xi^a\frac{d\xi}{ds}=\mp\mathrm{Re}\left\{ad_{0\mu j}\xi^{a-1}\frac{\xi}{|\xi|}e^{i\Phi(\varphi)}\right\}$$

$$=\mp|ad_{0\mu j}|\cdot|\xi|^{a-1}\cos R_j(\varphi)$$

according sa $\theta_{j0}\leqq\varphi\leqq\theta$ or $\theta\leqq\varphi\leqq\theta_{j0}$, where

$$R_j(\varphi)=\arg d_{0\mu j}+a\varphi+\Phi(\varphi). \tag{5. 11}$$

Then, in order to obtain (5. 10), $R_j(\varphi)$ must satisfy

$$\begin{aligned}&\frac{\pi}{2}\leqq R_j(\varphi)\leqq\frac{3}{2}\pi \qquad \text{for} \quad \theta_{j0}\leqq\varphi\leqq\theta,\\ &-\frac{\pi}{2}\leqq R_j(\varphi)\leqq\frac{\pi}{2} \qquad \text{for} \quad \theta\leqq\varphi\leqq\theta_{j0}.\end{aligned} \tag{5. 12}$$

From (5. 1), (5. 5), (5. 11) and (5. 12), $\Phi(\varphi)$ must satisfies the inequalities

$$\max_{j\in J_2,J_4}\{a(\theta_j^- -\varphi)+\pi,\, a\delta\}\leqq\Phi(\varphi)\leqq\min_{j\in J_2,J_4}\{a(\theta_j^+ -\varphi)+\pi,\, \pi-a\delta\} \qquad \text{for} \quad \theta\geqq\varphi\geqq\theta_{j0}.$$

$$\max_{j\in J_3,J_4}\{a(\theta_j^- -\varphi),\, a\delta\}\leqq\Phi(\varphi)\leqq\min_{j\in J_3,J_4}\{a(\theta_j^+ -\varphi),\, \pi-a\delta\} \qquad \text{for} \quad \theta\leqq\varphi\leqq\theta_{j0}.$$

Hence the function $\Phi(\varphi)$ satisfying the above inequalities will exist if we have

$$\max\left[\max_{jh}\{a(\theta_j^- -\varphi)+\pi,\, a(\theta_h^- -\varphi)\},\, a\delta\right]\leqq\min\left[\min_{j,h}\{a(\theta_j^+ -\varphi)+\pi,\, a(\theta_h^+ -\varphi)\},\, \pi-a\delta\right]$$

$$\text{for} \quad \frac{\Theta_4^+ +\Theta_4^-}{2}\leqq\varphi\leqq\Theta^+ \qquad (j\in J_2, J_4, \text{ and } h\in J_3),$$

$$\max\left[\max_{jh}\{a(\theta_j^- -\varphi),\, a(\theta_h^- -\varphi)+\pi\},\, a\delta\right] \leqq \min\left[\min_{jh}\{a(\theta_j^+ -\varphi),\, a(\theta_h^+ -\varphi)+\pi\},\, \pi - a\delta\right]$$

$$\text{for}\quad \frac{\Theta_4^+ + \Theta_4^-}{2} \geqq \varphi \geqq \Theta^- \qquad (j\in J_3, J_4, \text{ and } h\in J_2).$$

By using the notation (5. 3), these inequalities are reduced to

(5. 13)

$$\max\left[a\left\{\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_4^- + \frac{\pi}{a}, \Theta_3^-\right) - \varphi\right\}, a\delta\right]$$

$$\leqq \min\left[a\left\{\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_4^+ + \frac{\pi}{a}, \Theta_3^+\right) - \varphi\right\}, \pi - a\delta\right]$$

$$\text{for}\quad \frac{\Theta_4^+ + \Theta_4^-}{2} \leqq \varphi \leqq \Theta^+,$$

(5. 14)

$$\max\left[a\left\{\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_3^-, \Theta_4^-\right) - \varphi\right\}, a\delta\right]$$

$$\leqq \min\left[a\left\{\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+, \Theta_4^+\right) - \varphi\right\}, \pi - a\delta\right]$$

$$\text{for}\quad \frac{\Theta_4^+ + \Theta_4^-}{2} \geqq \varphi \geqq \Theta^-.$$

Since we can easily prove the following inequalities

$$\theta_j^- < \Theta^- < \theta_k^- < \theta_j^+ < \theta_k^+ < \Theta^+ < \theta_k^+ + \frac{\pi}{a} \qquad \text{for } j\in J_2 \text{ and } k\in J_4,$$

$$\Theta^- < \theta_k^- < \theta_j^- < \theta_k^+ < \Theta^+ < \theta_j^+ < \theta_k^+ + \frac{\pi}{a} \qquad \text{for } j\in J_3 \text{ and } k\in J_4,$$

we have

(5. 15)

$$\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_4^- + \frac{\pi}{a}, \Theta_3^-\right) = \Theta_4^- + \frac{\pi}{a},$$

$$\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_4^+ + \frac{\pi}{a}, \Theta_3^+\right) = \min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+\right),$$

$$\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_3^-, \Theta_4^-\right) = \max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_3^-\right),$$

$$\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+, \Theta_4^+\right) = \Theta_4^+,$$

and

$$\Theta^+ < \Theta_2^+ + \frac{\pi}{a}, \qquad \Theta^+ < \Theta_3^+, \qquad \Theta_2^- < \Theta^-, \qquad \Theta_3^- - \frac{\pi}{a} < \Theta^-. \tag{5.16}$$

Hence (5. 13) and (5. 14) become

$$\max\left[a\left(\Theta_4^- + \frac{\pi}{a} - \varphi\right), a\delta\right] \leqq \min\left[a\left\{\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+\right) - \varphi\right\}, \pi - a\delta\right] \quad \text{for} \quad \frac{\Theta_4^+ + \Theta_4^-}{2} \leqq \varphi \leqq \Theta^+, \tag{5.17}$$

$$\max\left[a\left\{\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_3^-\right) - \varphi\right\}, a\delta\right] \leqq \min\left[a\left(\Theta_4^+ - \varphi\right), \pi - a\delta\right] \quad \text{for} \quad \frac{\Theta_4^+ + \Theta_4^-}{2} \geqq \varphi \geqq \Theta^-. \tag{5.18}$$

But a simple calculation shows that the above inequalities are satisfied respectively in the intervals

$$\Theta_4^- + \delta \leqq \varphi \leqq \min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+\right) - \delta \tag{5.19}$$

and

$$\max\left(\Theta_2^-, \Theta_3^- - \frac{\pi}{a}\right) + \delta \leqq \varphi \leqq \Theta_4^+ - \delta. \tag{5.20}$$

If δ is sufficiently small the interval (5. 19) contains the interval $[(\Theta_4^+ + \Theta_4^-)/2, \Theta^+]$ and the interval (5. 20) contains the interval $[\Theta^-, (\Theta_4^+ + \Theta_4^-)/2]$.

Then if we put, for example

$$\Phi(\varphi) = \begin{cases} \max\left[a\left\{\max\left(\Theta_2^- + \frac{\pi}{a}, \Theta_3^-\right) - \varphi\right\}, a\delta\right] & \text{for} \quad \Theta^- \leqq \varphi \leqq \frac{\Theta_4^+ + \Theta_4^-}{2}, \\ \min\left[a\left\{\min\left(\Theta_2^+ + \frac{\pi}{a}, \Theta_3^+\right) - \varphi\right\}, \pi - a\delta\right] & \text{for} \quad \frac{\Theta_4^+ + \Theta_4^-}{2} \leqq \varphi \leqq \Theta^+, \end{cases}$$

we can define the desired function $\Phi(\varphi)$ and so the pathes of integration.

Now in the definition (5. 8) of the domain $\tilde{D}_1^{(k)}$, the constants c_1', $c_2'^{-1}$ and ε_1' must be taken so small that the integral equation (4. 6) has a solution and also it contains a domain of annulus $D_1^{(k)}$ for appropriately chosen constants c_1, $\tilde{c}_2$ and ε_1, and this is clearly possible. Thus we have proved Proposition 4. 1.

§ 6. Formal inner solution.

