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# ON CONCURRENT STRUCTURES

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## § 1. Concurrent algebras.

Let $V=\mathrm{R}^n$ and $V^*$ its dual. Let $x^1, \cdots, x^n$ be the natural coordinate system of $\mathrm{R}^n$. A vector field $X=\sum_{i=1}^n X^i\partial/\partial x^i$ on $\mathrm{R}^n$ is called an *infinitesimal concurrent transformation* if it satisfies

$$\frac{\partial X^i}{\partial x^j}=\rho\delta_j^i,$$

where $\rho$ is a constant on $\mathrm{R}^n$.

Let $\mathcal{L}$ be the sheaf of germs of all infinitesimal concurrent transformations of $\mathrm{R}^n$. Then $\mathcal{L}$ is a transitive sheaf of Lie algebra. Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in\mathrm{R}^n$. Then the linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{\begin{pmatrix} \lambda & & & \\ & \lambda & & 0 \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}\right\}.$$

Let $\mathfrak{g}^{(1)}$ be the first prolongation of $\mathfrak{g}$. Then $\mathfrak{g}^{(1)}=0$. By Theorem 4. 1 in [2], $\mathcal{L}(0)$ is isomorphic with $\mathrm{R}^n+\mathfrak{g}$:

$$\mathcal{L}(0)\cong\mathrm{R}^n+\mathfrak{g}.$$

The bracket operation is defined as follows: If $\xi, \eta\in\mathrm{R}^n$ and $A, B\in\mathfrak{g}$, then

$$[\xi, \eta]=0,$$

$$[A, \xi]=A\xi,$$

$$[A, B]=0.$$

Let $G$ be the Lie subgroup of $GL(n, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$ and let $\tilde{G}$ be the semidirect product of $\mathrm{R}^n$ and $G$. Let $\omega^i, \omega_j^i, i, j=1, \cdots, n$, be the left invariant 1-forms on $\tilde{G}$. Then the equations of Maurer-Cartan of $\tilde{G}$ are given by

$$d\omega^i=-\Sigma\omega_k^i\wedge\omega^k,$$

$$d\omega_j^i=0.$$

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## § 2. Concurrent structures.

Let $M$ be a differentiable manifold of dimension $n$ and $F(M)$ the bundle of linear frames of $M$. Let $G$ be the subgroup of $GL(n, \mathrm{R})$ defined in § 1. A *concurrent structure* on $M$ is, by definition, a reduction of the bundle of linear frames $F(M)$ to $G$, that is, a $G$-structure $P_G(M)$ on $M$.

$\mathrm{R}^n$ carries a natural $G$-structure $P_G(\mathrm{R}^n)$ which will be called the *standard G-structure.*

Let $\theta=(\theta^i)$ be the canonical form of $F(M)$ restricted to $P_G(M)$. A linear connection on $P_G(M)$ is called a *G-connection.* Let $\omega=(\omega^i_j)$ be a *G-connection.* Then the *structure equations* of $\omega$ are given by

$$d\theta^i=-\Sigma\omega^i_k\wedge\theta^k+\Theta^i, \tag{2. 1}$$

$$d\omega^i_j=\Omega^i_j. \tag{2. 2}$$

For the sake of simplicity, we shall take these equations as a definition of the 2-forms $\Theta^i$ and $\Omega^i_j$. We call $(\Theta^i)$ the *torsion form* of the connection $\omega$ and $(\Omega^i_j)$ the *curvature form* of $\omega$. Let $c^1$ and $c^2$ be the cohomology classes determined by $(\Theta^i)$ and $(\Omega^i_j)$ respectively. Then $c^1$ and $c^2$ are called the *first* and the *second order structure tensor* of $P_G(M)$ respectively.

PROPOSITION 2. 1. *If* $\Theta^i=0$, *then* $\Omega^i_j=0$.

*Proof.* Since $\omega=(\omega^i_j)$ is a $\mathfrak{g}$-valued 1-form on $P_G(M)$, $\omega^i_j$ can be written as

$$\omega^i_j=\delta^i_j\alpha,$$

where $\alpha$ is a 1-form on $P_G(M)$. Hence the equations (2. 1) and (2. 2) reduce to

$$d\theta^i=-\alpha\wedge\theta^i+\Theta^i \tag{2. 3}$$

and

$$\delta^i_j d\alpha=\Omega^i_j. \tag{2. 4}$$

If $\Theta^i=0$, then $d\theta^i=-\alpha\wedge\theta^i$. Taking the exterior differentiation of the both sides of this equation, we have

$$d\alpha\wedge\theta^i=0$$

for all $i$. Hence $d\alpha=0$. This, together with (2. 4), implies that $\Omega^i_j=0$. (Q.E.D.)

COROLLARY. *If* $c^1=0$, *then* $c^2=0$.

