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TOTALLY UMBILICAL SURFACES IN NORMAL CONTACT RIEMANNIAN MANIFOLDS

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Introduction. It is well known that any complete hypersurface of Euclidean space is isometric with a sphere if it is umbilical and has non-vanishing mean curvature. Recently, Okumura [4, 5] has studied totally umbilical surfaces in a Kaehlerian manifold, and in a locally product space. He proved that they are isometric with a sphere under some additional conditions. In this paper, we shall study surfaces in a normal contact manifold by devices similar to those used in [4] and [5].

We recall in § 1 the fundamental properties of normal contact manifolds. We obtain § 2 some formulas for surfaces in those manifolds. We mention in § 3 some properties of infinitesimal concircular transformations and state Obata's theorem for the later use. § 4 is mainly devoted to prove theorem 4.3, that is, the fact that in a normal contact manifold a complete and connected hypersurface is isometric with a sphere if it is totally umbilical and is of constant mean curvature. We prove in § 5 theorem 5.7, that is, the fact that in a normal contact manifold a complete and connected surface of codimension 2 is isometric with a sphere if it satisfies some special conditions. Finally, in § 6, non-existence of totally umbilical surfaces in a certain normal contact manifold will be proved.

§ 1. Normal contact Riemannian manifolds.

In a differentiable manifold of n dimensions, a set (ϕ, ξ, η) of three tensor fields ϕ, ξ and η of type (1, 1), (1, 0) and (0, 1) respectively is called an *almost contact structure* (cf. [8, 9]) if it satisfies the following conditions:[1]

$$(1.1)\qquad \xi^\lambda\eta_\lambda=1, \qquad \phi_\lambda{}^\kappa\xi^\lambda=0, \qquad \phi_\lambda{}^\kappa\eta_\kappa=0, \qquad \phi_\lambda{}^\kappa\phi_\nu{}^\lambda=-\delta_\nu^\kappa+\eta_\nu\xi^\kappa,$$

where $\phi_\lambda{}^\kappa$ is of rank $n-1$.

When a manifold admits an almost contact structure, it is called an *almost contact manifold* and is necessarily odd-dimensional. As is proved in [8], there exists in any almost contact manifold a Riemannian metric $G_{\lambda\kappa}$ such that

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1) For a tensor field, say, T of type (1, 2), we denote by $T_{\lambda\kappa}{}^\mu$ its components with respect to local coordinates $\{X^\kappa\}$ defined in each coordinate neighborhood of the manifold, where the indices κ, λ, μ, $\cdots$ run over the range $(1, 2, \cdots, n)$. The so-called Einstein's summation convention is used with respect to this system of indices.

(1. 2) $$G_{\lambda\kappa}\xi^{\lambda}=\eta_{\kappa}, \qquad G_{\lambda\kappa}\phi_{\mu}{}^{\lambda}\phi_{\nu}{}^{\kappa}=G_{\mu\nu}-\xi_{\mu}\xi_{\nu},$$

and such a Riemannian metric $G_{\lambda\kappa}$ is called a *Riemannian metric* associated with the given almost contact structure (ϕ, ξ, η). An almost contact manifold is called an *almost contact Riemannian manifold* when it is endowed with an associated Riemannian metric $G_{\lambda\kappa}$.

An almost contact Riemannian manifold is said to be *normal* if a certain tensor field constructed from the structure (ϕ, ξ, η, G) vanishes (cf. [9]). However, an almost contact Riemannian manifold is normal if and only if the conditions

(1. 3) $$\begin{aligned}&\nabla_{\lambda}\xi_{\kappa}=\phi_{\lambda\kappa},\\ &\nabla_{\nu}\phi_{\lambda\kappa}=\xi_{\lambda}G_{\kappa\nu}-\xi_{\kappa}G_{\lambda\nu}, \qquad \phi_{\lambda\kappa}=G_{\kappa\nu}\phi_{\lambda}{}^{\nu},\end{aligned}$$

are satisfied (cf. [1, 10]).

We know the following theorem, due to Okumura [6]:

THEOREM A. *Let M be a $(2n-1)$-dimensional Riemannian manifold with Riemannian metric $G_{\kappa\lambda}$. If M admits a Killing vector field V_{κ} of constant length satisfying*

$$C^{2}\nabla_{\lambda}\nabla_{\kappa}V_{\nu}=V_{\kappa}G_{\lambda\nu}-V_{\nu}G_{\lambda\kappa},$$

then the metric $G_{\kappa\lambda}$ of M is homothetic to the associated Riemannian metric of a normal contact Riemannian manifold.

