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ON CERTAIN CONDITIONS FOR A K-SPACE TO BE ISOMETRIC TO A SPHERE

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1. Introduction.

THEOREM A (Yano and Nagano [11]). *If M is a complete Einstein space of dimension $n>2$ and $C_0(M)\neq I_0(M)$, then M is isometric to a sphere, where $C_0(M)$ is the largest connected group of conformal transformations of a Riemannian manifold M and $I_0(M)$ the largest connected group of isometries of M.*

THEOREM B (Lichnerowicz [3], Yano and Obata [12]). *If a compact Riemannian manifold M with $R=$const. of dimension $n>2$ admits an infinitesimal conformal transformation v^i which is not an isometry: $\mathcal{L}\, g_{ji}=2\rho g_{ji}$, $\rho\neq$const., and if one of the following conditions is satisfied, then M is isometric to a sphere.*

(1) *The vector v^h is a gradient of a scalar.*

(2) *$R_i{}^h\rho^i=k\rho^h$, k being a constant.*

(3) *$\mathcal{L}R_{ji}=\alpha g_{ji}$, α being a scalar field, where $\mathcal{L}$ is the operator of Lie derivation with respect to v^i, g_{ji} the fundamental metric tensor, R_{ji} the Ricci tensor of M, $R=g^{ji}R_{ji}$ and ρ^h the gradient of the scalar ρ.*

These theorems support the following well known conjecture, that is, a compact Riemannian manifold with constant scalar curvature admitting a one-parameter group of conformal transformations which is not that of isometries is isometric to a sphere.

The purpose of the present paper is to obtain certain conditions for a K-space with constant scalar curvature to be isometric to a sphere. First let M be a connected Riemannian manifold of dimension n and ∇_i the operator of covariant differentiation with respect to the Levi-Civita connection. Indices run over the range $1, 2, \cdots, n$.

If M admits an infinitesimal conformal transformation v^h, then we have

$$\mathcal{L}g_{ji}=\nabla_j v_i+\nabla_i v_j=2\rho g_{ji}, \qquad \mathcal{L}g^{ji}=-2\rho g^{ji} \tag{1.1}$$

for a certain scalar field ρ.

For an infinitesimal conformal transformation v^h in M, we have

$$\mathcal{L}\, R_{kji}{}^h=-\delta_k^h\nabla_j\rho_i+\delta_j^h\nabla_k\rho_i-\nabla_k\rho^h\cdot g_{ji}+\nabla_j\rho^h\cdot g_{ki}, \tag{1.2}$$

$$\mathcal{L}\, R_{ji}=-(n-2)\nabla_j\rho_i-\Delta\rho\cdot g_{ji}, \tag{1.3}$$

$$\mathcal{L}\, R=-2(n-1)\Delta\rho-2\rho R, \tag{1.4}$$

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where $R_{kji}{}^h$ is Riemannian curvature tensor and $\Delta\rho=g^{ji}\nabla_j\nabla_i\rho$.

Thus in M with $R=$const., we have

$$\Delta\rho=-\frac{R}{n-1}\rho. \tag{1.5}$$

We also have

$$\pounds\, G_{ji}=-(n-2)\left(\nabla_j\rho_i-\frac{1}{n}\Delta\rho\cdot g_{ji}\right) \tag{1.6}$$

where $G_{ji}=R_{ji}-(R/n)g_{ji}$.

Hence in M with $R=$const., we have

$$\pounds\, G_{ji}=-(n-2)\left(\nabla_j\rho_i+\frac{R}{n(n-1)}\rho g_{ji}\right). \tag{1.7}$$

In this paper we need the following theorem and integral formula.

THEOREM C (Obata [4]). *If a complete Riemannian manifold of dimension* $n\geqq 2$ *admits a nonconstant function* ρ *such that*

$$\nabla_j\nabla_i\rho=-c^2\rho g_{ji} \tag{1.8}$$

where c *is a positive constant, then* M *is isometric to a sphere of radius* $1/c$ *in* $(n+1)$-*dimensional Euclidean space.*

For a vector field v^h in a compact orientable Riemannian manifold M of dimension $n\geqq 2$, we have the following known integral formula which is verified by a straightforward computation:

$$\begin{aligned}&\int_M\left(g^{ji}\nabla_j\nabla_i v^h+R_i{}^h v^i+\frac{n-2}{n}\nabla^h\nabla_i v^i\right)v_h dV\\ &+\frac{1}{2}\int_M\left(\nabla^j v^i+\nabla^i v^j-\frac{2}{n}\nabla_t v^t g^{ji}\right)\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{n}\nabla_s v^s g_{ji}\right)dV=0\end{aligned} \tag{1.9}$$ [1)]

where dV is the volume element of M.