9. At first we transform the equation (2. 1) by the stretching and shearing transformations:

$$x=\rho^{n-m}s, \qquad \varepsilon=\rho^{n-m+q},$$

$$Y=\Omega_1(\rho^{n-m})U, \qquad Z=\Omega_2(\rho^{n-m})V,$$

where the diagonal matrices $\Omega_1(x)$ and $\Omega_2(x)$ are defined in (3. 1), then we have a differential system of the form

$$\frac{dU}{ds}=A_1U+B_1V, \qquad \frac{dV}{ds}=C_1U+D_1V, \tag{6. 1}$$

where $A_1=A$, $B_1=B$ and

$$C_1(s,\rho)=\rho^{-q}\Omega_2(\rho^{n-m})^{-1}C(x,\varepsilon)\Omega_1(\rho^{n-m})=\begin{bmatrix} & 0 & \\ c_{11}(s,\rho), & c_{12}(s,\rho),\cdots, & c_{1m}(s,\rho)\end{bmatrix},$$

(6. 2)

$$D_1(s,\rho)=\rho^{-q}\Omega_2(\rho^{n-m})^{-1}D(x,\varepsilon)\Omega_2(\rho^{n-m})=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \\ d_{1m+1}(s,\rho), & d_{1m+2}(s,\rho), & \cdots, & d_{1n}(s,\rho)\end{bmatrix}.$$

Here and in below we use symbols $A_1, B_1, \cdots, c_{ij}(s,\rho), d_{ij}(s,\rho), \cdots$ which are different from those in § 3. Now the functions $c_{1j}(s,\rho)$ and $d_{1j}(s,\rho)$ satisfy the relations

$$c_{1j}(s,\rho)=\rho^{(n-m)(m+1-j-q)}(P_j+\varepsilon R_j)\cong\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}p_{j\nu\mu}s^{\mu}\rho^{(n-m)(\mu+m+1-q-j)+(n-m+q)\nu}$$
$$(j=1,2,\cdots,m),$$

$$d_{1m+1}(s,\rho)=\rho^{-(n-m)q}(x^q+\varepsilon R_{m+1})\cong s^q+\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}p_{j\nu\mu}s^{\mu}\rho^{(n-m)(\mu-q)+(n-m+q)\nu}, \tag{6. 3}$$

$$d_{1j}(s,\rho)=\rho^{-q(n-j+1)}(\varepsilon^{n-j+1}R_j)\cong\rho^{(n-m)(n-j+1)}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}p_{j\nu\mu}s^{\mu}\rho^{(n-m)\mu+(n-m+q)\nu}$$
$$(j=m+2,\cdots,n).$$

From the one segment condition, all of the powers of ρ in the above expressions are nonnegative, then the matrix functions $C_1(s,\rho)$ and $D_1(s,\rho)$ can be expanded in power series of ρ whose coefficients are polynomials of s.

Now let the equation (6. 1) be written by the combined form such as

(6. 4) $$\frac{dW}{ds}=G(s,\rho)W, \qquad W=\begin{bmatrix} U \\ V \end{bmatrix},$$

where

$$G(s,\rho)=\begin{bmatrix} A_1 & B_1 \\ C_1(s,\rho) & D_1(s,\rho) \end{bmatrix},$$

and let the asymptotic expansion of $G(s,\rho)$ be

$$G(s,\rho)\cong\sum_{\nu=0}^{\infty}G_\nu(s)\rho^\nu$$

where $G_\nu(s)$ are polynomials of s, in particular

(6. 5) $$\begin{bmatrix} A_1 & B_1 \\ C_{10}(s) & D_{10}(s) \end{bmatrix}=\left[\begin{array}{cccc:cccc} 0 & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ \hdashline & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ c_{11}(s,0), & c_{12}(s,0), & \cdots, & & d_{1m+1}(s,0), & 0, & \cdots, & 0 \end{array}\right].$$

Here we want to construct a formal solution of (6. 4) by the form

$$W\sim\sum_{\nu=0}^{\infty}W_\nu(s)\rho^\nu,$$

then each of the matrices $W_\nu(s)$ satisfies

(6. 6) $$\frac{dW_\nu(s)}{ds}=G_0(s)W_\nu(s)+\sum_{\mu=1}^{\nu}G_\mu(s)W_{\nu-\mu} \qquad (\nu=0,1,2,\cdots).$$

10. Firstly we analyze the above equation for $\nu=0$

(6. 7) $$\frac{dW(s)}{ds}=G_0(s)W(s), \qquad W(s)=\begin{bmatrix} U(s) \\ V(s) \end{bmatrix}.$$

Clearly there exists a fundamental solution of (6. 7) in the arbitrary neighborhood of the origin, then the problem is to discuss the asymptotic behavior of $W(s)$ in the neighborhood of $s=\infty$. To do this, we transform the equation (6. 7) by

(6.8)
$$\xi = s^{a}(a=(n-m+q)/(n-m)),$$
$$W=\Omega(s)W^{(1)}=\begin{bmatrix} s^{m} & & & & & \\ & \ddots & & & 0 & \\ & & s & & & \\ & & & 1 & & \\ & 0 & & & s^{q/(n-m)} & \\ & & & & & \ddots \\ & & & & & & s^{(n-m-1)q(n-m)} \end{bmatrix}\begin{bmatrix} U^{(1)} \\ \\ V^{(1)} \end{bmatrix}$$

then it becomes

(6.9)
$$\frac{dW^{(1)}}{d\xi}=\tilde{G}_{1}(\xi)W^{(1)}=\frac{1}{a}\xi^{-q/(n-m+q)}\left\{\Omega(s)^{-1}G_{0}(s)\Omega(s)-\Omega(s)^{-1}\frac{d\Omega(s)}{ds}\right\}W^{(1)}.$$

From (6.5) and (6.8), $\tilde{G}_{1}(\xi)$ can be written

$$\tilde{G}_{1}(\xi)=\begin{bmatrix} \tilde{A}_{1} & \tilde{B}_{1} \\ \tilde{C}_{1}(\xi) & \tilde{D}_{1}(\xi) \end{bmatrix}, \qquad \tilde{A}_{1}=\frac{1}{a\xi}\begin{bmatrix} -m & 1 & & 0 \\ & -(m-1) & \ddots & \\ & & \ddots & 1 \\ 0 & & & -1 \end{bmatrix},$$

(6.10)
$$\tilde{B}_{1}=\frac{1}{a\xi}\begin{bmatrix} & 0 \\ 1 & \end{bmatrix}, \qquad \tilde{C}_{1}(\xi)=\frac{1}{a}\begin{bmatrix} 0 \\ \tilde{c}_{11}(\xi), \cdots, \tilde{c}_{1m}(\xi) \end{bmatrix},$$

$$\tilde{D}_{1}(\xi)=\frac{1}{a}\begin{bmatrix} & 1 & & 0 \\ 0 & & \ddots & \\ & & & 1 \\ \tilde{d}_{1m+1}(\xi), & & & 0 \end{bmatrix}-\frac{q}{(n-m+q)\xi}\begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ 0 & & & (n-m+1) \end{bmatrix},$$

where

(6.11)
$$\tilde{c}_{1j}(\xi)=\xi^{(m+1-j-q)/a}c_{1j}(s,0) \qquad (j=1,2,\cdots,m),$$
$$\tilde{d}_{1m+1}(\xi)=\xi^{-q/a}d_{1m+1}(s,0).$$

Now from the asymptotic expansions (6.3) of $c_{1j}(s,\rho)$ and $d_{1m+1}(s,\rho)$, the polynomials $c_{1j}(s,0)$ and $d_{1m+1}(s,0)$ have the forms

$$c_{1j}(s,0)=\sum_{\mu}\sum_{\nu=0}p_{j\nu\mu}s^{\mu} \qquad (j=1,2,\cdots,m),$$

$$d_{1m+1}(s,0)=s^{q}+\sum_{\mu}\sum_{\nu=1}p_{m+1\nu\mu}s^{\mu},$$

where for nonzero coefficients $p_{j\nu\mu}$, following relations must be satisfied

$$(n-m)(\mu+m+1-q-j)+(n-m+q)\nu=0 \qquad (j=1, 2, \cdots, m+1).$$

Then the order μ of nonzero terms s^{μ} of (6. 12) must satisfies

$$\mu=q+j-(m+1)-a\nu \qquad (\nu=0, 1, \cdots),$$

and so the functions (6. 11) can be written

$$(6.12)\qquad \begin{aligned} \tilde{c}_{1j}(\xi)&=\sum_{\nu=0} p_{j\nu\mu_\nu}\xi^{-\nu} \qquad (j=1, 2, \cdots, m),\\ \tilde{d}_{1m+1}(\xi)&=1+\sum_{\nu=1} p_{m+1\nu\mu_\nu}\xi^{-\nu},\end{aligned}$$

where the summations are taken for a finite terms of ν for which $\mu_\nu=q+j-(m+1)-a\nu$.

