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G-STRUCTURES DEFINED BY TENSOR FIELDS

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Introduction.

In this paper we shall give systematic approaches to some pseudogroup structures and G-structures defined by tensor fields. We consider the following correspondences between structures on even dimensional manifolds and those on odd dimensional ones:

(*)	complex structure	(*)	cocomplex structure
(♯)	almost complex structure	(♯)	almost cocomplex structure
(*)	symplectic structure	(*)	cosymplectic structure
(♯)	almost symplectic structure	(♯)	almost cosymplectic structure
(*)	homogeneous contact structure	(*)	contact structure
(♯)	almost homogeneous contact structure	(♯)	almost contact structure.

The (*)ed structures are pseudogroup structures and the (♯)ed structures are G-structures.

§ 1. Preliminaries.

A pseudogroup is a collection of transformations which is closed under inverse and composition whenever these are defined.

DEFINITION 1. 1. Let V be a differentiable manifold. A *pseudogroup*, Γ, is a collection of local diffeomorphisms of V satisfying the following axioms:

(1) If $\varphi\in\Gamma$ and $\psi\in\Gamma$, and the domain of φ equals the range of ψ, then $\varphi\circ\psi\in\Gamma$.

(2) If $\varphi\in\Gamma$, then $\varphi^{-1}\in\Gamma$.

(3) If $\varphi\in\Gamma$ and U is an open set contained in the domain of φ, then $\varphi|U\in\Gamma$.

(4) If φ is a local diffeomorphism with domain U, and $U=\cup_\alpha U_\alpha$ with $\varphi|U_\alpha\in\Gamma$, then $\varphi\in\Gamma$.

(5) The identity diffeomorphism is in Γ.

Let Γ be a pseudogroup of differentiable transformations of a manifold V (say R^n) and let M be a differentiable manifold. A Γ-*atlas* on M is a collection of local diffeomorphisms $\{\lambda_i; U_i\}$ of M into V which satisfies $\cup U_i=M$ and $\lambda_i\circ\lambda_j^{-1}\in\Gamma$ for all

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i and j such that $U_i \cap U_j \neq \phi$.

Two Γ-atlases are said to be *equivalent* if their union is a Γ-atlas.

DEFINITION 1.2. An equivalence class of Γ-atlases is called a *Γ-structure* on M.

By an *almost Γ-structure* on a manifold M we mean, roughly speaking, a structure on M which is identified with a Γ-structure up to a certain order of contact at each point.

DEFINITION 1.3. An almost Γ-structure is said to be *integrable* if it determines a Γ-structure.

Let M be a differentiable manifold of dimension n and $F(M)$ the *bundle of linear frames* of M. Then $F(M)$ is a principal fibre bundle over M with structure group $GL(n, \mathrm{R})$.

DEFINITION 1.4. Let G be a subgroup of $GL(n, \mathrm{R})$. A *G-structure* $P_G(M)$ on M is a *reduction* of $F(M)$ to the group G.

In this paper, every almost Γ-structure is a G-structure for some G.

Let $V=\mathrm{R}^n$ and V^* its dual and G a subgroup of $GL(n, \mathrm{R})$. Let $\mathfrak{g}$ be the Lie algebra of G. We define the coboundary operator ∂: $\mathfrak{g}\otimes V^* \to V\otimes\wedge^2(V^*)$ by

$$(\partial t)(x, y)=t(x)\cdot y-t(y)\cdot x$$

for $x, y \in V$. We denote the cohomology group $V\otimes\wedge^2(V^*)/\partial(\mathfrak{g}\otimes V^*)$ by $H^{0,2}(G)$. Let $P_G(M)$ be a G-structure on M. We call a connection on $P_G(M)$ a *G-connection*. The torsion form of a local G-connection determines a function on $P_G(M)$ with value in $H^{0,2}(G)$. We call it the *first order structure tensor* of $P_G(M)$ and denote by c.

We shall give answers to the integrability problems for almost Γ-structures in terms of the first order structure tensor of the corresponding G-structure.

DEFINITION 1.5. Let L be a Lie algebra with a decreasing sequence of subalgebras $L=L_{-2}=L_{-1}\supset L_0\supset L_1\supset L_2\supset\cdots$. We call L a *filtered Lie algebra* if the following conditions are satisfied:

(1) $\cap L_p=\{0\}$,
(2) $[L_p, L_q]\subset L_{p+q}$,
(3) $\dim L_p/L_{p+1}<\infty$,
(4) $L_p=\{t\in L_{p-1}\mid [t, L]\subset L_{p-1}\}$.

Suppose we are given a Γ-structure on M. Let $\mathcal{L}$ be the sheaf of germs of infinitesimal automorphisms of the Γ-structure. Let $x_0\in M$ and $\mathcal{L}(x_0)$ the stalk of $\mathcal{L}$ at x_0. Let $\mathcal{L}_p(x_0)$ be the subset of $\mathcal{L}(x_0)$ consisting of the elements vanishing to order p at x_0. Then $\mathcal{L}(x_0)$ is a filtered Lie algebra with filtration $\mathcal{L}(x_0)\supset\mathcal{L}_0(x_0)\supset\mathcal{L}_1(x_0)\supset\mathcal{L}_2(x_0)\supset\cdots$.

DEFINITION 1. 6. A filtered Lie algebra L is said to be *flat* if it is isomorphic with $\Pi_{p=-1}^{\infty}(L_p/L_{p+1})$.

Let L be a filtered Lie algebra. We call L_0 the *isotropy algebra* and L_0/L_1 the *linear isotropy algebra*.

Let $\mathfrak{g}$ be a Lie algebra of linear endomorphisms of a vector space V. We call

$$\mathfrak{g}^{(p)}=\mathfrak{g}\otimes S^p(V^*)\cap V\otimes S^{p+1}(V^*)$$

the *p-th prolongation* of $\mathfrak{g}$.

DEFINITION 1. 7. Let V be a vector space of dimension n and $\mathfrak{g}$ a subalgebra of $\mathfrak{gl}(V)$. $\mathfrak{g}$ is said to be *involutive* if there exists a basis $e_1, \cdots, e_n$ of V such that

$$\dim \mathfrak{g}^{(1)}=\dim \mathfrak{g}+\sum_{k=1}^{n-1} d_k,$$

where

$$d_k=\dim \{t\in\mathfrak{g} \mid t(e_1)=\cdots=t(e_k)=0\}.$$

§ 2. Complex structures and almost complex structures.