2. Identities and lemmas in a K-space.

Let M be an n-dimensional almost-Hermitian manifold which admits an almost complex structure tensor $\varphi_j{}^i$ and a positive definite Riemannian metric tensor g_{ji} satisfying

$$\varphi_j{}^l\varphi_l{}^i=-\delta_j^i, \tag{2.1}$$

$$g_{ab}\varphi_j{}^a\varphi_i{}^b=g_{ji}. \tag{2.2}$$

Then from (2.1) and (2.2), we have

$$\varphi_{ji}=-\varphi_{ij} \tag{2.3}$$

1) See Yano [13].

where $\varphi_{ji}=g_{li}\varphi_j{}^l$.

An almost-Hermitian manifold is called a K-space if it satisfies

(2.4) $$\nabla_j\varphi_{ih}+\nabla_i\varphi_{jh}=0$$

from which we have easily

(2.5) $$\nabla_j\varphi_i{}^j=0.$$

In a K-space, we know the following identities obtained by Tachibana [10]:

(2.6) $$R_{ji}^*=R_{ij}^*,$$

(2.7) $$R_{ji}-R_{ji}^*=(\nabla_j\varphi_{rs})\nabla_i\varphi^{rs},$$

(2.8) $$R-R^*=\text{constant}\geqq 0$$

where $R_{ji}^*=(1/2)\varphi^{ab}R_{absi}\varphi_j{}^s$, $R^*=g^{ji}R^*_{ji}$,

(2.9) $$\nabla^h N(v)_h=0 \qquad \text{for any vector } v^i$$

where $N(v)_h=\varphi_h{}^s(\nabla_s\varphi_{rt})\nabla^r v^t$.

In a Riemannian manifold, we have

(2.10) $$\nabla^i R_{ji}=\frac{1}{2}\nabla_j R$$

and in a K-space

(2.11) $$\nabla^i R_{ji}^*=\frac{1}{2}\nabla_j R^*.$$[2]

Thus from (2.8), (2.10) and (2.11), we have

(2.12) $$\nabla^i R_{ji}=\nabla^i R_{ji}^*.$$

Putting

$$T_{ji}=\nabla_j\xi_i+\nabla_i\xi_j+\varphi_j{}^a\varphi_i{}^b(\nabla_a\xi_b+\nabla_b\xi_a)$$

for any vector ξ^i, we have the following

LEMMA 2.1.[3] *In a compact K-space M with constant scalar curvature, if $T_{ji}=0$ and η_i is a vector field such that $\eta_i=\nabla_i\eta$ for a certain scalar η,* then we have

(2.13) $$\int_M \xi^j\eta^i R_{ji}dV=0.$$

LEMMA 2.2.[4] *In a compact K-space M, we have*

(2.14) $$\int_M\left[\frac{1}{4}T_{ji}T^{ji}+\xi^j\{\nabla^i(\nabla_j\xi_i+\nabla_i\xi_j)+\varphi_j{}^a\varphi_i{}^b\nabla^i(\nabla_a\xi_b+\nabla_b\xi_a)\}\right]dV=0.$$

2) See Sawaki [5].
3) See Takamatsu [8], p. 76.
4) See Takamatsu [8], p. 77.

LEMMA 2.3. *In a compact K-space M with constant scalar curvature, a conformal Killing vector v^i can be decomposed as*

$$v^i = p^i + \eta^i \tag{2.15}$$

where $\nabla_i p^i = 0$ and $\eta^i = \nabla^i \eta$ for a certain scalar function η, and

$$\int_M \left[\frac{1}{4} T_{ji}T^{ji} + 2(R_{ji} - R^*_{ji})\eta^j\eta^i - \frac{2}{n}(R - R^*)\eta_i\eta^i\right] dV = 0 \tag{2.16}$$

where

$$T_{ji} = \nabla_j p_i + \nabla_i p_j + \varphi_j{}^a \varphi_i{}^b (\nabla_a p_b + \nabla_b p_a).$$

Proof. According to the theory of harmonic integrals, we have (2.15). Next we consider (2.14) in which $\xi^i = p^i$, that is,

$$\int_M \left[\frac{1}{4} T_{ji}T^{ji} + p^j\{\nabla^i(\nabla_j p_i + \nabla_i p_j) + \varphi_j{}^a \varphi_i{}^b \nabla^i(\nabla_a p_b + \nabla_b p_a)\}\right] dV = 0. \tag{2.17}$$