Thus the matrix function $\tilde{G}_1(\xi)$ is a polynomial of ξ^{-1} and if we write it by the form

$$\tilde{G}_1(\xi)\cong\sum_{\nu=0}\tilde{G}_{1\nu}\xi^{-\nu}=\sum_{\nu=0}\begin{bmatrix}\tilde{A}_{1\nu} & \tilde{B}_{1\nu}\\ \tilde{C}_{1\nu} & \tilde{D}_{1\nu}\end{bmatrix}\xi^{-\nu},$$

then from (6. 10) and (6. 12) we have

$$\tilde{G}_{10}=\begin{bmatrix}0 & 0\\ \tilde{C}_{10} & \tilde{D}_{10}\end{bmatrix} \quad \text{with} \quad \tilde{C}_{10}=\frac{1}{a}\begin{bmatrix} & 0 & \\ \tilde{c}_{101}, & \cdots, & \tilde{c}_{10m}\end{bmatrix}, \qquad \tilde{D}_{10}=\frac{1}{a}\begin{bmatrix}0 & 1 & & 0\\ & 0 & \ddots & \\ & & & 1\\ 1, & 0, & \cdots, & 0\end{bmatrix},$$

$$(6.13)\quad \tilde{G}_{11}=\begin{bmatrix}\tilde{A}_{11} & \tilde{B}_{11}\\ \tilde{C}_{11} & \tilde{D}_{11}\end{bmatrix} \quad \text{with} \quad A_{11}=\frac{1}{a}\begin{bmatrix}-m & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & -1\end{bmatrix}, \qquad B_{11}=\frac{1}{a}\begin{bmatrix} & 0 & \\ 1 & & \end{bmatrix},$$

$$\tilde{C}_{11}=\frac{1}{a}\begin{bmatrix} & 0 & \\ \tilde{c}_{111}, & \tilde{c}_{112}, \cdots, & \tilde{c}_{11m}\end{bmatrix}, \qquad \tilde{D}_{11}=\frac{1}{a}\begin{bmatrix} & 0 & \\ d_{11m+1}, & 0\cdots 0 & \end{bmatrix}-\frac{q}{n-m+q}\begin{bmatrix}0 & & 0\\ & 1 & \\ 0 & \ddots & n-m-1\end{bmatrix},$$

where the constants $\tilde{c}_{10j}$, $\tilde{c}_{11j}$ and $\tilde{d}_{11m+1}$ are equal to the numbers $p_{j0\mu_0}$, $p_{j1\mu_1}$, and $p_{m+1,1\mu_1}$ in (6. 3) respectively provided the indices of these numbers satisfy the relations $\mu_\nu=q+j-(m+1)-a\nu$ $(\nu=0, 1, j=1, 2, \cdots, m+1)$. Here we remark that if we compare the above coefficient matrices with those of (3. 2) and (3. 3), then it is found that $\tilde{A}_{11}=A_1/a$, $\tilde{B}_{11}=B_1/a$, $\tilde{C}_{10}=C_{10}(0)/a$, $\tilde{D}_{10}=D_{10}/a$ and $\tilde{D}_{11}=D_{11}(0)/a$.

For the differential system (6. 9), we prove a following lemma which is analogous to the lemma 3. 2. In order to calculate the connection matrix between the inner solution and the outer solution in the last section, we must take always the relation between the coefficient matrices in (3. 4), (3. 5) and those in the following lemma into our considerations.

LEMMA 6. 1. *There exists a linear nonsingular transformation*

$$(6.14)\qquad \begin{aligned} U^{(1)} &= \tilde{Q}_{11}^{(0)} U^{(2)} + \{\tilde{Q}_{12}^{(1)}\xi^{-1} + \tilde{Q}_{12}^{(2)}\xi^{-2}\} V^{(2)}, \\ V^{(1)} &= \{\tilde{Q}_{21}^{(0)} + \tilde{Q}_{21}^{(1)}\xi^{-1}\} U^{(2)} + \{\tilde{Q}_{22}^{(0)} + \tilde{Q}_{22}^{(1)}\xi^{-1}\} V^{(2)} \end{aligned}$$

where $\tilde{Q}_{ij}^{(k)}$ *are some constant matrices, and this transformation changes* (6. 9) *into*

$$(6.15)\qquad \frac{dU^{(2)}}{d\xi} = \tilde{A}_2 U^{(2)} + \tilde{B}_2 V^{(2)}, \qquad \frac{dV^{(2)}}{d\xi} = \tilde{C}_2 U^{(2)} + \tilde{D}_2 V^{(2)},$$

where the coefficient matrices are convergent power series of ξ^{-1} *such that*

$$(6.16)\qquad \begin{aligned} \tilde{A}_2 &\cong \sum_{\nu=1}^{\infty} \tilde{A}_{2\nu}\xi^{-\nu}, & \tilde{B}_2 &= \sum_{\nu=2}^{\infty} \tilde{B}_{2\nu}\xi^{-\nu}, \\ \tilde{C}_2 &= \sum_{\nu=2}^{\infty} \tilde{C}_{2\nu}\xi^{-\nu}, & \tilde{D}_2 &= \sum_{\nu=0}^{\infty} \tilde{D}_{2\nu}\xi^{-\nu}. \end{aligned}$$

If we compare the coefficient matrices of (6. 14) *and* (6. 16) *with those of* (3. 4) *and* (3. 5) *we have*

$$(6.17)\qquad \begin{aligned} &\tilde{Q}_{11}^{(0)} = Q_{11}(0), \qquad \tilde{Q}_{12}^{(1)} = Q_{12}^{(1)}(0), \qquad \tilde{Q}_{21}^{(0)} = Q_{21}^{(0)}(0), \qquad \tilde{Q}_{22}^{(0)} = Q_{22}^{(0)}(0), \\ &\tilde{A}_{21} = A_{20}/a, \qquad \tilde{D}_{20} = D_{20}/a, \qquad \tilde{D}_{21} = D_{21}(0)/a. \end{aligned}$$

Proof. At first we transform the equation (6. 9) by

$$U^{(1)} = \tilde{U}^{(1)} + \tilde{Q}_1\xi^{-1}\tilde{V}^{(1)}, \qquad U^{(1)} = (\tilde{R}_0 + \tilde{R}_1\xi^{-1})\tilde{U}^{(1)} + \tilde{V}^{(1)},$$

where the matrices $\tilde{Q}_1$, $\tilde{R}_0$ and $\tilde{R}_1$ are determined by the equations

$$(6.18)\qquad \begin{aligned} &\tilde{Q}_1\tilde{D}_{10} - \tilde{B}_{11} = 0, \qquad \tilde{C}_{10} + \tilde{D}_{10}\tilde{R}_0 = 0, \\ &\tilde{C}_{11} + \tilde{D}_{11}\tilde{R}_0 + \tilde{D}_{10}\tilde{R}_1 - \tilde{R}_0\tilde{A}_{11} - \tilde{R}_0\tilde{B}_{11}\tilde{R}_0 = 0. \end{aligned}$$

Then after a little calculations as used in the proof of Lemma 3. 2, it becomes

$$\frac{d\tilde{U}^{(1)}}{d\xi} = \{(\tilde{A}_{11}\tilde{B}_{11}\tilde{R}_0)\xi^{-1} + O(\xi^{-2})\}\tilde{U}^{(1)} + O(\xi^{-2})\tilde{V}^{(1)},$$

$$\frac{d\tilde{V}^{(1)}}{d\xi}=\{\tilde{D}_{10}+(\tilde{D}_{11}+\tilde{C}_{10}\tilde{Q}_{1})\xi^{-1}+O(\xi^{-2})\}\tilde{V}^{(1)}+O(\xi^{-2})\tilde{U}^{(1)},$$

and furthermore if we diagonalize the principal parts of the above equation, we have a differential system which has a form (6. 15) with (6. 16). The relations (6. 17) can be easily verified by a careful comparison of each step of transformation of the above procedure with the one in the proof of Lemma 3. 2. This completes the proof.