Let $y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n} and let

$$F=\sum_{i=1}^{n}\frac{\partial}{\partial y^i}\otimes dy^{i+n}-\sum_{i=1}^{n}\frac{\partial}{\partial y^{i+n}}\otimes dy^i.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n} which satisfy

$$L_XF=0,$$

where L_X denotes the Lie differentiation with respect to X. Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in R^{2n}$. Then $\mathcal{L}(0)$ is a flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{\begin{pmatrix} A & B \\ -B & A\end{pmatrix} \middle| A, B\in\mathfrak{gl}(n, R)\right\}\subset\mathfrak{gl}(2n, R),$$

which is isomorphic with $\mathfrak{gl}(n, C)$. The Lie algebra $\mathfrak{g}$ is involutive. $\mathcal{L}(0)$ is isomorphic with $R^{2n}+\mathfrak{g}+\mathfrak{g}^{(1)}+\cdots$.

A diffeomorphism f: $U\rightarrow U'$, where U and U' are open subsets of R^{2n}, is called a *complex* (*analytic*) *transformation* if it satisfies

$$f_*\circ F=F\circ f_*,$$

where f_* denotes the differential map of f. The collection, Γ, of all such complex (analytic) transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n$. A Γ-structure on M is called a *complex structure*.

Giving a complex structure is the same as giving a tensor field J of type (1, 1) which can locally be written as

$$J=\sum_{i=1}^{n}\frac{\partial}{\partial x^{i}}\otimes dx^{i+n}-\sum_{i=1}^{n}\frac{\partial}{\partial x^{i+n}}\otimes dx^{i}.$$

Let Γ_0 be the subset of Γ consisting of the elements which leave the origin 0 invariant. Let j: $\Gamma_0\to GL(2n, \mathrm{R})$ be defined as follows: for $f\in\Gamma_0$, $j(f)$ is the 1-jet determined by f, that is, the *Jacobian* of f at 0. Let $G=j(\Gamma_0)$. Then G is a subgroup of $GL(2n, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(0)$. G is isomorphic with $GL(n, \mathrm{C})$.

Let M be a differentiable manifold of dimension $2n$. An *almost complex structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Giving a G-structure $P_G(M)$ on M is the same as giving a tensor field J of type (1, 1) on M which satisfies

$$J^2=-I,$$

where I denotes the field of identity endomorphisms,

The answer to the integrability problem for an almost complex structure is the following

THEOREM 2. 1. (Newlander-Nirenberg [4]). *An almost complex structure whose structure tensor of the first order vanishes is complex.*

§ 2′. Cocomplex structures and almost cocomplex structures.

Suppose we are given in R^{2n+1} an involutive differential system of codimension one and a complex structure on its integral manifolds. To fix our notations, let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1} and let

$$\alpha=dy^0$$

and

$$F=\sum_{i=1}^{n}\frac{\partial}{\partial y^{i}}\otimes dy^{i+n}-\sum_{i=1}^{n}\frac{\partial}{\partial y^{i+n}}\otimes dy^{i}.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

$$L_X\alpha=0$$

and

$$L_XF=0.$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in\mathrm{R}^{2n+1}$. Then $\mathcal{L}(0)$ is a flat filtered

Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{\left(\begin{array}{c|cc} 0 & 0 \cdots & 0 \\ \hline 0 & A & B \\ \vdots & & \\ 0 & -B & A \end{array}\right) \middle| A, B \in \mathfrak{gl}(n, \mathrm{R})\right\} \subset \mathfrak{gl}(2n+1, \mathrm{R})$$

which is isomorphic with $\mathfrak{gl}(n, \mathrm{C})$. The Lie algebra $\mathfrak{g}$ is involutive. $\mathcal{L}(0)$ is isomorphic with $\mathrm{R}^{2n+1}+\mathfrak{g}+\mathfrak{g}^{(1)}+\cdots$.

A local diffeomorphism f of R^{2n+1} is called a *cocomplex transformation* if it satisfies

$$f^*\alpha=\alpha$$

and

$$f_*\circ F=F\circ f_*.$$

The collection, Γ, of all such cocomplex transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *cocomplex structure.*

A cocomplex structure is the same as a $2n$-dimensional involutive *complex* differential system. In other words, giving a cocomplex structure on M is the same as giving a *closed* 1-form ω and a tensor field J of type (1, 1) on M which satisfy

$$\omega\circ J=0,$$

$$J^2=-I+Z\otimes\omega,$$

where Z is a unique vector field on M defined by

$$\omega(Z)=1 \qquad \text{and} \qquad J(Z)=0,$$

and

$$[JX, JY]-J[JX, Y]-J[X, JY]+J^2[X, Y]=0$$

for any vector fields X and Y which satisfy $\omega(X)=\omega(Y)=0$.

ω, J and Z can locally be written as

$$\omega=dx^0,$$

$$J=\sum_{i=1}^n \frac{\partial}{\partial x^i}\otimes dx^{i+n}-\sum_{i=1}^n \frac{\partial}{\partial x^{i+n}}\otimes dx^i,$$

$$Z=\frac{\partial}{\partial x^0}.$$

Let Γ_0 be the subset of Γ consisting of the elements which leave the origin 0 invariant. Let $j: \Gamma_0 \to GL(2n+1, R)$ be defined as follows: for $f \in \Gamma_0$, $j(f)$ is the 1-jet determined by f. Let $G=j(\Gamma_0)$. Then G is a subgroup of $GL(2n+1, R)$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(0)$. G is isomorphic with $GL(n, C)$.