By (2.15) and (1.1), we have

$$\begin{aligned} p^j\nabla^i(\nabla_j p_i + \nabla_i p_j) &= p^j\nabla^i(\nabla_j v_i + \nabla_i v_j - 2\nabla_j \eta_i) \\ &= p^j \nabla^i(2\rho g_{ji} - 2\nabla_j \eta_i) \\ &= 2p^j(\rho_j - \nabla^i\nabla_j \eta_i), \end{aligned}$$

or by Ricci's identity,

$$\begin{aligned} p^j\nabla^i(\nabla_j p_i + \nabla_i p_j) &= 2p^j(\rho_j - \nabla_j\nabla^i\eta_i - R_j{}^s\eta_s) \\ &= 2\nabla_j(\rho p^j - p^j\nabla^i\eta_i) - 2R_{js}p^j\eta^s. \end{aligned} \tag{2.18}$$

Similarly

$$\begin{aligned} \varphi_j{}^a\varphi_i{}^b\nabla^i(\nabla_a p_b + \nabla_b p_a) &= \varphi_j{}^a\varphi_i{}^b\nabla^i(\nabla_a v_b + \nabla_b v_a - 2\nabla_a \eta_b) \\ &= 2\varphi_j{}^a\varphi_i{}^b\rho^i g_{ab} - 2\varphi_j{}^a\varphi_i{}^b\nabla^i\nabla_a\eta_b \\ &= 2\rho_j - \varphi_j{}^a\varphi^{ib}(\nabla_i\nabla_b\eta_a - \nabla_b\nabla_i\eta_a) \\ &= 2\rho_j + \varphi_j{}^a\varphi^{ib}R_{iba}{}^s\eta_s \\ &= 2\rho_j + 2R^*{}_j{}^s\eta_s \end{aligned}$$

from which we have

$$p^j\varphi_j{}^a\varphi_i{}^b\nabla^i(\nabla_a p_b + \nabla_b p_a) = 2\nabla_j(\rho p^j) + 2R_{js}{}^*p^j\eta^s. \tag{2.19}$$

By (2.18) and (2.19), (2.17) turns to

$$\int_M \left[\frac{1}{4} T_{ji}T^{ji} - 2(R_{js} - R^*_{js})p^j\eta^s\right] dV = 0$$

or by (2.15)

$$\int_M \left[\frac{1}{4} T_{ji}T^{ji} + 2(R_{js} - R^*_{js})\eta^j\eta^s - 2(R_{js} - R^*_{js})v^j\eta^s\right] dV = 0. \tag{2.20}$$

Taking account of (2.8), (2.12) and $\rho=(1/n)\nabla_i\eta^i$, we have

$$\begin{aligned}\int_M (R_{js}-R^*_{js})v^j\eta^s dV &= -\int_M (R_{js}-R^*_{js})\nabla^s v^j\cdot\eta dV \\ &= -\frac{1}{2}\int_M (R_{js}-R^*_{js})(\nabla^s v^j+\nabla^j v^s)\eta dV \\ &= -\frac{1}{n}\int_M (R-R^*)\eta\nabla_i\eta^i dV \\ &= \frac{1}{n}\int_M (R-R^*)\eta_i\eta^i dV.\end{aligned} \tag{2.21}$$

Thus, from (2.20) and (2.21), we obtain (2.16).

LEMMA 2.4.[5] *If a K-space with* $R=$const. *of dimension n admits a conformal Killing vector* v^h: $\pounds g_{ji}=2\rho g_{ji}$, *then we have*

$$\left(\frac{1}{n-1}R-R^*\right)\rho=0. \tag{2.22}$$

3. Extended contravariant almost analytic vectors.

In an almost complex manifold M, v^i is called an *extended contravariant almost analytic vector* if it satisfies

$$\pounds\ \varphi_j{}^i+\lambda\varphi_j{}^r N_{rl}{}^i v^l=0 \tag{3.1}$$

where $N_{rl}{}^i$ is the Nijenhuis tensor, that is, $N_{rl}{}^i=\varphi_r{}^s(\partial_s\varphi_l{}^i-\partial_l\varphi_s{}^i)-\varphi_l{}^s(\partial_s\varphi_r{}^i-\partial_r\varphi_s{}^i)$ and λ a scalar function [7]. This extended contravariant almost analytic vector is characterized as a cross-section of the tangent bundle $T(M)$ with a suitable almost complex structure [9].