Now we proceed to construct an asymptotic solution of the system (6. 15), but since this is easily realized by the usual methods, we give only the results in the following proposition.

PROPOSITION 6. 1. *The differential equation* (6. 15) *has a fundamental system of formal solutions of the form*

$$W^{(2)}\sim\left\{\sum_{\nu=0}^{\infty}W_{\nu}^{(2)}\xi^{-\nu}\right\}\tilde{F}(\xi), \tag{6. 19}$$

where the matrices $W_{\nu}^{(2)}$ *are constant, in particular* $W_{0}^{(2)}=I_n$ (n-dim *nnit matrix*), *and*

$$\tilde{F}(\xi)=\begin{bmatrix}\xi^{\tilde{A}_{21}}, & 0\\ 0, & \{\exp\tilde{D}_{20}\xi\}\cdot\xi^{\tilde{D}_{21}}\end{bmatrix}\qquad\left(\xi=s^{a}=\frac{1}{t}\right).$$

Corresponding to this formal solution, there exists a fundamental system of actual solutions $W^{(2)}(\xi)$ *which has it as the asymptotic expansion in the domain*:

$$\tilde{D}_2\colon\ |\xi|>\xi_0,\qquad \arg\xi\in\tilde{S},$$

where ξ_0 *is some positive constant, and the sector* $\tilde{S}$ *is defined below.*

$$\tilde{S}\colon\ -\frac{\pi}{2}+\alpha+\gamma\leqq\arg\xi\leqq\frac{\pi}{2}+\alpha-\gamma, \tag{6. 20}$$

where γ *is positive and arbitrary, and* $\alpha\neq\arg(d_{0j},-d_{0j},d_{0j}-d_{0k})$ $(j,k=m+1,\cdots,n,j\neq k)$.

A connection formula between the convergent solution of the differential equation (6. 7) in the neighborhood of $s=0$ and the asymptotic solution of it in the neighborhood of $s=\infty$ which is described in the above proposition can be determined by the method of convergent matching because the asymptotic solution of (6. 9) has a convergent expression by a factorial series from a theorem of Turritten [8].

PROPOSITION 6. 2. *Let* α *be any angle for which*

$$\alpha\neq+\arg(d_{0j},-d_{0j},d_{0j}-d_{0k})\qquad(j,k=m+1,\cdots,n,j\neq k).$$

Then there exists positive numbers $\omega_0 \geqq 1$ *and* κ *such that for* $\omega \geqq \omega_0$ *the differential equation* (6. 15) *possesses in the half plane*

$$\operatorname{Re}(\xi e^{-\alpha}) > \kappa \tag{6.21}$$

a fundamental solution $W^{(2)}(\xi)$ *of the form*

$$W^{(2)}(\xi) = \{w_{jk}(\xi)\}\tilde{F}(\xi),$$

$$w_{jk}(\xi) = \delta_{jk} + \sum_{r=0}^{\infty} \frac{c_{r,ij}}{[\xi e^{-i\alpha}/\omega]\cdot[\xi e^{-i\alpha}/\omega+1]\cdots[\xi e^{-i\alpha}/\omega+r]}.$$

The series converges in the half-plane (6. 21). *Moreover* $W^{(2)}(\xi)$ *can also be represented asymptotically by the formal series* (6. 19) *in the domain* $\tilde{D}_2$.

In the above definition of the sector $\tilde{S}$, we assume that γ is sufficiently small and take the angle α so that the boundary lines of $\tilde{S}$ do not coincide with any singular direction

$$\operatorname{Re}(d_{0j})\xi = 0, \qquad \operatorname{Re}(d_{0j} - d_{0k})\xi = 0, \qquad j, k = m+1, \cdots, n, j \neq k,$$

and contain them in the interior $\tilde{S}$. Furthermore when we calculate a matching matrix between the outer and the inner solutions in § 9, the sector S defined by

$$\frac{n-m}{n-m+q}\left\{-\frac{\pi}{2}+\alpha+\gamma\right\} \leqq \arg s \leqq \frac{n-m}{n-m+q}\left\{\frac{\pi}{2}+\alpha-\gamma\right\}$$

is assumed to be contained in the sector S_1 defined in Theorem 4. 1.

§ 7. Solution of nonhomogeneous equations.

11. In this section we consider the nonhomogeneous equation (6. 6) for $\nu \geqq 1$,

$$\begin{aligned} \frac{dW_\nu}{ds} &= G_0(s) W_\nu + H(s) \\ H(s) &= \sum_{\mu=1}^{\nu} G_\mu(s) W_{\nu-\mu}(s) \end{aligned} \tag{7.1}$$

At first we examine the asymptotic behavior of solutions when s tends to infinity. The solution of (7. 1) is represented by

$$W_\nu(s) = \int_\Gamma W_0(s) W_0(\tau)^{-1} H(\tau) d\tau \tag{7.2}$$

under the assumption that W_μ $(\mu = 0, 1, \cdots, \nu-1)$ are already known, where $W_0(s)$ is the fundamental solution of the homogeneous equation (6. 7) constructed in § 6, and Γ denotes a set of paths of integrations for each function in the integrand.

Let us define matrix functions $\tilde{G}_\mu(s)$, $\tilde{W}_0(s)$ and $\tilde{W}_\mu(s)$ by the relations

$$\begin{aligned} G_\mu(s) &= \Omega(s)\tilde{G}_\mu(s)\Omega(s)^{-1}, \\ W_0(s) &= \Omega(s)\tilde{W}_0(s)F(s), \\ W_\mu(s) &= \Omega(s)\tilde{W}_\mu(s)F(s), \end{aligned} \tag{7. 3}$$

where $\Omega(s)$ is defined in (6. 8) and $F(s)\equiv\tilde{F}(\xi)$ $(\xi=s^a)$. Then the integral (7. 2) becomes

$$\tilde{W}_\nu(s)=\int_\Gamma \tilde{W}_0(s)F(s)F(\tau)^{-1}\tilde{W}_0(\tau)^{-1}\tilde{H}(\tau)F(\tau)F(s)^{-1}d\tau, \tag{7. 4}$$

where

$$\tilde{H}(\tau)=\sum_{\mu=1}^{\nu}\tilde{G}_\mu(\tau)\tilde{W}_{\nu-\mu}(\tau).$$

Now we prove a few lemmas in the sequel.

LEMMA 7. 1. *The growth order of the matrix* $\tilde{G}_\mu(s)$ $(\mu\geqq 1)$ *when* s *grows into infinity is* $s^{(\mu+q)/(n-m)}$, *and* $\tilde{G}_\mu(s)$ *is a polynomial of* $s^{1/(n-m)}$ *and* $s^{-1/(n-m)}$.

Proof. From (6. 3) and the definitions of $G_\mu(s)$ and $\tilde{G}_\mu(s)$, this is obvious.

Here we assume for the moment that $H(s)$ has the growth order of s^b when $|s|$ is large, that is, we can write that $\tilde{H}(s)=s^bH^*(s)$ with bounded matrix $H^*(s)$, and assume that $H^*(s)$ has an asymptotic expansion in power series of $s^{-1/(n-m)}$ whose coefficients are polynomials of $\log s$ in the neighborhood of $s=\infty$. From the proposition 6. 1, $\tilde{W}_0(s)$ and $\tilde{W}(s)^{-1}$ are bounded and nonsingular in the neighborhood of $s=\infty$ and have asymptotic power series of $\xi^{-1}=s^{-a}$ when $\xi\to\infty$ in the sector $\tilde{S}$.