Let M be a differentiable manifold of dimension $2n+1$. An *almost cocomplex structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Given a G-structure $P_G(M)$ on M, we can define a 1-form η and a tensor field ϕ of type (1, 1) on M which satisfy

$$\eta \circ \phi = 0 \tag{2'.1}$$

and

$$\phi^2 = -I + \xi \otimes \eta, \tag{2'.2}$$

where ξ is a unique vector field on M defined by

$$\eta(\xi)=1 \quad \text{and} \quad \phi(\xi)=0.$$

In fact, for each $x \in M$, let u be a point of $P_G(M)$ with $\pi(u)=x$, where $\pi: P_G(M) \to M$ is the projection. For any tangent vector X at x, we set

$$\eta_x(X)=\alpha(u^{-1}X) \tag{2'.3}$$

and

$$\phi_x(X)=u(F(u^{-1}X)), \tag{2'.4}$$

where we regard a frame u at x as a linear isomorphism of R^{2n+1} onto $T_x(M)$. From the properties of G, this definition is independent of the choice of u.

Conversely, given a pair of a 1-form η and a tensor field ϕ of type (1, 1) on M, let $P_G(M)$ be the set of all linear frames u which satisfy (2'. 3) and (2'. 4) for any tangent vector X at $x=\pi(u)$. Then $P_G(M)$ is a G-structure on M.

Thus giving a G-structure on M is the same as giving a pair of a 1-form η and a tensor field ϕ of type (1, 1) which satisfy (2'. 1) and (2'. 2).

Then the answer to the integrability problem for an almost cocomplex structure is the following

THEOREM 2'. 1 ([7]). *An almost cocomplex structure whose structure tensor of the first order vanishes is cocomplex.*

Proof. Let $P_G(M)$ be an almost cocomplex structure on M and (ϕ, η) the associated pair. Let Π be a linear connection and ∇ the covariant differentiation with respect to Π. Then Π is a G-connection if and only if

$$\nabla\eta=0 \quad \text{and} \quad \nabla\phi=0.$$

Since the first order structure tensor of $P_G(M)$ vanishes, there exists a torsionfree

G-connection.

In general, let ∇ be a torsionfree G-connection and α a differential form. Then

$$d\alpha = \mathcal{A}(\nabla\alpha),$$

where $\mathcal{A}$ is the alternation operator. Hence, let ∇ be a torsionfree G-connection. Then we have

$$d\eta = 0.$$

Hence the differential system defined by η is involutive.

We have to prove that ϕ gives rise to a complex structure on each integral manifold of η.

The equation (2′. 2) implies that ϕ is an almost complex structure on each integral manifold of η. Let N be the Nijenhuis torsion tensor field of ϕ and let X and Y be vector fields on an integral manifold. Then

$$\begin{aligned} N(X, Y) &= [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] \\ &= [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] - [X, Y], \end{aligned}$$

since $\eta([X, Y]) = 0$.

On the other hand, since ∇ is a torsionfree G-connection, we have

$$\nabla\phi = 0$$

and

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

for any X and Y. Therefore

$$\begin{aligned} N(X, Y) &= \nabla_{\phi X}(\phi Y) - \nabla_{\phi Y}(\phi X) - \phi(\nabla_{\phi X} Y - \nabla_Y(\phi X)) - \phi(\nabla_X(\phi Y) - \nabla_{\phi Y} X) - \nabla_X Y + \nabla_Y X \\ &= \phi^2(\nabla_Y X - \nabla_X Y) - (\nabla_X Y - \nabla_Y X) \\ &= -[Y, X] + \eta([Y, X]) \cdot \xi - [X, Y] \\ &= 0. \end{aligned}$$

This implies that ϕ defines a complex structure on each integral manifold of η. Hence (ϕ, η) determines a cocomplex structure. (Q.E.D.)

Since $\mathfrak{g}$ contains no elements of rank 1, *the automorphism group of an almost cocomplex structure on a compact manifold is a Lie group.*

§ 3. Symplectic structures and almost symplectic structures.

Let $y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n} and let

$$\beta = \sum_{i=1}^{n} dy^i \wedge dy^{i+n}.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n} which satisfy

$$L_X\beta=0. \tag{3.1}$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in\mathrm{R}^{2n}$. Then $\mathcal{L}(0)$ is a flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra of $\mathcal{L}(0)$ is

$$\mathfrak{sp}(n)=\left\{A\in\mathfrak{gl}(2n,\mathrm{R})\,\middle|\,{}^tAJ+JA=0 \quad \text{for } J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\right\}.$$

The Lie algebra $\mathfrak{sp}(n)$ is involutive and $\mathcal{L}(0)$ is isomorphic with

$$\mathrm{R}^{2n}+\mathfrak{sp}(n)+\mathfrak{sp}(n)^{(1)}+\cdots.$$

A local diffeomorphism f of R^{2n} is called a *symplectic transformation* if it satisfies

$$f^*\beta=\beta. \tag{3.2}$$

The collection, Γ, of all such symplectic transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n$. A Γ-structure on M is called a *symplectic structure*.

Giving a symplectic structure is the same as giving a *closed* 2-form Ω which satisfies $\Omega^n\neq 0$.

Let $Sp(n)$ be the subgroup of $GL(2n,\mathrm{R})$ with Lie algebra $\mathfrak{sp}(n)$. An *almost symplectic structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to $Sp(n)$, that is, a $Sp(n)$-structure $P_{Sp(n)}(M)$ on M.

Giving a $Sp(n)$-structure $P_{Sp(n)}(M)$ on M is the same as giving a 2-form Ω on M which satisfies

$$\Omega^n\neq 0.$$

The answer to the integrability problem for an almost symplectic structure is the following

THEOREM 3.1. *An almost symplectic structure whose structure tensor of the first order vanishes is symplectic.*

Proof. Let $P_{Sp(n)}(M)$ be an almost symplectic structure on M and Ω the associated 2-form. Let Π be a linear connection. Then Π is a $Sp(n)$-connection if and only if

$$\nabla\Omega=0.$$

Since the first order structure tensor of $P_{Sp(n)}(M)$ vanishes, there exists a torsionfree $Sp(n)$-connection. Hence we have

$$d\Omega=\mathcal{A}(\nabla\Omega)=0,$$

This implies that $P_{Sp(n)}(M)$ determines a symplectic structure. (Q.E.D.)