In a K-space, since $N_{ji}{}^h=4\varphi_j{}^s\nabla_s\varphi_i{}^h$, we have

$$v^t\varphi_j{}^l N_{lt}{}^i=4\varphi_j{}^l\varphi_l{}^s(\nabla_s\varphi_t{}^i)v^t=4v^t\nabla_t\varphi_j{}^i.$$

Hence, when $\lambda=-1/4$, (3.1) turns to

$$\begin{aligned}\pounds\ \varphi_j{}^i-\frac{1}{4}\varphi_j{}^r N_{rl}{}^i v^l &= v^r\nabla_r\varphi_j{}^i-\varphi_j{}^r\nabla_r v^i+\varphi_r{}^i\nabla_j v^r-\frac{1}{4}\varphi_j{}^r N_{rl}{}^i v^l \\ &= -\varphi_j{}^r\nabla_r v^i+\varphi_r{}^i\nabla_j v^r=0\end{aligned} \tag{3.2}$$

or

$$\nabla_j v_i-\varphi_j{}^a\varphi_i{}^b\nabla_a v_b=0, \tag{3.3}$$

when $\lambda=0$, (3.1) is the equation defining usual contravariant almost analytic vector. For an extended contravariant almost analytic vector, we have the following

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5) See Sawaki [6].

LEMMA 3. 1.[6] *In a K-space, if v^i is an extended contravariant almost analytic vector for $\lambda=-1/4$, then we have*

(3. 4) $$\nabla^r\nabla_r v_i + R_{ir}{}^* v^r = 0.$$

Recently Takamatsu [9] proved the following

LEMMA 3. 2. *In a compact K-space with $R=$const., if v^i is an extended contravariant almost analytic vector for a constant λ such that $-3/4 \leqq \lambda \leqq 0$, then v^i is decomposed into the form*

(3. 5) $$v^i = p^i + \eta^i$$

where p^i is a Killing vector and $\eta^i = \nabla^i \eta$ for a certain scalar function η.

4. Theorems.

THEOREM 4. 1.[7] *If a complete proper K-space M with $R=$const. of dimension $n>2$ admits an infinitesimal nonhomothetic conformal transformation v^i: $\mathfrak{L} g_{ji} = 2\rho g_{ji}$, $\rho \neq 0$ and*

(4. 1) $$\mathfrak{L} G_{ji} = 0,$$

then M is isometric to a sphere.

Proof. By Lemma 2. 4 and the assumption of the theorem, we have

$$\frac{1}{n-1} R - R^* = 0$$

from which it follows

(4. 2) $$(n-2)R = (n-1)(R-R^*).$$

From (1. 7), by (4. 1), we have

$$\nabla_j \rho_i + \frac{R}{n(n-1)} \rho g_{ji} = 0.$$

On the other hand, from (2. 8) and (4. 2), we have $R-R^*>0$, that is, $R>0$. Because if $R-R^*=0$, then from (2. 7), we have $\nabla_j \varphi_{rs} = 0$ and therefore M becomes a Kählerian manifold [13].

Consequently, by Theorem C, M is isometric to a sphere.

REMARK 4. 1. Since $\mathfrak{L} R_{ji} = \alpha g_{ji}$ implies $\mathfrak{L} G_{ji} = 0$ and we consider a complete space, in a proper K-space, Theorem 4. 1 generalizes Theorem A and B (3).

THEOREM 4. 2. *If a compact K-space M with $R=$const. of dimension $n>2$ such that*

(4. 3) $$R_{ji} - R^*_{ji} = k g_{ji} \qquad (k=\text{const.}),$$

admits an infinitesimal nonhomothetic conformal transformation v^i: $\mathfrak{L} g_{ji} = 2\rho g_{ji}$,

6) See Sawaki and Takamatsu [7].

7) Cf. Theorem A. In an Einstein space $\mathfrak{L} G_{ji} = 0$.

$\rho \neq 0$, *then M is isometric to a sphere.*

Proof. v^i is decomposed into the form (2.15) and from (4.3), we see easily $k=(1/n)(R-R^*)$.