If we replace the matrix $\tilde{H}(s)$ by $s^aH^*(s)$ and change the variables s and τ by

$$\xi=s^a, \qquad \eta=\tau^a$$

then the integral (7. 4) becomes

$$\tilde{W}_\nu(s)=\frac{n-m}{n-m+q}\tilde{W}_0(s)\int_\Gamma \tilde{F}(\xi)\tilde{F}(\eta)^{-1}\tilde{W}_0(\tau)^{-1}H^*(\eta)\tilde{F}(\eta)\tilde{F}(\xi)^{-1}\eta^{b/a-q/(n-m+q)}d\eta. \tag{7. 5}$$

Since the matrix function $\tilde{W}_0(\tau)^{-1}H^*(\eta)$ is bounded and has an asymptotic expansion in power series of $\eta^{-1/(n-m+q)}$, and from the definition of the matrix $\tilde{F}(\eta)$, the above integral for each component of integrand has a form

$$\int_{\lambda_{jk}} \left(\frac{\xi}{\eta}\right)^{\tilde{a}_j-\tilde{a}_k} h_{jk}(\eta)\eta^{b/a-q/(n-m+q)} d\eta \qquad (j, k=1, 2, \cdots, m, j\neq k),$$

$$\int_{\lambda_{jk}} \left(\frac{\xi}{\eta}\right)^{\tilde{a}_j-\tilde{d}_{1k}} \{\exp(-\tilde{d}_{0k})(\xi-\eta)\} h_{jk}(\eta)\eta^{b/a-q/(n-m+q)} d\eta$$
$$(j=1, \cdots, m, k=m+1, \cdots, n),$$

(7. 6)
$$\int_{\lambda_{jk}} \left(\frac{\xi}{\eta}\right)^{\tilde{d}_{1j}-\tilde{d}_{1k}} \{\exp(\tilde{d}_{0j}-\tilde{d}_{0k})(\xi-\eta)\} h_{jk}(\eta)\eta^{b/a-q/(n-m+q)} d\eta$$
$$(j, k=m+1, \cdots, n, j\neq k),$$

$$\int_{\lambda_{jk}} \left(\frac{\xi}{\eta}\right)^{\tilde{d}_{1j}-\tilde{a}_k} \{\exp \tilde{d}_{0j}(\xi-\eta)\} h_{jk}(\eta)\eta^{b/a-q/(n-m+q)} d\eta$$
$$(j=m+1, \cdots, n, k=1, \cdots, m),$$

$$\int_{\lambda_{jj}} h_{jj}(\eta)\eta^{b/a-q/(n-m+q)} d\eta \qquad (j=k=1, 2, \cdots, n)$$

where $\tilde{a}_j$, $\tilde{d}_{0k}$ and $\tilde{d}_{1k}$ are diagonal elements of the matrices $\tilde{A}_{21}$, $\tilde{D}_{20}$ and $\tilde{D}_{21}$ respectively. Here $h_{jk}(\eta)$ is a bounded function and has an asymptotic power series of $\eta^{-1/(n-m+q)}$ in the sense that

(7. 7)
$$h_{jk}(\eta)=\sum_{\nu=0}^{r} h_\nu(\log \eta)\eta^{-\nu/(n-m+q)}+o(\eta^{-r/(n-m+q)})$$

for all positive integers r, where $h_\nu(z)$ are polynomials of z and in particular $h_0(z)$ is constant.

Now under the assumption that none of the quantities $(n-m)\ \{a_j-a_k\}$ $(j, k=1, 2, \cdots, m, j\neq k)$ are integers, we can prove a following lemma.

LEMMA 7. 2. *By choosing an appropriate path of integration or by taking an appropriate indefinite integral for each integral of* (7. 6), *we have*

$$\tilde{W}_\nu(s)=s^{b+1}\tilde{W}_\nu^*(s),$$

where $\tilde{W}_\nu^*(s)$ *is bounded and has an asymptotic power series of* $s^{-1/(n-m)}$ *in the same sense as* (7. 7) *when* $s\to\infty$ *in the sector* S

$$S\colon \frac{n-m}{n-m+q}\left\{-\frac{\pi}{2}+\alpha+\gamma\right\}\leqq \arg s \leqq \frac{n-m}{n-m+q}\left\{\frac{\pi}{2}+\alpha-\gamma\right\}.$$

Proof. Case 1. $k, j=1, 2, \cdots, m, j\neq k$. Let $\tilde{a}_j-\tilde{a}_k=\lambda+i\mu$ (λ, μ real) and let the integrand divide into three parts such that

(7. 8)
$$h(\eta)\equiv\eta^{-(\tilde{a}_j-\tilde{a}_k)} h_{jk}(\eta)\eta^{b/a-q/(n-m+q)}=h_1(\eta)+h_2(\eta)+h_3(\eta),$$

where

$$h_1(\eta)=\eta^{-\lambda-i\mu}\sum_{\nu=r_0}^{r_1}\tilde{h}_\nu(\log\eta)\eta^{\{-\nu+(n-m)b-q\}/(n-m+q)}\qquad\text{with}\quad -\lambda-\frac{r_1+(n-m)b-q}{n-m+q}>-1,$$

$$h_2(\eta)=\eta^{-\lambda-i\mu}h_{r_2}(\log\eta)\eta^{\{-r_2+(n-m)b-q\}/(n-m+q)}\qquad\text{with}\quad -\lambda-\frac{r_2+(n-m)b-q}{n-m+q}=-1,$$

$$h_3(\eta)=\eta^{-\lambda-i\mu}h_{jk}(\eta)\eta^{b/a-q/(n-m+q)}-h_1(\eta)-h_2(\eta).$$

Here we remark that the imaginary part of $\tilde{a}_j-\tilde{a}_k$ is not zero if $h_2(\eta)\not\equiv 0$ from the assumption made above the Lemma 7. 2.

Now if we define the integral of (7. 8) by

$$\int_0^\xi h_1(\eta)d\eta+\int_1^\xi h_2(\eta)d\eta+\int_\infty^\xi h_3(\eta)d\eta, \tag{7. 9}$$

then we can easily see that the statements of the lemma hold.

Case 2. $j=1,2,\cdots,m$, $k=m+1,\cdots,n$. From the shape of the sector $\tilde{S}$ in the η-plane there exists a vector l_{jk} in $\tilde{S}$ which satisfies

$$\cos(\arg d_{0k}+\arg l_{jk})<0,$$

then as the paths of integration $\lambda_{jk}(\xi)$, we choose the line parallel to l_{jk}, starting from ξ and extending to infinity in $\tilde{S}$. Clearly for all η on this path of integration, there exists a positive constant δ_{jk} such that

$$\operatorname{Re}\{-d_{0k}(\xi-\eta)\}\leqq-\delta_{jk}|\xi-\eta|. \tag{7. 10}$$

Since we have from the integration by parts

$$\int_\xi^\infty e^{\tilde{d}_{0k}\eta}\eta^\alpha(\log\eta)^\beta d\eta=-\frac{1}{\tilde{d}_{0k}}\xi^\alpha(\log\xi)^\beta e^{\tilde{d}_{0k}\xi}-\int_\xi^\infty\frac{1}{\tilde{d}_{0k}}e^{\tilde{d}_{0k}}\eta^{\alpha-1}\{a(\log\eta)^\beta+\beta(\log\eta)^{\beta-1}\}d\eta$$

for all number α and $\beta>0$, then if we substitute the asymptotic expression (7. 7) of $h_{jk}(\eta)$ into the integrand of (7. 6) and write it by

$$\begin{aligned}&\left(\frac{\xi}{\eta}\right)^{(\tilde{a}_j-\tilde{d}_{1k})}\{\exp-\tilde{d}_{0k}(\xi-\eta)\}h_{jk}(\eta)\eta^{b/a-q/(n-m+q)}\\&\cong\left(\frac{\xi}{\eta}\right)^{(\tilde{a}_j-\tilde{d}_{1k})}\{\exp-\tilde{d}_{0k}(\xi-\eta)\}\sum_{\nu=r_0}^{\infty}h_\nu(\log\eta)\eta^{-\nu/(n-m+q)},\end{aligned} \tag{7. 11}$$

where r_0 may be negative integer such that $b/a-q/(n-m+q)=-r_0/(n-m+q)$, then we have by repeated integrations by parts,

$$\begin{aligned}(7.12)\qquad &\int_{\xi}^{\infty}\left(\frac{\xi}{\eta}\right)^{(\tilde{a}_j-\tilde{d}_{1k})}\{\exp -\tilde{d}_{0k}(\xi-\eta)\}h_{jk}(\eta)\eta^{b/a-q/(n-m+q)}d\eta\\ &=\sum_{\nu=r_0}^{r}\tilde{\tilde{h}}_\nu(\log\eta)\eta^{-\nu/(n-m+q)}+R_r\end{aligned}$$

where

$$R_r=\int_{\xi}^{\infty}\left(\frac{\xi}{\eta}\right)^{(\tilde{a}_j-\tilde{d}_{1k})}\{\exp -d_{0k}(\xi-\eta)\}\tilde{\tilde{h}}_{r+1}(\log\eta)\eta^{-(r+1)/(n-m+q)}d\eta.$$