If we replace (3. 1) and (3. 2) respectively by

(3. 1)′ $$L_X\beta=\lambda\beta,$$

where λ is a function and

(3. 2)′ $$f^*\beta=\rho\beta,$$

where ρ is a non-zero function, then the linear isotropy algebra is

$$\mathfrak{csp}(n)=\{A\in\mathfrak{gl}(2n, \mathrm{R})\,|\,{}^tAJ+JA=\lambda J\}$$

and $\mathcal{L}(0)$ is isomorphic with

$$\mathrm{R}^{2n}+\mathfrak{csp}(n)+\mathfrak{sp}(n)^{(1)}+\cdots.$$

The resulting structures are called a *conformal symplectic structure* and an *almost conformal symplectic structure.*

§ 3′. Cosymplectic structures and almost cosymplectic structures.

Let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1} and let

$$\alpha=dy^0$$

and

$$\beta=\sum_{i=1}^{n}dy^i\wedge dy^{i+n}.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

$$L_X\alpha=0 \quad \text{and} \quad L_X\beta=0.$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in\mathrm{R}^{2n+1}$. Then $\mathcal{L}(0)$ is a flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is

$$\left\{\left(\begin{array}{c|c} 0 & 0\cdots 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right)\,\middle|\, A\in\mathfrak{sp}(n)\right\}.$$

The Lie algebra $\mathfrak{g}$ is involutive and $\mathcal{L}(0)$ is isomorphic with

$$\mathrm{R}^{2n+1}+\mathfrak{g}+\mathfrak{g}^{(1)}+\cdots.$$

A local diffeomorphism f of R^{2n+1} is called a *cosymplectic transformation* if it satisfies

$$f^*\alpha=\alpha \quad \text{and} \quad f^*\beta=\beta.$$

The collection, Γ, of all such cosymplectic transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *cosymplectic structure.*

Giving a cosymplectic structure is the same as giving a pair of a *closed* 1-form ω and a *closed* 2-form Ω which satisfy $\omega\wedge\Omega^n\neq 0$.

Let G be a subgroup of $GL(2n+1, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$. An *almost cosymplectic structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is G-structure $P_G(M)$ on M.

Giving a G-structure on M is the same as giving a pair of a 1-form ω and a 2-form Ω which satisfy $\omega\wedge\Omega^n\neq 0$.

The answer to the integrability problem for an almost cosymplectic structure is the following

THEOREM 3′. 1. *An almost cosymplectic structure whose structure tensor of the first order vanishes is cosymplectic.*

Proof. Let $P_G(M)$ be an almost cosymplectic structure on M and (ω, Ω) the associated pair. Let Π be a linear connection. Then Π is a G-connection if and only if

$$\nabla\omega=0 \quad \text{and} \quad \nabla\Omega=0.$$

Since the first order structure tensor of $P_G(M)$ vanishes, there exists a torsion-free G-connection. Hence we have

$$d\omega=\mathcal{A}(\nabla\omega)=0$$

and

$$d\Omega=\mathcal{A}(\nabla\Omega)=0.$$

This imlpies the $P_G(M)$ determines a cosymplectic structure (Q.E.D.)

§ 4. Homogeneous contact structure and almost homogeneous contact structures.[1]

Let $y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n} and let

$$\alpha=-\frac{1}{2}\sum_{i=1}^{n}(y^{i+n}dy^i-y^i dy^{i+n}).$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n} which satisfy

$$L_X\alpha=\rho\alpha, \tag{4. 1}$$

1) Perhaps, "exact symplectic structure" is more appropriate. But in conformity with other authors, we use the term "homogeneous contact structure".

where ρ is a function depending on X.

Let x_0 be a point of R^{2n} different from the origin and let $\mathcal{L}(x_0)$ be the stalk of $\mathcal{L}$ at x_0. Then $\mathcal{L}(x_0)$ is a filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(x_0)$ is the direct sum of the linear Lie algebra $\mathfrak{sp}(n)$ and its center, that is, $\mathfrak{g}=\mathfrak{csp}(n)$. By theorem 4. 3 in [2], $\mathcal{L}(x_0)$ is a flat filtered Lie algebra, that is, $\mathcal{L}(x)$ is isomorphic with $R^{2n}+\mathfrak{g}+\mathfrak{g}^{(1)}+\cdots$.

A local diffeomorphism f of R^{2n} is called a *homogeneous contact transformation* if it satisfies

$$f^*\alpha=\rho\alpha, \tag{4. 2}$$

where ρ is a non-zero function.

The collection, Γ, of all such homogeneous contact transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n$. A Γ-structure on M is called a *homogeneous contact structure.*

Giving a homogeneous contact structure on M is the same as giving a 1-form ω *up to a scalar factor* on M which satisfies

$$(d\omega)^n\neq 0.$$

The theorem of Darboux states that a 1-form satisfying $(d\omega)^n\neq 0$ can locally be written as

$$\omega=-\frac{1}{2}\sum_{i=1}^{n}(x^{i+n}dx^i-x^i dx^{i+n}).$$

A local coordinate system in which the form ω is written as above will be called an *admissible* coordinate.

Let Γ_0 be the subset of Γ consisting of the elements which leave the point x_0 invariant. Let j: $\Gamma_0\to GL(2n, R)$ be defined as follows: for $f\in\Gamma_0$, $j(f)$ is the 1-jet at x_0 determined by f. Let $G=j(\Gamma_0)$. Then G is a subgroup of $GL(2n, R)$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(x_0)$.

Let M be a differentiable manifold of dimension $2n$. An *almost homogeneous contact structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Given a G-structure $P_G(M)$ on M, we can define a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$ up to scalar factors which satisfy $\{\Omega\}^n\neq 0$. In fact, for each $x\in M$, let u be a point of $P_G(M)$ with $\pi(u)=x$. For any tangent vector X and Y at x, set

$$\omega_x(X)=\rho\cdot\alpha_{x_0}(u^{-1}X),$$

$$\Omega_x(X, Y)=\sigma\cdot(d\alpha)_{x_0}(u^{-1}X, u^{-1}Y),$$

where α_{x_0} and $(d\alpha)_{x_0}$ denote, respectively, the values of α and $d\alpha$ at x_0, ρ and σ are scalars, and u can be considered as a linear isomorphism of $T_{x_0}(R^{2n})$ onto $T_x(M)$. From the properties of G, this definition is independent of the choice of u.