§ 2. Surfaces in normal contact Riemannian manifolds.

We first of all study hypersurfaces in a normal contact Riemannian manifold $\bar{M}$. Let M be a hypersurface differentiably immersed in $\bar{M}$. We suppose that M is represented by equations

$$X^{\lambda}=X^{\lambda}(x^{i})$$

in each coordinate neighborhood $\bar{U}$ of $\bar{M}$, where $\{X^{\lambda}\}$ are local coordinates in $\bar{U}$, and $\{x^{i}\}$ are local coordinates in $M\cap\bar{U}$.[2)] If we put

$$B_{j}{}^{\lambda}=\partial_{j}X^{\lambda}$$

then, $B_{j}{}^{\lambda}$ define, for each fixed index j, a local vector field in $\bar{U}$ and $2n$ vector fields $B_{j}{}^{\lambda}$ span the tangent plane of M at each point of $\bar{U}$, where ∂_{j} denotes the operator defined $\partial_{j}=\partial/\partial x^{j}$. On putting

$$g_{ji}=G_{\lambda\kappa}B_{j}{}^{\lambda}B_{i}{}^{\kappa},$$

2) The indices h, i, j, k, run over the range $(1, 2, \cdots, 2n)$ and the so-called Einstein's summation convention is used with respect to this system of indices.

we see that g_{ji} define in M a Riemannian metric which is called the *induced metric*.

As is well known [2], a contact manifold is always orientable. We assume that the hypersurface M is also orientable and $2n$ tangent vectors $B_j{}^\lambda$ are chosen in such a way that $B_1{}^\lambda, \cdots, B_{2n}{}^\lambda$ form a frame of positive orientation in M. Then we can choose a globally defined field of unit normal vectors C^λ in such a way that $2n+1$ vectors $C^\lambda, B_1{}^\lambda, \cdots, B_{2n}{}^\lambda$ form a frame of positive orientation in $\bar{M}$. Then we find

$$G_{\lambda\kappa}B_i{}^\lambda C^\kappa=0, \qquad C^\lambda C_\lambda=1,$$
$$B^i{}_\lambda B_j{}^\lambda=\delta_j^i, \qquad B^i{}_\lambda B_i{}^\kappa=\delta_\lambda^\kappa-C_\lambda C^\kappa, \tag{2. 1}$$

where we have put $B^j{}_\kappa=G_{\lambda\kappa}B_i{}^\lambda g^{ji}$, $(g^{ji})=(g_{ji})^{-1}$, $C_k=G_{\lambda k}C^\lambda$. Therefore, we can put

$$\phi_\lambda{}^\kappa B_i{}^\lambda=f_i{}^jB_j{}^\kappa+f_iC^\kappa, \qquad \phi_\lambda{}^\kappa C^\lambda=-f^iB_i{}^\kappa, \tag{2. 2}$$

f^i being defined by $f^i=g^{ij}f_j$. Taking account of (2. 1), we find

$$f_i{}^j=B^j{}_\kappa\phi_\lambda{}^\kappa B_i{}^\lambda, \qquad f_i=C_\kappa\phi_\lambda{}^\kappa B_i{}^\lambda \tag{2. 3}$$

by virtue of (2. 2). Furthermore we introduce following tensors for the later use:

$$a=C_\kappa\xi^\kappa, \tag{2. 4}$$

$$u_i=B_i{}^\kappa\xi_\kappa. \tag{2. 5}$$

We then get equations

$$f_if^i=1-a^2, \qquad u_if^i=0, \qquad u^iu_i=1-a^2, \qquad f_{ji}u^j=af_i, \tag{2. 6}$$

because of (2. 1), (2. 4) and skew symmetry of $\phi_{\lambda\kappa}$.