Now we estimate the remainder therm R_r. Let

$$\eta-\xi=\beta e^{i\alpha}\qquad(\alpha=\arg l_{jk}),$$

then from (7.10) we have

$$\begin{aligned}|R_r|&\leqq|\xi|^{-(r+1)/(n-m+q)}h'_{r+1}(|\log\xi|)\int_0^{\infty}\left|\left(1+\frac{\beta}{|\xi|}e^{i\alpha}\right)\right|^{-(\tilde{a}_j-\tilde{d}_{1k})-(r+1)/(n-m+q)}\\ &\qquad\cdot h''_{r+1}\left(\left|\log 1+\frac{\beta e^{i\alpha}}{\xi}\right|\right)e^{-\delta_{jk}\beta}d\beta\\ &\leqq K\tilde{\tilde{h}}_{r+1}(|\log\xi|)|\xi|^{-(r+1)/(n-m+q)},\end{aligned}$$

where $h'_{r+1}(z)$ and $h''_{r+1}(z)$ are polynomials of z, and K is some positive constant. This inequality implies that the integral of (7.11) along $\lambda_{jk}(\xi)$ can be represented by an asymptotic expansion in power series of $\eta^{-1/(n-m+q)}$ in the sense of (7.7) and in particular has a growth order of $\xi^{b/a-q/(n-m+q)}=s^{b-q/(n-m)}$ as $\xi\to\infty$ in the sector $\tilde{S}$, and then in this case we proved the desired properties.

For other cases of j, k, we can prove by the same method as in the case 1 or case 2 that the integrals (7.6) have properties stated in the Lemma 6.2. Thus we have the Lemma 7.2.

LEMMA 7.3. *The nonhomogeneous differential equation* (7.1) *possesses a particular solution such that*

$$(7.13)\qquad W_\nu(s)=s^{e\nu}\Omega(s)\tilde{W}^*_\nu(s)F(s).$$

Here the matrix $\Omega(s)$ *is defined in* (6.8), *the matrices* $F(s)\equiv\tilde{F}(\xi)$ *and the matrix* $\tilde{W}^*_\nu(s)$ *is bounded at* $s=\infty$ *and has an asymptotic expansion in power series of* $s^{-1/(n-m)}$ *when* $s\to\infty$ *in the sector* S. *Here the number* e *denotes*

$$(7.14)\qquad e=\frac{1+q}{n-m}+1.$$

Proof. For $\nu=0$, the equation (7. 1) becomes homogeneous equation (6. 7), then the statements of the lemma is satisfied from the Proposition 6. 1. Assume it to be true for $\nu<r$. Then by using the Lemma 7. 1 the μ-th term of the summation in (7. 1) has a form

$$G_{\mu}(s)W_{r-\mu}(s)=\Omega(s)\tilde{G}_{\mu}(s)\tilde{W}_{r-\mu}(s)F(s)$$

$$=s^{f(r,\mu)}\Omega(s)G_{\mu}^{*}(s)W_{r-\mu}^{*}(s)F(s),$$

where $\tilde{G}_{\mu}^{*}(s)$ and $\tilde{W}_{r-\mu}^{*}(s)$ are bounded, and

$$f(r,\mu)=\frac{q-\mu}{n-m}+e(r-\mu).$$

The exponent $f(r,\mu)$ is the largest for $\mu=1$, and then if we apply the Lemma 7. 2 to the integral (7. 5) with $b=f(r,1)$, we have the Lemma 7. 3.

12. Now we want to determine the values of the solutions $W_{\nu}(s)$ of (6. 6) in the neighborhood of $s=0$. This is essential to solve the connection problems, that is to understand an asymptotic behavior of an outer solution at the turning point itself. For $\nu=0$, we have already stated at the last of §6 that the value at $s=0$ of the asymptotic solution $\Omega(s)\tilde{W}^{(2)}(\xi)$ of (6. 7) whose existence was proved in Proposition 6. 1 can be obtained by the method of convergent matching. Then we consider here the equation (7. 1).

Let $W_0(s)$ be a fundamental solution of the homogeneous equation (6. 7) in the neighborhood of $s=0$, and assume that the solutions $W_{\mu}(s)$ $(\mu<\nu)$ of (6. 6) are determined in the neighborhood af $s=0$, then the solution $W_{\nu}(s)$ can be written as

$$W_{\nu}(s)=\int_0^s W_0(s)W_0(\tau)^{-1}H(\tau)d\tau+W_0(s)C \tag{7. 15}$$

where $H(\tau)$ is an entire function whose asymtotic behavior in some neighborhood of $s=\infty$ is known. The problem is to determine the constant matrix $C=(c_{jk})$.

The values of the matrix $W_{\nu}(s)$ in the neighborhood of $s=\infty$ are determined by taking some special integrals of the integrand of (7. 15) as stated in the Lemma 7. 2, and then corresponding to those, the matrix C must be determined as follow.

Case 1. $j,k=1,2,\cdots,m(j\neq k)$.

$$c_{jk}=-\int_0^1 h_2(\eta)d\eta+\int_{\infty}^0 h_3(\eta)d\eta=-\int_0^1\{h(\eta)-h_1(\eta)\}d\eta+\int_{\infty}^1\{h(\eta)-h_1(\eta)-h_2(\eta)\}d\eta.$$

Case 2. $j=1,2,\cdots,m,\ k=m+1,\cdots,n$.

$$c_{jk}=\int_{\infty}^0 \eta^{\tilde{a}_j-\tilde{a}_{1k}}\{\exp\tilde{d}_{0k}\eta\}h_{jk}(\eta)\eta^{b/a-q/(n-m-q)}d\eta.$$

The definite integrals which define c_{jk} are clearly exist from the natures of the integrand and the choice of the paths of integrations. For other cases of j, k we can determine c_{jk} by the same method as above.

Here we summarize the results of § 6 and § 7 in the following proposition.

PROPOSITION 7. 1. *The differential equation* (6. 4) *has a fundamental system of formal solutions in power series of* ρ *such that if* $|s|\leqq s_0$ *for some positive constant* s_0,

$$W\sim\sum_{\nu=0}^{\infty} W_\nu(s)\rho^\nu \qquad \text{if} \quad |s|\leqq s_0, \tag{7. 16}$$

where $W_\nu(s)$ *are holomorphic in the domain* $|s|\leqq s_0$, *and if* $|s|>s_0$ *and* $\arg s\in S$,

$$W\sim\Omega(s)\left\{\sum_{\nu=0}^{\infty}\widetilde{W}_\nu^*(s)[s^e\rho]^\nu\right\}F(s) \qquad \left(e=\frac{q+1}{n-m}+1\right). \tag{7. 17}$$

Here the sector S *is defined in the Lemma* 7. 2, *the matrix* $\Omega(s)$ *is in* (6. 8), *the matrix* $F(s)$ *is of the form*

$$F(s)=\begin{bmatrix} s^{a_1} & & & & \\ & \ddots & & & 0 \\ & & s^{a_m} & & \\ & & & (\exp d_{0m+1}s^a)s^{d_{1m+1}} & \\ 0 & & & & \ddots \\ & & & & (\exp d_{0n}s^a)s^{d_{1n}} \end{bmatrix} \qquad \left(a=\frac{n-m+q}{n-m}\right),$$

and the matrices $W^*(s)$ *are bounded and have asymptotic expansions such that*

$$\widetilde{W}_\nu^*(s)\cong\sum_{\mu=0}^{\infty} W_{\nu\mu}(\log s)s^{-\mu/(n-m)}, \tag{7. 18}$$

where $W_{\nu\mu}(\log s)$ *are polynomials of* $\log s$ *of degree at most* ν.