Conversely, given, up to scalar factors, a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$, let $P_G(M)$ be the set of all linear frames u satisfying

$$\{\omega\}_x(X)=\alpha_{x_0}(u^{-1}X),$$

$$\{\Omega\}_x(X, Y)=(d\alpha)_{x_0}(u^{-1}X, u^{-1}Y)$$

for any vectors X and Y at $x=\pi(u)$. Then $P_G(M)$ is a G-structure on M.

Thus giving a G-structure on M is the same as giving a pair of a 1-form $\{\omega\}$ up to a scalar factor and a 2-form $\{\Omega\}$ up to a scalar factor which satisfies $\{\Omega\}^n \neq 0$ at every point of M.

Let M_0 be a manifold with a homogeneous contact structure. Since every Γ-structure gives rise canonically to an almost Γ-structure, M_0 has a G-structure, an almost homogeneous contact structure.

THEOREM 4.1. *Let $P_G(M_0)$ be the almost homogeneous contact structure associated with a homogeneous contact structure on M_0. Then the first order structure tensor c has the following representative*:

$$(c^i_{\alpha\beta})=\frac{1}{ny^{i+n}}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$$(c^{i+n}_{\alpha\beta})=-\frac{1}{ny^i}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.^{2)}$$

Proof. A representative of c is given by the torsion tensor of a G-connection. Let Π be a connection. Then Π is a G-connection if and only if

$$\nabla\omega=0.$$

Let T be the torsion tensor of Π and $T^\alpha_{\beta\gamma}$ the components of T with respect to an admissible coordinate system $(x^0, x^1, \cdots, x^{2n})$. Then the equation $\nabla\omega=0$ implies

$$-\frac{1}{2}\sum_{i=1}^n x^{i+n}T^i_{jk}+\frac{1}{2}\sum_{i=1}^n x^iT^{i+n}_{jk}=0,$$

$$-\frac{1}{2}\sum_{i=1}^n x^{i+n}T^i_{j+n,k}+\frac{1}{2}\sum_{i=1}^n x^iT^{i+n}_{j+n,k}=-\delta_{jk},$$

$$-\frac{1}{2}\sum_{i=1}^n x^{i+n}T^i_{j+n,k+n}+\frac{1}{2}\sum_{i=1}^n x^iT^{i+n}_{j+n,k+n}=0.$$

We can take T as follows:

$$T^i_{j+n,k}=-T^i_{k,j+n}=-\frac{1}{nx^{i+n}}\delta_{jk},$$

$$T^{i+n}_{j+n,k}=-T^{i+n}_{k,j+n}=\frac{1}{nx^i}\delta_{jk}$$

---

2) $\alpha, \beta, \gamma=1, 2, \cdots, 2n$.

and the other components are all zero.

Since the first order structure tensor c is independent of the choice of a G-connection, our assertion is now clear. (Q.E.D.)

Let c_0 be an element of $H^{0,2}(G)=V\otimes\wedge^2(V^*)/\partial(\mathfrak{g}\otimes V^*)$, $V=\mathrm{R}^{2n}$, whose representative is given by

$$(c^i_{\alpha\beta})=\frac{1}{ny^{i+n}}\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix},$$

$$(c^{i+n}_{\alpha\beta})=-\frac{1}{ny^i}\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$

The answer to the integrability problem for an almost homogeneous contact structure is the following

THEOREM 4. 2. *An almost homogeneous contact structure whose structure tensor of the first order is c_0 is homogeneous contact.*

Proof. Let $P_G(M)$ be an almost homogeneous contact structure on M whose structure tensor of the first order is c_0.

Since $\mathfrak{g}$ is reductive, there is an invariant complement C to $\partial(\mathfrak{g}\otimes V^*)$ in $V\otimes\wedge^2(V^*)$, $V=\mathrm{R}^{2n}$. Let $\tilde{c}_0$ be the element in C which corresponds to c_0 under the isomorphism $C\cong H^{0,2}(G)$. Then there exists a G-connection whose torsion is $\tilde{c}_0$. More precisely, let τ be an element of $V\otimes\wedge^2(V^*)$ whose components $(\tau^{\gamma}_{\alpha\beta})$ are given by

$$(\tau^i_{\alpha\beta})=\frac{1}{ny^{i+n}}\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix},$$

$$(\tau^{i+n}_{\alpha\beta})=-\frac{1}{ny^i}\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$

Then it is easily seen that τ belongs to C. This implies that τ is just $\tilde{c}_0$.

Let σ: $U\to P_G(M)$, $u=\sigma(x)$, be a local cross section. If we set

$$\Theta_x(X, Y)=\tau(u^{-1}X, u^{-1}Y),$$

where $X, Y\in T_x(M)$, then Θ is a R^{2n}-valued 2-form on M defined in U. Let $\tilde{\sigma}$: $U\to P_G(M)$, $\tilde{u}=\tilde{\sigma}(x)$, be an another local cross section and set

$$\tilde{\Theta}_x(X, Y)=\tau(\tilde{u}^{-1}X, \tilde{u}^{-1}Y).$$

Then $\tilde{\Theta}$ differs from Θ by a scalar factor. Hence we have a global 2-form Θ up to a scalar factor.

Let T be a tensor field of type (1, 2) on M determined by Θ. The dimension of the space of G-connections with torsion tensor T is equal to $\dim\mathfrak{g}^{(1)}=(2/3)n(n+1)(2n+1)$. On the other hand, let ϕ be a 1-form on M. Then the dimension of the space of

G-connections satisfying $\nabla\phi=0$ is equal to $\dim\{t\in\mathfrak{g}\otimes V^* \mid \phi\circ t=0\}=(2n-1)(2n^2+n+1)$. Since $\dim\mathfrak{g}\otimes V^*=2n(2n^2+n+1)$, there exists a G-connection, with torsion tensor T, which satisfies $\nabla\phi=0$.