We denote by $\{{}_\mu{}^\lambda{}_\nu\}$ the Christoffel symbols constructed from the given Riemannian metric $G_{\lambda\kappa}$ in $\bar{M}$ and by $\{{}_j{}^h{}_i\}$ those constructed from the metric g_{ji} induced in the hypersurface M. Denote by H_{ji} the second fundamental tensor of the hypersurface M and put $H^i{}_j=g^{ik}H_{kj}$. Then, the Gauss and the Weingarten equations for M are given respectively by

$$\nabla_jB_i{}^\lambda=H_{ji}C^\lambda, \qquad \nabla_jC^\lambda=-H^i{}_jB_i{}^\lambda, \tag{2. 7}$$

where we have put

$$\nabla_jB_i{}^\lambda=\partial_jB_i{}^\lambda+\{{}_\mu{}^\lambda{}_\nu\}B_j{}^\nu B_i{}^\mu-\{{}_j{}^h{}_i\}B_h{}^\lambda,$$

$$\nabla_jC^\lambda=\partial_jC^\lambda+\{{}_\mu{}^\lambda{}_\nu\}B_j{}^\nu C^\mu.$$

Differentiating covariantly the both sides of (2. 3), (2. 4) and (2. 5), and taking account of (1. 3), (2. 3), (2. 4), (2. 5) and (2. 7), we have

$$\nabla_j f_i = -a g_{ji} - H_{kj} f_i{}^k,$$
(2.8)
$$\nabla_j u_i = f_{ji} + a H_{ji},$$
$$\nabla_j a = f_j - u^i H_{ji},$$

by virtue of skew-symmetry of $\phi_{\lambda\kappa}$, where $\nabla_j f_i$, $\nabla_j u_i$, $\nabla_j a$ etc. denote the covariant derivatives of f_i, u_i, a etc., respectively, with respect to $\{_j{}^h{}_i\}$. In the sequel, we denote by ∇_j the covariant derivative with respect to $\{_j{}^h{}_i\}$ in M.

Next, we shall study surfaces of codimension 2 in a normal contact Riemannian manifold $\bar{M}$. Let M be a surface of codimension 2 which is differentiably immersed in $\bar{M}$. We suppose that M is represented by equation

$$X^\lambda = X^\lambda(x^a)$$

in each coordinate neighborhood U of $\bar{M}$, $\{X^\lambda\}$ being coordinates defined in U, and $\{x^a\}$ local coordinates defined in $M \cap U$.[3] If we put

$$B_c{}^\lambda = \partial_c X^\lambda,$$

then $B_c{}^\lambda$ define, for each fixed index c, a local vector field in U and $2n-1$ vector fields $B_c{}^\lambda$ span the tangent plane of M at each point of U, where ∂_c denotes the operator $\partial_c = \partial/\partial x^c$. On putting

$$g_{ab} = G_{\lambda\kappa} B_a{}^\lambda B_b{}^\kappa,$$

we see that g_{ab} define in M a Riemannian metric which is called the induced metric.

The contact manifold $\bar{M}$ being orientable, we assume that the surface M is also orientable and that $B_1{}^\lambda, \cdots, B_{2n-1}{}^\lambda$ are chosen in such a way that they form a frame of positive orientation. We then choose two local fields of mutually orthogonal unit vectors C^λ and D^λ in such a way that $C^\lambda, D^\lambda, B_1{}^\lambda, \cdots, B_{2n-1}{}^\lambda$ form a frame of positive orientation in $\bar{M}$. If $'C^\lambda$ and $'D^\lambda$ are another set of normals satisfying the same condition, then we have

(2.9)
$$'C^\lambda = \cos\theta\, C^\lambda - \sin\theta\, D^\lambda, \qquad 'D^\lambda = \sin\theta\, C^\lambda + \cos\theta\, D^\lambda.$$

Then we find

$$G_{\lambda\kappa} B_a{}^\lambda C^\kappa = G_{\lambda\kappa} B_a{}^\lambda D^\kappa = G_{\lambda\kappa} C^\lambda D^\kappa = 0,$$
(2.10)
$$G_{\lambda\kappa} C^\lambda C^\kappa = G_{\lambda\kappa} D^\lambda D^\kappa = 1,$$
$$B^a{}_\lambda B_b{}^\lambda = \delta_b^a,\ B^a{}_\lambda B_a{}^\kappa = \delta_\lambda^\kappa - C_\lambda C^\kappa - D_\lambda D^\kappa.$$

Therefore, we can put

3) The indices a, b, c, d run over the range $(1, 2, \cdots, 2n-1)$ and the so-called Einstein summation convention is also used with respect to this system of indices.