§ 8. Existence theorem of inner solution.

13. In the older treatments of a turning point problem, the existence domain of an inner solution is limited only in the neighborhood of the origin and then the existence domains of an outer and an inner solution do not overlap for small ε which makes it impossible to calculate a connection matrix between the inner and the outer solutions. The consideration of an asymptotic nature of the inner solution at $s=\infty$ is due originally to Wasow [11].

Corresponding to the formal solution of the Proposition 7. 1 we have a following existence theorem.

THEOREM 8.1. *Let r be any positive integer. Then there exists an actual solution $W(s, \rho)$ of* (6.4) *and a domain D_2 of s, ρ-plane defined by*

(8.1) $$D_2\colon \arg s \in S, \qquad 0<\rho\leqq\rho_2, \qquad |s^e\rho|\leqq c_3$$

(ρ_2 and c_3 are some constants independent of ρ) such that for s and ρ in D_2 it holds that

(8.2) $$\begin{aligned} &W(s, \rho)-\sum_{\nu=0}^{r} W_\nu(s)\rho^\nu=E_{r+1}(s, \rho)\rho^{r+1} \qquad \text{for} \quad |s|\leqq s_0, \\ &W(s, \rho)-\Omega(s)\sum_{\nu=0}^{r} \widetilde{W}^*(s)[s^e\rho]^\nu F(s)=\Omega(s)E_{r+1}(s, \rho)[s^e\rho]^{r+1}F(s) \qquad \text{for} \quad |s|>s_0, \end{aligned}$$

where $E_{r+1}(s, \rho)$ is bounded.

Proof. This is almost the same as that of, for example, the Theorem 5.1 in Nishimoto [5], and then is omitted.

§ 9. Matching matrix.

14. If we rewrite the domain D_2 in terms of x, ε-plane, it becomes

$$D_2\colon \arg x\in S, \qquad 0<\varepsilon\leqq\varepsilon_2, \qquad |x|\leqq c_3'\varepsilon^{1/a-1/(n-m+q)e}.$$

Then the domain D_1 defined in Theorem 4.1 and the above domain D_2 are overlapped for all sufficiently small parameter ε. From this fact we want to identify two solutions at some suitable point belonging to both domains D_1 and D_2, and for such a point we choose the most symmetrically located point x_η such that

(9.1) $$x_\eta=\eta^{(n-m)}\rho^{(n-n)-1/2e}, \qquad s_\eta=\eta^{(n-m)}\rho^{-1/2e}$$

and then

(9.2) $$\begin{aligned} &x_\eta^{1/(n-m)}=\eta\rho^{(\delta-1)/\delta}, \qquad t_\eta=\eta^{-(n-m+q)}\rho^{(n-m+q)/\delta}, \\ &s_\eta^{-1/(n-m)}=\eta^{-1}\rho^{1/\delta}, \qquad s_\eta^e\rho=\eta^{e(n-m)}\rho^{1/2}, \end{aligned}$$

where $\delta=2e(n-m)$ and η is a parameter such that $\arg\eta^{n-m}\in S$.

Since the value of s_η becomes infinite when $\rho\to 0$ for any fixed η, we use the asymptotic representation of the inner solution for $|s|>s_0$ in D_2. The outer solution $Y_1(x, \varepsilon)$ of the differential equation has from (3.1), Lemma 3.2 and Theorem 5.1 an asymptotic representation in D_1 of the form

$$
(9.3) \qquad Y_1(x,\varepsilon)=\begin{bmatrix} x^m & & & & & & \\ & x^{m-1} & & & & 0 & \\ & & \ddots & & & & \\ & & & x & & & \\ & & & & 1 & & \\ & 0 & & & & x^{q/(n-m)} & \\ & & & & & & \ddots \\ & & & & & & & x^{q(n-m-1)/(n-m)} \end{bmatrix}\cdot\sum_{\nu=0}^{\infty} W_\nu(x)t^\nu
$$

$$
\cdot\begin{bmatrix} x^{a_1} & & & & \\ & \ddots & & 0 & \\ & & x^{a_m} & & \\ & 0 & & (\exp d_{0m+1}/t)x^{d_{1m+1}} & \\ & & & & \ddots \\ & & & & & (\exp d_{0n}/t)x^{d_{1n}} \end{bmatrix},
$$

where $W_\nu(x)$ are polynomials of $\log x$ of degree at most ν whose coefficients are holomorphic in $x^{1/(n-m)}$, in particular $W_0(0)$ is nonsingular. And the inner solution $Y_2(s,\rho)$ in D_2 and $|s|>s_0$ can be written from Theorem 8.1 such that

$$
(9.4) \qquad Y_2(s,\rho)=\begin{bmatrix} (\rho^{n-m}s)^m & & & & & \\ & \ddots & & & 0 & \\ & & (\rho^{n-m}s) & & & \\ & & & 1 & & \\ & 0 & & & (\rho^{n-m}s)^{q/(n-m)} & \\ & & & & & \ddots \\ & & & & & & (\rho^{n-m}s)^{q(n-m-1)/(n-m)} \end{bmatrix}\sum_{\nu=0}^{\infty}\widetilde{W}_\nu^*(s)[s^e\rho]^\nu
$$

$$
\cdot\begin{bmatrix} s^{a_1} & & & & \\ & \ddots & & 0 & \\ & & s^{a_m} & & \\ & & & (\exp d_{0m+1}s^a)s^{d_{1m+1}} & \\ & 0 & & & \ddots \\ & & & & & (\exp d_{0n}s^a)s^{d_{1n}} \end{bmatrix},
$$

where $\widetilde{W}_\nu^*(s)$ are bounded for $|s|>s_0$ and have asymptotic expansions in power series of $s^{-1/(n-m)}$ whose coefficients are polynomials of $\log s$, in particular $W_0^*(s)$ is nonsingular and from Lemma 3. 2 and (6. 17) we have

$$
(9.5) \qquad W_0(0)=\widetilde{W}_0^*(0).
$$

Now let the connection matrix $C(\rho)$ between $Y_1(x,\varepsilon)$ and $Y_2(s,\rho)$ be such that

$$
(9.6) \qquad Y_1(x,\varepsilon)=Y_2(s,\rho)C(\rho),
$$

and let

$$\bar{Y}_1(x, \varepsilon) = \sum_{\nu=0}^{\infty} W_\nu(x) t^\nu,$$

$$\bar{Y}_2(s, \rho) = \sum_{\nu=0}^{\infty} \widetilde{W}_\nu^*(s)[s^e \rho]^\nu.$$

If we substitute (9. 1) for x and s in (9. 6), then we have from (9. 3) and (9. 4)

$$C(\rho) = F(s_\eta)^{-1} \bar{Y}_2(s_\eta, \rho)^{-1} \bar{Y}_1(x_\eta, \varepsilon) F(x_\eta, t_\eta). \tag{9. 7}$$

Now from the asymptotic natures of $\bar{Y}_1(x, t)$ and $\bar{Y}_2(s, \rho)$, we have following lemmas.

LEMMA 9. 1.

$$\bar{Y}_1(x_\eta, \varepsilon) \cong \sum_{\nu=0}^{\infty} \bar{Y}_1^{(\nu)}(\eta\rho) \rho^{\nu/\delta} \qquad (\rho \to 0),$$

$$\bar{Y}_1^{(0)}(\eta, \rho) = W_0(0),$$

$$\bar{Y}_1^{(\nu)}(\eta, \rho) = \sum_\mu \bar{Y}_{1,\mu}^{(\nu)} (\log \eta \rho^{(\delta-1)/\delta}) \eta^\mu,$$

where the summation with respect to μ *consists of a finite number of terms for which* $\mu \equiv -\nu \pmod \delta$ *and* $\bar{Y}_{1\mu}^{(\nu)}(z)$ *are polynomials of* z.

LEMMA 9. 2.

$$\bar{Y}_2(s_\eta, \rho)^{-1} \cong \sum_{\nu=0}^{\infty} \bar{Y}_2^{(\nu)}(\eta, \rho) \rho^{\nu/\delta} \qquad (\rho \to 0),$$

$$\bar{Y}_2^{(0)}(\eta, \rho) = \widetilde{W}_0^*(0)^{-1},$$

$$\bar{Y}_2^{(\nu)}(\eta, \rho) = \sum_\mu \bar{Y}_{2\mu}^{(\nu)} (\log \eta \rho^{-1/\delta}) \eta^\mu,$$

where the summation with respect to μ *is taken over a finite number of integers* μ *such that* $\mu \equiv -\nu \pmod \delta$ *and* $Y_{2\mu}^{(\nu)}(z)$ *are polynomials of* z.