Let $\{\omega\}$ and $\{\Omega\}$ be the classes of 1-forms and 2-forms on M determined by $P_G(M)$. Then we can find locally a 1-form ω in $\{\omega\}$ and a G-connection with torsion tensor T which satisfy

$$\nabla\omega=0.$$

The 1-form ω satisfies

$$2d\omega(X, Y)=\omega(T(X, Y))$$

for any X and Y. In fact, for any X and Y, we have

$$0=(\nabla_X\omega)(Y)=X\cdot\omega(Y)-\omega(\nabla_X Y)$$

and

$$0=(\nabla_Y\omega)(X)=Y\cdot\omega(X)-\omega(\nabla_Y X).$$

Hence we obtain

$$X\cdot\omega(Y)-Y\cdot\omega(X)-\omega([X, Y])=\omega(\nabla_X Y)-\omega(\nabla_Y X)-\omega([X, Y]),$$

that is,

$$2d\omega(X, Y)=\omega(T(X, Y)).$$

Let U be a coordinate neighborhood in M with a local coordinate system $x^1, \cdots, x^{2n}$. We denote by X_α the vector field $\partial/\partial x^\alpha$, $\alpha=1, \cdots, 2n$, defined in U. Every linear frame at a point x of U can be uniquely expressed by

$$\left(\sum_{\alpha=1}^{2n} X_1^\alpha(X_\alpha)_x, \cdots, \sum_{\alpha=1}^{2n} X_{2n}^\alpha(X_\alpha)_x\right),$$

where (X_β^α) is a non-singular matrix.

We take $(x^\alpha, X_\beta^\alpha)$ as a local coordinate system in $\pi^{-1}(U)$. Let (Y_β^α) be the inverse matrix of (X_β^α). Let $e_1, \cdots, e_{2n}$ be the natural basis for R^{2n}. Let u be a point of $P_G(M)$ with coordinates $(x^\alpha, X_\beta^\alpha)$ so that u maps e_α into $\sum_{\beta=1}^{2n} X_\alpha^\beta(X_\beta)_x$, where $x=\pi(u)$.

If $X=\sum_{\alpha=1}^{2n}\xi^\alpha X_\alpha$ and $Y=\sum_{\alpha=1}^{2n}\eta^\alpha X_\alpha$, then

$$u^{-1}X=\sum_{\alpha,\beta=1}^{2n} Y_\beta^\alpha\xi^\beta e_\alpha \qquad \text{and} \qquad u^{-1}Y=\sum_{\alpha,\beta=1}^{2n} Y_\beta^\alpha\eta^\beta e_\alpha.$$

Hence we have

$$\omega(T(X, Y))=\alpha_{x_0}(\Theta(X, Y))$$
$$=\rho\cdot\alpha_{x_0}(\tau(u^{-1}X, u^{-1}Y))$$
$$=-\rho\sum_{\alpha,\beta=1}^{2n}\sum_{i=1}^{n}(Y_\alpha^i Y_\beta^{i+n}-Y_\alpha^{i+n}Y_\beta^i)\xi^\alpha\eta^\beta.$$

On the other hand,

$$2(d\alpha)_{x_0}(u^{-1}X, u^{-1}Y)=2\sum_{i=1}^{n}(dy^i\wedge dy^{i+n})(u^{-1}X, u^{-1}Y)$$
$$=\sum_{i=1}^{n}\{dy^i(u^{-1}X)\cdot dy^{i+n}(u^{-1}Y)-dy^{i+n}(u^{-1}X)\cdot dy^i(u^{-1}Y)\}$$
$$=\sum_{\alpha,\beta=1}^{2n}\sum_{i=1}^{n}(Y_\alpha^i Y_\beta^{i+n}-Y_\alpha^{i+n}Y_\beta^i)\xi^\alpha\eta^\beta.$$

Therefore we have

$$d\omega(X, Y)=-\rho\cdot(d\alpha)_{x_0}(u^{-1}X, u^{-1}Y).$$

This implies that $d\omega\in\{\Omega\}$ and hence ω satisfies

$$(d\omega)^n\neq 0.$$

Hence $\{\omega\}$ defines a homogeneous contact structure on M. (Q.E.D.)

If we replace (4. 1) and (4. 2) by

(4. 1)′ $$L_X\alpha=0$$

and

(4. 2)′ $$f^*\alpha=\alpha$$

respectively, then the resulting structures are called a *strict homogeneous contact structure* and an *almost strict homogeneous contact structure.*

§ 4′. Contact structures and almost contact structures.

Suppose we are given a differential system of codimension one which is of maximal rank. To fix our notations, let $y^0, y^1, \cdots, y^{2n}$ be the natural coordinate system of R^{2n+1} and let

$$\alpha=dy^0-\frac{1}{2}\sum_{i=1}^{u}(y^{i+n}dy^i-y^i dy^{i+n}).$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^{2n+1} which satisfy

(4'. 1) $$L_X\alpha=\rho\cdot\alpha,$$

where ρ is a function depending on X.

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0\in R^{2n+1}$. Then $\mathcal{L}(0)$ is a *non-flat* filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the sum of the linear Lie algebra

$$\left\{\left(\begin{array}{c|c} 0 & 0 \cdots 0 \\ \hline * & \\ \vdots & A \\ * & \end{array}\right) \middle| A\in\mathfrak{sp}(n)\right\}$$

and

$$\left\{\begin{pmatrix} 2\lambda & & & \\ & \lambda & & 0 \\ 0 & & \ddots & \\ & & & \lambda \end{pmatrix}\right\}.$$

PROPOSITION 4'. 1. *$\mathfrak{g}$ is involutive.*

Proof. Let $e_0, e_1, \cdots, e_{2n}$ be the natural basis for R^{2n+1}. Let

$$d_k=\dim\{t\in\mathfrak{g}\,|\,t(e_0)=\cdots=t(e_k)=0\}.$$

Then we have

$$d_k=(n+1)(2n+1)-(k+1)(2n+1)+\frac{k(k+1)}{2},$$

and hence

$$\sum_{k=0}^{2n-1} d_k=\frac{2}{3}n(n+1)(2n+1).$$

On the other hand, since $\mathfrak{g}^{(1)}\cong\mathfrak{sp}(n)^{(1)}+\mathfrak{g}$, we have $\dim\mathfrak{g}^{(1)}=(1/3)(n+1)(2n+1)(2n+3)$. Therefore $\dim\mathfrak{g}^{(1)}=\dim\mathfrak{g}+\sum_{k=0}^{2n-1} d_k$.