$$\phi_\lambda{}^\kappa B_a{}^\lambda = f_a{}^b B_b{}^\kappa + f_a C^\kappa + g_a D^\kappa,$$
(2.11)
$$\phi_\lambda{}^\kappa C^\lambda = -f^a B_a{}^\kappa + r D^\kappa, \qquad \phi_\lambda{}^\kappa D^\lambda = -g^a B_a{}^\kappa - r C^\kappa,$$

f^a and g^a being defined by $f^a = g^{ab} f_b$ and $g^a = g^{ab} g_b$ respectively. By virtue of (2.10), we find

$$f_a{}^b = B^b{}_\kappa \phi_\lambda{}^\kappa B_a{}^\lambda, \qquad f_{ab} = B_{b\kappa} \phi_\lambda{}^\kappa B_a{}^\lambda,$$
(2.12)
$$f_a = C_\kappa \phi_\lambda{}^\kappa B_a{}^\lambda, \qquad g_a = D_\kappa \phi_\lambda{}^\kappa B_a{}^\lambda, \qquad r = D_\kappa \phi_\lambda{}^\kappa C^\lambda,$$

as consequence of (2.11). Furthermore we introduce following tensors for the later use:

(2.13) $$u_a = B_a{}^\lambda \xi_\lambda,$$

(2.14) $$a = \xi^\lambda C_\lambda, \qquad b = \xi^\lambda D_\lambda.$$

We denote by $\{{}_b{}^a{}_c\}$ the Christoffel symbols constructed from the metric g_{ab} induced in the surface M. Denote by H_{ab} and K_{ab} the second fundamental tensors of the surface M and by L_a the third fundamental tensor of the surface M with respect to the normals C^λ and D^λ and put $H^a{}_b = g^{ac} H_{cb}$, $K^a{}_b = g^{ac} K_{cb}$. Then, the Gauss and the Weingarten equations for M are given respectively by

$$\nabla_a B_b{}^\kappa = H_{ab} C^\kappa + K_{ab} D^\kappa,$$
(2.15)
$$\nabla_a C^\kappa = -H^b{}_a B_b{}^\kappa + L_a D^\kappa, \qquad \nabla_a D^\kappa = -K^b{}_a B_b{}^\kappa - L_a C^\kappa,$$

where we have put

$$\nabla_a B_b{}^\lambda = \partial_a B_b{}^\lambda + \{{}_\mu{}^\lambda{}_\nu\} B_a{}^\nu B_b{}^\mu - \{{}_a{}^c{}_b\} B_c{}^\lambda,$$

$$\nabla_a C^\lambda = \partial_a C^\lambda + \{{}_\mu{}^\lambda{}_\nu\} B_a{}^\nu C^\mu, \qquad \nabla_a D^\lambda = \partial_a D^\lambda + \{{}_\mu{}^\lambda{}_\nu\} B_a{}^\nu D^\mu.$$

Differentiating covariantly the both sides of (2.12), (2.13) and (2.14), and taking account of (1.3), (2.12), (2.13), (2.14) and (2.15), we find

$$\nabla_a f_{bc} = H_{ac} f_b + K_{ac} g_b + g_{ac} u_b - (H_{ab} f_c + K_{ab} g_c + g_{ab} u_c)$$

$$\nabla_a f_b = H^c{}_a f_{cb} + L_a g_b - a g_{ab} - r K_{ab}, \qquad \nabla_a g_b = K^c{}_a f_{cb} - L_a f_b - b g_{ab} + r H_{ab},$$
(2.16)
$$\nabla_a r = K^b{}_a f_b - H^b{}_a g_b, \qquad \nabla_a u_b = a H_{ab} + b K_{ab} + f_{ab},$$

$$\nabla_a a = f_a - H^b{}_a u_b + b L_a, \qquad \nabla_a b = g_a - K^b{}_a u_b - a L_a,$$

by virtue of skew symmetry of $\phi_{\lambda\kappa}$, where $\nabla_a f_{bc}$, $\nabla_a f_b$, $\cdots$ etc. denote the covariant derivatives of $f_{bc}, f_b, \cdots$ etc., respectively, with respect to $\{{}_b{}^a{}_c\}$. In the sequel, we denote by ∇_a the covariant derivative with respect to $\{{}_b{}^a{}_c\}$ in M.