Hence (2.16) becomes

$$\int_M \frac{1}{4} T_{ji}T^{ji}dV=0$$

from which it follows $T_{ji}=0$. Consequently, by Lemma 2.1, we have

$$\int_M p^j\eta^i R_{ji}dV=0. \tag{4.4}$$

To prove that p^i is a Killing vector, we put

$$U_{ji}=\nabla_j p_i+\nabla_i p_j$$

and operate ∇^i to $p^j U_{ji}$, then we have, by $p_i=v_i-\eta_i$,

$$\nabla^i(p^j U_{ji})=\frac{1}{2}U_{ji}U^{ji}+p^j\nabla^i(\nabla_j p_i+\nabla_i p_j). \tag{4.5}$$

For the last term, from (2.18), we have

$$p^j\nabla^i(\nabla_j p_i+\nabla_i p_j)=2\nabla_j(\rho p^j-p^j\nabla^i\eta_i)-2p^j\eta^i R_{ji}.$$

Thus integrating (4.5) and using (4.4), we have

$$\int_M \frac{1}{2}U_{ji}U^{ji}dV=0$$

from which it follows $U_{ji}=0$, i.e., p^i is a Killing vector.

Consequently $\eta_i=v_i-p_i$ is a gradient conformal Killing vector such that $\rho=(1/n)\nabla^i\eta_i\neq 0$ and therefore, by Theorem B(1), M is isometric to a sphere.

REMARK 4.2. A K-space of constant curvature satisfies the condition (4.3).

The same remark applies to Theorem 4.3 and Theorem 4.4.

THEOREM 4.3. *If a compact K-space M of dimension $n>2$ such that*

$$\frac{1}{n-1}R_{ji}=R^*_{ji} \tag{4.6}$$

admits a gradient extended contravariant almost analytic vector η^i for $\lambda=-1/4$:

$$\nabla_j\eta_i-\varphi_j{}^a\varphi_i{}^b\nabla_a\eta_b=0, \qquad \nabla_i\eta^i\neq 0, \tag{4.7}$$

then M is isometric to a sphere.

Proof. Operating ∇^j to $\nabla_j\eta_i=\nabla_i\eta_j$ and using Ricci's identity, we have

$$\begin{aligned}\nabla^j\nabla_j\eta_i&=\nabla^j\nabla_i\eta_j\\&=\nabla_i\nabla^j\eta_j+R_i{}^s\eta_s\end{aligned}$$

and substituting this equation into (3.4) in which $v^i=\eta^i$, we have

$$\nabla_i \nabla^r \eta_r = -(R_i{}^r + R^{*r}_i)\eta_r. \tag{4.8}$$

Again from (3.4) in which $v^i=\eta^i$, we have

$$\nabla^r \nabla_r \eta^i = -R^*{}_r{}^i \eta^r. \tag{4.9}$$

Substituting (4.8) and (4.9) into (1.9) in which $v^i=\eta^i$, we have

$$\int_M \left[(R_r{}^i - R^{*i}_r)\eta^r\eta_i - \frac{n-2}{n}(R_r{}^i + R^{*i}_r)\eta^r\eta_i\right] dV$$
$$+2\int_M \left(\nabla^j\eta^i - \frac{1}{n}\nabla_t\eta^t\cdot g^{ji}\right)\left(\nabla_j\eta_i - \frac{1}{n}\nabla_t\eta^t\cdot g_{ji}\right) dV = 0$$

or

$$\frac{2(n-1)}{n}\int_M \left(\frac{1}{n-1}R_{ji} - R^*_{ji}\right)\eta^j\eta^i dV$$
$$+2\int_M \left(\nabla^j\eta^i - \frac{1}{n}\nabla_t\eta^t g^{ji}\right)\left(\nabla_j\eta_i - \frac{1}{n}\nabla_t\eta^t g_{ji}\right) dV = 0 \tag{4.10}$$

from which we find $\nabla^j\eta^i=(1/n)\nabla_t\eta^t g_{ji}$, that is, η^i is a gradient conformal Killing vector.

On the other hand, operating ∇^j to (4.6) and making use of (2.10), (2.11) and (2.12), we have

$$\frac{2-n}{n-1}\nabla_j R = 0$$

that is, R is constant.

Consequently, by Theorem B(1), M is isometric to a sphere.

THEOREM 4.4. *Let M be a compact K-space of dimension $n>2$ such that*

$$\frac{1}{n-1}R_{ji} = R^*_{ji}.$$

If M admits an extended contravariant almost analytic vector v^i for $\lambda=-1/4$ and $\nabla_i v^i \neq 0$, then M is isometric to a sphere.