This implies that $\mathfrak{g}$ is involutive. (Q.E.D.)

A local diffeomorphism f of R^{2n+1} is called a *contact transformation* if it satisfies

(4'. 2) $$f^*\alpha=\rho\cdot\alpha,$$

where ρ is a non-zero function.
The collection, Γ, of all such contact transformations forms an infinite, continuous pseudogroup.

Let M be a differentiable manifold of dimension $2n+1$. A Γ-structure on M is called a *contact structure*.

Giving a contact structure on M is the same as giving a 1-form ω *up to a scalar factor* on M which satisfies

$$\omega\wedge(d\omega)^n\neq 0.$$

The theorem of Darboux states that a 1-form ω satisfying $\omega\wedge(d\omega)^n\neq 0$ can locally be written as

$$\omega=dx^0-\frac{1}{2}\sum_{i=1}^{n}(x^{i+n}dx^i-x^i dx^{i+n}).$$

A local coordinate system in which the form ω is written as above will be called an *admissible* coordinate.

Let Γ_0 be the subset of Γ consisting of the elements which leave the origin 0 invariant. Let j: $\Gamma_0\to GL(2n+1, \mathrm{R})$ be defined as follows: for $f\in\Gamma_0$, $j(f)$ is the 1-jet determined by f. Let $G=j(\Gamma_0)$. Then G is a subgroup of $GL(2n+1, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$, the linear isotropy algebra of $\mathcal{L}(0)$.

Let M be a differentiable manifold of dimension $2n+1$. An almost contact structure on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M.

Given a G-structure $P_G(M)$ on M, we can define, up to scalar factors, a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$ which satisfy $\{\omega\}\wedge\{\Omega\}^n\neq 0$. In fact, for each $x\in M$, let u be a point of $P_G(M)$ with $\pi(u)=x$. For any tangent vectors X and Y at x, set

$$\omega_x(X)=\rho\cdot\alpha_0(u^{-1}X),$$

$$\Omega_x(X, Y)=\sigma\cdot(d\alpha)_0(u^{-1}X, u^{-1}Y),$$

where α_0 and $(d\alpha)_0$ denote, respectively, the values of α and $d\alpha$ at the origin 0, and ρ and σ are scalars. From the properties of G, this definition is independent of the choice of u.

Conversely, given, up to scalar factors, a pair of a 1-form $\{\omega\}$ and a 2-form $\{\Omega\}$, let $P_G(M)$ be the set of all linear frames u satisfying

$$\{\omega\}_x(X)=\alpha(u^{-1}X),$$

$$\{\Omega\}_x(X, Y)=(d\alpha)_0(u^{-1}X, u^{-1}Y)$$

for any vectors X and Y at $x=\pi(u)$. Then $P_G(M)$ is a G-structure.

Thus giving a G-structure on M is the same as giving a pair of a 1-form $\{\omega\}$ up to a scalar factor and a 2-form $\{\Omega\}$ up to a scalar factor which satisfies $\{\omega\}\wedge\{\Omega\}^n\neq 0$ at every point of M.

Let c_0 be an element of $H^{0,2}(G)=V\otimes\wedge^2(V^*)/\partial(\mathfrak{g}\otimes V^*)$, $V=\mathrm{R}^{2n+1}$, whose representative is given by

$$(c^0_{\alpha\beta})=\left(\begin{array}{c|cc} 0 & 0 & \cdots\ 0 \\ \hline 0 & 0 & I_n \\ \vdots & & \\ 0 & -I_n & 0 \end{array}\right),$$

$$(c^i_{\alpha\beta})=(c^{i+n}_{\alpha\beta})=0.^{3)}$$

The answer to the integrability problem for an almost contact structure is the following

THEOREM 4′. 1 ([6]). *An almost contact structure whose structure tensor of the first order is* c_0 *is contact.*

Proof. Let $P_G(M)$ be an almost contact structure on M whose structure tensor of the first order is c_0.

Since $\mathfrak{g}$ is reductive, there is an invariant complement C to $\partial(\mathfrak{g}\otimes V^*)$ in $V\otimes\wedge^2(V^*)$. Let $\tilde{c}_0$ be the element in C which corresponds to c_0 under the isomorphism $C\cong H^{0,2}(G)$. Then there exists a G-connection on $P_G(M)$ whose torsion is $\tilde{c}_0$. More precisely, let τ be an element of $V\otimes\wedge^2(V^*)$ whose componeuts $(\tau^\gamma_{\alpha\beta})$ are given by

$$(\tau^0_{\alpha\beta})=\left(\begin{array}{c|cc} 0 & 0 & \cdots\ 0 \\ \hline 0 & 0 & I_n \\ \vdots & & \\ 0 & -I_n & 0 \end{array}\right),$$

$$(\tau^i_{\alpha\beta})=(\tau^{i+n}_{\alpha\beta})=0.$$

Then it is easily seen that τ belongs to C. This implies that τ is just $\tilde{c}_0$.

Let σ: $U\rightarrow P_G(M)$, $u=\sigma(x)$, be a local cross section. If we set

$$\Theta_x(X, Y)=\tau(u^{-1}X, u^{-1}Y),$$

where $X, Y\in T_x(M)$. Then Θ is a R^{2n+1}-valued 2-form on M defined in U. Let $\tilde{\sigma}$: $U\rightarrow P_G(M)$, $\tilde{u}=\tilde{\sigma}(x)$, be an another local cross section and set

$$\tilde{\Theta}_x(X, Y)=\tau(\tilde{u}^{-1}X, \tilde{u}^{-1}Y).$$

Then $\tilde{\Theta}$ differs from Θ by a scalar factor.

Hence we have a global 2-form Θ up to a scalar factor.