Then, we have the following

LEMMA 2.1. *The scalar function r defined by* (2.12) *is determined inde-*

pendently of the choice of mutually orthogonal unit normal vectors C^λ and D^λ to the surface M, consequently, r is a globally defined function in M.

Proof. Let $'C^\lambda$ and $'D^\lambda$ be mutually orthogonal unit normal vectors to the manifold M at a point p. Then we find that between a pair of unit normal vectors (C^λ, D^λ) and $('C^\lambda, 'D^\lambda)$ chosen above at each point of M, the relations (2. 9) hold. So we find

$$r' = 'D_\kappa \phi_\lambda{}^\kappa \, 'C^\lambda = (\sin\theta\, C_\kappa + \cos\theta\, D_\kappa)\phi_\lambda{}^\kappa(\cos\theta\, C^\lambda - \sin\theta\, D^\lambda)$$

$$= -\sin^2\theta\, C_\kappa \phi_\lambda{}^\kappa D^\lambda + \cos^2\theta\, D_\kappa \phi_\lambda{}^\kappa C^\lambda = (\sin^2\theta + \cos^2\theta) D_\kappa \phi_\lambda{}^\kappa C^\lambda = r,$$

which shows that r is independent of the choice of unit normal vectors C^λ and D^λ and that r is a globally defined function. Q.E.D.

§ 3. Infinitesimal concircular transformations.

An infinitesimal conformal transformation is by definition an infinitesimal transformation u^κ satisfying the equation

$$\nabla_\lambda u_\kappa + \nabla_\kappa u_\lambda = 2\phi G_{\lambda\kappa}, \tag{3. 1}$$

where ϕ is a certain function. When an infinitesimal conformal transformation u^κ is a gradient vector field, i.e. $u_\kappa = \nabla_\kappa v$, where v is a certain function in M and $u_\kappa = G_{\lambda\kappa} u^\lambda$, the equation (3. 1) reduces to

$$\nabla_\lambda \nabla_\kappa v + \nabla_\kappa \nabla_\lambda v - 2\phi G_{\lambda\kappa}, \tag{3. 2}$$

which is equivalent to

$$\nabla_\lambda \nabla_\kappa v = \phi G_{\lambda\kappa}, \tag{3. 3}$$

because of $\nabla_\lambda \nabla_\kappa v = \nabla_\kappa \nabla_\lambda v$. When a function v satisfies (3. 3), that is, when the gradient vector field of a function v is an infinitesimal conformal transformation, the transformation $u^\kappa = G^{\kappa\lambda} \nabla_\lambda v$ is called an infinitesimal concircular transformation [12]. If the function appearing in (3. 3) is of the form $\phi = -kv$ with positive constant coefficient k, the infinitesimal concircular transformation u^κ is called an infinitesimal special concircular transformation [11]. As to a Riemannian manifold admitting an infinitesimal special concircular transformation, we know the following Obata's theorem.

THEOREM B. [3] [11] *Let M be a complete connected Riemannian manifold of dimension n ($\geqq 2$). In order that M admits a non-trivial solution of the system of differential equations*

$$\nabla_\lambda \nabla_\kappa \phi + k\phi G_{\lambda\kappa} = 0, \qquad k > 0,$$

it is necessary and sufficient that M is isometric with a sphere S^n of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space.

§ 4. Totally umbilical hypersurfaces in normal contact Riemannian manifolds.

When, at each point of the hypersurface M, the second fundamental tensor is proportional to the first fundamental tensor of M, that is, when the relation

$$H_{ji}=Hg_{ji} \tag{4.1}$$

is always valid at each point, the hypersurface is called a totally umbilical hypersurface, the proportional factor H being the mean curvature of the hypersurface.

First of all, we shall prove

LEMMA 4.1. *If M is a totally umbilical hypersurface with constant mean curvature in a normal contact Riemannian manifold $\bar{M}$, then the scalar function a defined by* (2.4) *is not constant in M.*

Proof. Suppose that the function a is constant in M. Then we have

$$f_j=Hu_j, \tag{4.2}$$

because of the equation

$$\nabla_j a=f_j-Hu_j \tag{4.3}$$

which is a direct consequence of the last equation in (2.8) and (4.1). Making use of (2.6), we have from (4.2)

$$f_jf^j=1-a^2=Hu_jf^j=0,$$

which implies

$$f_j=0, \tag{4.4}$$

$$1-a^2=0. \tag{4.5}$$

Substituting (4.4) and (4.1) into the first equation in (2.8), we get

$$ag_{ji}+Hf_{ji}=0, \tag{4.6}$$

from which, transvecting g^{ji}, we get easily $a=0$ because of skew symmetry of f_{ji}. However this is contradictory to (4.5). Thus a can not be constant in M. Q.E.D.