LEMMA 9. 3. *From above two lemmas we have*

$$\bar{Y}_2(s_\eta, \rho)^{-1} \bar{Y}_1(x_\eta, \varepsilon) \cong \sum_{\nu=0}^{\infty} A^{(\nu)}(\eta, \rho) \rho^{\nu/\delta} \qquad (\rho \to 0),$$

$$\Lambda^{(0)}(\eta, \rho)=I \text{ (unit matrix)},$$

$$\Lambda^{(\nu)}(\eta, \rho)=\sum \Lambda_{\mu}^{(\nu)}(\eta, \rho)\eta^{\mu},$$

where $\Lambda_{\mu}^{(\nu)}(\eta, \rho)$ *are polynomiols of* $\log \eta\rho^{(\delta-1)/\delta}$ *and* $\log \eta\rho^{-1/\delta}$. *The summation with respect to* μ *is over a finite number of integers for which* $\mu=-\nu \pmod{\delta}$.

Proof. We give a proof only for Lemma 9.1, and for others it is almost obvious. If we replaced x and t by x_η and t_η in the asymptotic expansion of $\bar{Y}_1(x, \varepsilon)$, we have formally a series

$$\bar{Y}_1(x_\eta, \varepsilon)\sim \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\widetilde{W}_{\mu}^{(\nu)}(\log \eta\rho^{(\delta-1)/\delta})\eta^{\mu-(n-m-q)\nu}\rho^{[(n-m+q)\nu+(\delta-1)\mu]/\delta}$$

where $\widetilde{W}_{\mu}^{(\nu)}(z)$ are polynonials of z. If we rearrange this series formally by collecting all the terms of same power of ρ, we have

$$\bar{Y}_1(x_\eta, \varepsilon)\sim \sum_{r=0}^{\infty}\bar{Y}_1^{(r)}(\eta, \rho)\rho^{r/\delta}, \tag{9.8}$$

where

$$\bar{Y}_1^{(r)}(\eta, \rho)=\sum_{(n-m+q)\nu+(\delta-1)\mu=r}\widetilde{W}_{\mu}^{(\nu)}(z)\eta^{\mu-(n-m+q)\nu} \qquad (z=\log \eta\rho^{(\delta-1)/\delta}),$$

in particular we have

$$\bar{Y}_1^{(0)}(\eta, \rho)=W_0(0).$$

We remark here that for every r, $\bar{Y}_1^{(r)}(\eta, \rho)$ contains only a finite number of terms $\widetilde{W}_{\mu}^{(\nu)}(z)\eta^{\lambda}$ for which $\lambda=-r \pmod{\delta}$. Next let us examine the asymptotic property of (9.8). From Theorem 5.1 and the properties of $W_\nu(x)$ we can write for every positive integer r,

$$\bar{Y}_1(x_\eta, \varepsilon)-\sum_{\nu=0}^{r}\bar{Y}_1^{(\nu)}(\eta, \rho)^{r/\delta}\cong \sum_{\nu>r/(n-m+q)}W_\nu(x_\eta)t_\eta^{\nu}+\sum_{\nu\leqq r/(n-m+q)}$$

$$\sum_{\mu>\{r-(n-m+q)\nu\}/(\delta-1)}\widetilde{W}_{\mu}^{(\nu)}(z)\eta^{\mu}t_\eta^{\nu}=o(\rho^{r/\delta}).$$

This proves our lemma.

We denote the each element of the connection matrix $C(\rho)$ by $c_{jk}(\rho)$. Then from (9.7) and Lemma 9.3 $c_{jk}(\rho)$ can be written as

$$c_{jk}(\rho)\cong\eta^{(n-m)(a_k-a_j)}\rho^{(n-m)\{a_j+(\delta-1)a_k\}/\delta}\sum_{\nu=0}^{\infty}c_{jk}^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}$$

$$(j,k=1,2,\cdots,m),$$

$$c_{jk}(\rho)\cong\{\exp -d_{0j}s_\eta^a\}\eta^{(n-m)(a_k-d_{1j})}\rho^{(n-m)\{d_{1j}+(\delta-1)a_k\}/\delta}\sum_{\nu=0}^{\infty}c_{jk}^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}$$

$$(j=m+1,\cdots,n,\ k=1,\cdots,m),$$

(9. 9)

$$c_{jk}(\rho)\cong\{\exp d_{0k}s_\eta^a\}\eta^{(n-m)(d_{1k}-a_j)}\rho^{(n-m)\{a_j+(\delta-1)d_{1k}\}/\delta}\sum_{\nu=0}^{\infty}c_{jk}^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}$$

$$(j=1,\cdots,m,\ k=m+1,\cdots,n),$$

$$c_{jk}(\rho)\cong\{\exp (d_{0k}-d_{0j})s_\eta^a\}\eta^{(n-m)(d_{1k}-d_{1j})}\rho^{(n-m)\{d_{1j}+(\delta-1)d_{1k}\}/\delta}\sum_{\nu=0}^{\infty}c_{jk}^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}$$

$$(j,k=m+1,\cdots,n),$$

where $c_{jk}^{(\nu)}(\eta,\rho)$ are of the same forms as the elements of $\Lambda^{(\nu)}$, and then we have

THEOREM 9. 1. *Let S be a sector of central angle less than $(n-m)\pi/(n-m+q)$ which was defined in Proposition 6. 1, and let $Y_1(x,\varepsilon)$ and $Y_2(s,\rho)$ be fundamental systems of outer solutions and inner solutions which are defined in D_1 and D_2 respectively under the assumptions* (2. 3), (2. 5) *and* (3. 15). *Then the connection matrix $C(\rho)$ between them has a form*

$$C(\rho)\cong\left\{\sum_{\nu=r_0}C_\nu\rho^\nu\right\}\varepsilon^{\pi/a}\qquad(r_0\leqq 0),$$

where C_ν are diagonal constant matrices, in particular $C_0=I$ (unit matrix) and

$$\pi=\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_m & & & \\ & & & d_{1m+1} & & \\ & 0 & & & \ddots & \\ & & & & & d_{1n}\end{bmatrix}.$$

Proof. Since the elements $c_{jk}(\rho)$ do not depend on η, so must be the right hand terms of relations (9. 9). Let j or k or both j and k be larger than m. Then the representations of $c_{jk}(\rho)$ for $j\neq k$ carry the exponential factors which imply that $c_{jk}(\rho)$ must be identically zero, otherwise $c_{jk}(\rho)$ must depend on η. For $c_{jj}(\rho)$

($j=m+1, \cdots, n$), it does not depend on η if and only if all of the coefficients $c_{jj}^{(\nu)}(\eta, \rho)$ are constants, but from the structures of them it is possible if and only if $\nu=0$ (mod δ). Then we have

$$c_{jj}(\rho)=\rho^{(n-m)d_{1j}}\sum_{\nu=0}^{\infty} c_{jj}^{(\nu)}\rho^{\nu} \qquad (j=m+1, \cdots, n).$$

For the case of $j, k=1, 2, \cdots, m$, since $(n-m)(a_j-a_k)$ ($j, k=1, 2, m, j\neq k$) is not an integer the same reasons as stated as above insure us that the statements of the theorem are satisfied and this completes our proof of the theorem.

REMARK 2. If the assumption (3. 15) is not satisfied, that is, if we have $(n-m)(a_j-a_k)=$integer for some j, k ($j, k=1, \cdots, m, j\neq k$) our theory is also true without any essential changes. In this case it may occur in the Theorem 9. 1 that some elements $c_{jk}(\rho)$ of the connection matrix $C(\rho)$ are not always identically zero for $j, k=1, 2, \cdots, m, j\neq k$. We need a little more careful constructions of the inner and outer formal solutions and comparison of the coefficients of them than that of § 3, § 6 and § 9 to obtain the exact informations about $c_{jk}(\rho)$ in this case.

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