Let T be a tensor field of type (1, 2) on M determined by Θ. The dimension

3) $\alpha, \beta, \gamma=0, 1, 2, \cdots, 2n$.

of the space of G-connections with torsion tensor T is equal to $\dim \mathfrak{g}^{(1)}=(1/3)(n+1)(2n+1)(2n+3)$. On the other hand, let ψ be a 1-form on M. Then the dimension of the space of G-connections satisfying $\nabla\psi=0$ is equal to $\dim\{t\in\mathfrak{g}\otimes V^* \mid \psi\circ t=0\} =2n(n+1)(2n+1)$. Since $\dim \mathfrak{g}\otimes V^*=(n+1)(2n+1)^2$, there exists a G-connection, with torsion tensor T, which satisfies $\nabla\psi=0$.

Let $\{\omega\}$ and $\{\Omega\}$ be the classes of 1-forms and 2-forms on M determined by $P_G(M)$. Then we can find locally a 1-form ω in $\{\omega\}$ and a G-connection with torsion T which satisfy

$$\nabla\omega=0.$$

The 1-form ω satisfies

$$2d\omega(X, Y)=\omega(T(X, Y))$$

for any X and Y.

Moreover, by the straightforward calculation, we have

$$d\omega(X, Y)=\rho\cdot(d\alpha)_0(u^{-1}X, u^{-1}Y).$$

This implies that $d\omega\in\{\Omega\}$ and hence ω satisfies

$$\omega\wedge(d\omega)^n\neq 0.$$

Hence $\{\omega\}$ defines a contact structure on M. (Q.E.D.)

If we replace (4′. 1) and (4′. 2) by

(4′. 1)′ $$L_X\alpha=0$$

and

(4′. 2)′ $$f^*\alpha=\alpha$$

respectively, then the resulting structures are called a *strict contact structure* and an *almost strict contact structure*.

§ 5. A concluding remark.

Let $\alpha, \beta, \cdots$ be tensor fields on R^n with *constant components*.

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^n which satisfy

$$L_X\alpha=0,$$

$$L_X\beta=0,$$

$$\cdots\cdots\cdots.$$

Let $\mathcal{L}(x_0)$ be the stalk of $\mathcal{L}$ at a point $x_0 \in \mathrm{R}^n$. Then $\mathcal{L}(x_0)$ is a filtered Lie algebra. Let $\mathfrak{g}$ be the linear isotropy algebra of $\mathcal{L}(x_0)$ and G a subgroup of $GL(n, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$.

Let Γ be the pseudogroup of local diffeomorphisms of R^n which preserve $\alpha, \beta, \cdots$. An almost Γ-structure on a manifold M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure on M.

The answer to the integrability problem for an almost Γ-structure is clearly the following

THEOREM 5. 1. *An almost* Γ-*structure whose structure tensor of the first order vanishes is a* Γ-*structure.*

Appendix. *f*-structures and framed *f*-structures.

Let $y^1, \cdots, y^k, y^{k+1}, \cdots, y^{2k}, y^{2k+1}, \cdots, y^n$ be the natural coordinate system of R^n and let

$$F=\sum_{i=1}^{k}\frac{\partial}{\partial y^i}\otimes dy^{i+k}-\sum_{i=1}^{k}\frac{\partial}{\partial y^{i+k}}\otimes dy^i.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^n which satisfy

$$L_X F=0.$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0 \in \mathrm{R}^n$. Then $\mathcal{L}(0)$ is a flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & C \end{pmatrix} \middle| \; A,\ B \in \mathfrak{gl}(k, \mathrm{R}),\ C \in \mathfrak{gl}(n-2k, \mathrm{R})\right\}$$

which is isomorphic with

$$\mathfrak{gl}(k, \mathrm{C})+\mathfrak{gl}(n-2k, \mathrm{R}).$$

Let G be a subgroup of $GL(n, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}$. Then G is isomorphic with $GL(k, \mathrm{C}) \times GL(n-2k, \mathrm{R})$. Let M be a differentiable manifold of dimension n. An *f-structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M. Giving a G-structure on M is the same as giving a tensor field f of type (1, 1) which satisfies

$$f^3+f=0$$

and

$$\operatorname{rank} f=2k.$$

Then the answer to the integrability problem for an f-structure is the following

THEOREM A. 1 ([1]). *An f-structure whose structure tensor of the first order vanishes is integrable.*

Let

$$\alpha_1 = dy^{2k+1}, \cdots, \alpha_{n-2k} = dy^n$$

and

$$F = \sum_{i=1}^{k} \frac{\partial}{\partial y^i} \otimes dy^{i+k} - \sum_{i=1}^{k} \frac{\partial}{\partial y^{i+k}} \otimes dy^i.$$

Let $\mathcal{L}$ be the sheaf of germs of all vector fields X on R^n which satisfy

$$L_X\alpha_1 = 0, \cdots, L_X\alpha_{n-2k} = 0$$

and

$$L_X F = 0.$$

Let $\mathcal{L}(0)$ be the stalk of $\mathcal{L}$ at the origin $0 \in R^n$. Then $\mathcal{L}(0)$ is a flat filtered Lie algebra of infinite dimensions. The linear isotropy algebra $\mathfrak{g}$ of $\mathcal{L}(0)$ is the linear Lie algebra

$$\left\{ \begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| A, B \in \mathfrak{gl}(k, R) \right\} \subset \mathfrak{gl}(n, R)$$

which is isomorphic with $\mathfrak{gl}(k, C)$.

Let G be a subgroup of $GL(n, R)$ whose Lie algebra is $\mathfrak{g}$. Then G is isomorphic with $GL(k, C)$. A *framed f-structure* on M is, by definition, a reduction of the bundle of linear frames $F(M)$ to G, that is, a G-structure $P_G(M)$ on M. Giving a G-structure on M is the same as giving $n-2k$ 1-forms $\omega_1, \cdots, \omega_{n-2k}$ and a tensor field f of type (1, 1) which satisfy

$$\omega_1 \circ f = 0, \cdots, \omega_{n-2k} \circ f = 0,$$

$$f^3 + f = 0$$

and

$$\text{rank } f = 2k.$$

Then the answer to the integrability problem for a framed f-structure is the following

THEOREM A. 2. *A framed f-structure whose structure tensor of the first order vanishes is integrable.*

Since $\mathfrak{g}$ contains no elements of rank 1, *the automorphism group of a framed f-structure on a compact manifold is a Lie group.*

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