Proof. As we have seen in the proof of Theorem 4.3, R is constant and hence by Lemma 3.2, v^i is decomposed into the form

$$v^i = p^i + \eta^i$$

where p^i is a Killing vector and $\eta^i=\nabla^i\eta$.

Consequently we have

$$\nabla^j v^i + \nabla^i v^j = 2\nabla^j\eta^i. \tag{4.11}$$

In §3, we have seen that (3.1) for $\lambda=-1/4$ can be written as

$$\nabla_j v_k - \varphi_j{}^a \varphi_k{}^b \nabla_a v_b = 0. \tag{4.12}$$

Interchanging j and k in (4.12) and adding thus obtained to (4.12), we have

$$\nabla_j v_k + \nabla_k v_j - \varphi_j{}^r \varphi_k{}^i (\nabla_r v_i + \nabla_i v_r) = 0.$$

Substituting (4.11) into this equation, we have

$$\nabla_j \eta_k - \varphi_j{}^a \varphi_k{}^b \nabla_a \eta_b = 0$$

which shows that η^i is a gradient extended contravariant almost analytic vector for $\lambda=-1/4$.

Moreover, by the assumption of the theorem, we have $\nabla_i v^i = \nabla_i \eta^i \neq 0$ and consequently by Theorem 4.3, M is isometric to a sphere.

REMARK 4.3. In Theorem 4.4, when M (dim. $n>2$) admits an extended contravariant almost analytic vector v^i for $\lambda=-1/4$, that is,

$$\pounds \varphi_j{}^i - \frac{1}{4} \varphi_j{}^r N_{rl}{}^i v^l = 0, \tag{4.13}$$

M is isometric to a sphere.

Consequently, as is well known, M which is an almost complex manifold must be S^6.

Conversely, in the following way we can see the fact that S^6 admits a vector field v^i satisfying (4.13).

Let S^6 be the sphere in 7-dimensional Euclidean space E^7 defined by

$$X^A = X^A(x^h), \qquad \sum X^A X^A = r^2 \qquad (r=\text{const.} >0) \tag{4.14}$$

where $h=1, 2, \cdots, 6$ and $A=1, 2, \cdots, 7$.

It is well known that on this sphere S^6 there exists a non-integrable almost complex structure [1] and that it is a K-space with the natural Riemannian metric induced from E^7 [2].

If we denote its almost complex structure tensor by $\varphi_j{}^i$ and its metric tensor by g_{ji}, then we have

$$g_{ji} = \sum B_j{}^A B_i{}^A, \tag{4.15}$$

$$g_{ji} = \varphi_j{}^a \varphi_i{}^b g_{ab} \tag{4.16}$$

where $B_i{}^A = \partial X^A / \partial x^i$. Let h_{ji} be the second fundamental tensor, then we have

$$\nabla_j B_i{}^A = h_{ji} C^A \tag{4.17}$$

where C^A are components of the unit normal to the sphere.

On the other hand, operating ∇_i to (4.14), we have

$$\sum X^A B_i{}^A = 0 \tag{4.18}$$

and again operating ∇_j to (4.18), we have

$$\sum X^A \nabla_j B_i^A = -\sum B_j^A B_i^A = -g_{ji}. \tag{4.19}$$

Since (4.18) shows that X^A is normal to S^6 and C^A is the unit normal to S^6 we have

$$C^A = \frac{X^A}{r}. \tag{4.20}$$

Substituting (4.17) into (4.19), by (4.14) and (4.20), we have

$$r h_{ji} = -g_{ji}. \tag{4.21}$$

Thus by (4.20) and (4.21), (4.17) becomes

$$\nabla_j B_i^A = -\frac{X^A}{r^2} g_{ji}. \tag{4.23}$$

Then if we put $v_i = \nabla_i f$ where

$$f = a_1 X^1 + \cdots + a_7 X^7, \qquad a_1, \cdots, a_7 \text{ being constants,}$$

by (4.23), we have

$$\begin{aligned} \nabla_j v_i &= a_1 \nabla_j B_i^1 + \cdots + a_7 \nabla_j B_i^7 \\ &= -\frac{f}{r^2} g_{ji}. \end{aligned} \tag{4.24}$$

Consequently, from (4.16) and (4.24), we have

$$\nabla_j v_i - \varphi_j{}^a \varphi_i{}^b \nabla_a v_b = 0,$$

that is, by (3.2),

$$\pounds \varphi_j{}^i - \frac{1}{4} \varphi_j{}^r N_{rl}{}^i v^l = 0.$$

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