## § 3. Concurrent transformations and integrable concurrent structures.

Let $P_G(M)$ and $P_G(M')$ be concurrent structures on manifolds $M$ and $M'$ of the same dimension $n$ respectively. A diffeomorphism $f$: $M\to M'$ is called *concurrent* (with respect to $P_G(M)$ and $P_G(M')$) if $f$, prolonged to a mapping of $F(M)$ onto $F(M')$, maps $P_G(M)$ onto $P_G(M')$. In particular, a transformation $f$ of $M$ is called concurrent (with respect to $P_G(M)$) if it maps $P_G(M)$ onto itself.

A concurrent structure $P_G(M)$ on a manifold $M$ is said to be *integrable* if, for each point of $M$, there exists a neighborhood $U$ and a concurrent diffeomorphism (with respect to $P_G(M)$ and $P_G(\mathrm{R}^n)$) of $U$ onto an open subset of $\mathrm{R}^n$. The answer to the integrability problem for a concurrent structure is the following

THEOREM 3. 1. *A concurrent structure whose structure tensor of the first order vanishes is integrable.*

*Proof.* Let $P_G(M)$ be a concurrent structure on $M$. Since $P_G(M)$ is a $G$-structure of type 1, our assertion follows immediately from Theorem 5. 1 in [1] and Corollary to Prososition 2. 1. (Q.E.D.)

Every vector field $X$ on $M$ generates a 1-parameter local group of local transformations. This local group, prolonged to $F(M)$, induces a vector field on $F(M)$, which will be denoted by $\tilde{X}$. We call $X$ an *infinitesimal concurrent transformation* (with respect to $P_G(M)$) if the local 1-parameter group of local transformations generated by $X$ in a neighborhood of each point of $M$ consists of local concurrent transformations. In other words, $X$ is an infinitesimal concurrent transformation if $\tilde{X}$ is tangent to $P_G(M)$ at each point of $P_G(M)$.

Let $\mathcal{L}$ be the sheaf of germs of infinitesimal concurrent transformations of $P_G(M)$ and $\mathcal{L}(x)$ the stalk of $\mathcal{L}$ at $x \in M$. Then

$$\dim \mathcal{L}(x) \leqq \dim P_G(M) = n+1.$$

THEOREM 3. 2. *Let $P_G(M)$ be a concurrent structure on $M$. Then $P_G(M)$ is integrable if and only if* $\dim \mathcal{L}(x)=n+1$ *at every point $x$ of $M$.*

*Proof.* Let $\theta=(\theta^i)$ be the canonical form of $F(M)$ restricted to $P_G(M)$ and let $\omega=(\omega^i_j)$ be an arbitrary $G$-connection on $P_G(M)$. Let $E$ be the identity element in $\mathfrak{g}$ and $E^*$ the vertical vector field on $P_G(M)$ corresponding to $E$. From the structure equations (2. 1) and (2. 2) we have

$$d\Theta^i = \Sigma\Omega^i_k \wedge \theta^k - \Sigma\omega^i_k \wedge \Theta^k.$$

If we denote by $L_X$ the Lie differentiation with respect to $X$, then we have

$$\begin{aligned} L_{E^*}\Theta^i &= (\iota_{E^*} \circ d + d \circ \iota_{E^*})\Theta^i \\ &= \iota_{E^*} d\Theta^i \\ &= -\Theta^i \end{aligned}$$

since $\Theta^i$ and $\Omega^i_j$ are horizontal form, where $\iota_{E^*}$ denotes the interior product with respect to $E^*$.

If $\tilde{X}$ is the vector field of $P_G(M)$ induced by an infinitesimal concurrent transformation $X$, then we have

$$L_{\tilde{X}}\Theta^i = L_{\tilde{X}}(d\theta^i + \Sigma\omega^i_k \wedge \theta^k)$$
$$= \Sigma(L_{\tilde{X}}\omega^i_k)\wedge\theta^k$$

since $L_{\tilde{X}}\theta^i = 0$.

On the other hand, since dim $\mathcal{L}(x)$=dim $P_G(M)$, for every point $u$ of $P_G(M)$, there exists an infinitesimal concurrent transformation $X$ such that $\tilde{X}_u = E^*_u$. We have therefore

$$\Theta^i = -L_{E^*}\Theta^i = -L_{\tilde{X}}\Theta^i$$
$$= -\Sigma(L_{\tilde{X}}\omega^i_k)\wedge\theta^k$$
$$= -\Sigma(L_{E^*}\omega^i_k)\wedge\theta^k$$

at $u$. This implies that

$$(u^*\Theta^i) \in \partial(\mathfrak{g}\otimes V^*),$$

where $u$ is considered as a linear isomorphism of $V = \mathbf{R}^n$ onto $T_x(M)$ with $x = \pi(u)$ and $\partial$: $\mathfrak{g}\otimes V^* \to V\otimes\wedge^2(V^*)$ is the usual coboundary operator. Thus the structure tensor of the first order $c^1$ of $P_G(M)$ vanishes. Our assertion follows from Theorem 3.1. (Q.E.D.)

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