Next, we shall prove

THEOREM 4.2. *If M is a totally umbilical hypersurface with constant mean curvature in a normal contact Riemannian manifold $\bar{M}$, the gradient vector field of the function a defined by* (2.4) *is in M an infinitesimal special concircular transformation.*

Proof. Differentiating (4.3) covariantly, and taking account of (2.8) and (4.1) we have

(4. 7) $$\nabla_k \nabla_j a + (1+H^2) a g_{kj} = 0,$$

where $1+H^2$ is a positive constant.

The function a is non-trivial because of Lemma 4. 1. Thus the gradient vector field of the function a is an infinitesimal special concircular transformation. Q.E.D.

Combining Theorem B and Theorem 4. 2, we have

THEOREM 4. 3. *Let M be a complete connected totally umbilical hypersurface in a normal contact Riemannian manifold $\bar{M}$. If M is of constant mean curvature H, M is isometric with a sphere of radius $1/\sqrt{1+H^2}$ in the Euclidean space.*

§ 5. Totally umbilical surfaces of codimension 2 in normal contact Riemannian manifolds.

When, at each point of the surface M of codimension 2, the second fundamental tensors are proportional to the first fundamental tensor of M, that is, when the relations

(5. 1) $$H_{ab} = H g_{ab}, \qquad K_{ab} = K g_{ab}$$

are always valid at each point, the surface is called a totally umbilical surface, H and K being given by $(1/(2n-1)) g^{ab} H_{ab}$ and $(1/(2n-1)) g^{ab} K_{ab}$ respectively. The mean curvature vector field H^λ of M in $\bar{M}$ is given by

(5. 2) $$H^\lambda = H C^\lambda + K D^\lambda,$$

which is independent of the choice of mutually orthogonal unit normal vectors C^λ and D^λ to the surface M. Differentiating (5. 2) covariantly and making use of (2. 15), we have

$$\nabla_a H^\lambda = -(H^2+K^2) B_a{}^\lambda + (\nabla_a H - K L_a) C^\lambda + (\nabla_a K + H L_a) D^\lambda,$$

which implies the following

LEMMA 5. 1. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional Riemannian manifold $\bar{M}$. In order that the covariant derivative $\nabla_a H^\lambda$ of the mean curvature vector field H^λ of M is tangent to M, it is necessary and sufficient that*

(5. 3) $$\nabla_a H = K L_a, \qquad \nabla_a K = -H L_a.$$

Next, we shall prove

THEOREM 5. 2. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional Riemannian manifold $\bar{M}$. If the covariant derivative $\nabla_a H^\lambda$ of the mean curvature vector field H^λ of M is tangent to M, then M is of constant* *mean curvature.*

Proof. The mean curvature h of M is given by

$$h^2=G_{\lambda\kappa}H^\lambda H^\kappa.$$

Substituting (5. 2) in the equation above, we have

$$h^2=G_{\lambda\kappa}(HC^\lambda+KD^\lambda)(HC^\kappa+KD^\kappa)=H^2+K^2. \tag{5. 4}$$

Differentiating (5. 4) covariantly and making use of Lemma 5. 1, we have

$$\nabla_a h^2=2(H\nabla_a H+K\nabla_a K)=2(HKL_a-KHL_a)=0,$$

which shows that the mean curvature of M is constant. Q.E.D.

As a consequence of equations (2. 16), we have

THEOREM 5. 3. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional normal contact Riemannian manifold $\bar{M}$. Suppose that the covariant derivative of the mean curvature vector of M is tangent to M, and that the scalar function r defined by* (2. 12) *is non-constant. Then the gradient of the scalar function r is an infinitesimal special concircular transformation.*

Proof. Differentiating the fourth equation in (2. 16) covariantly under the conditions (5. 1), and taking account of (2. 16) and (5. 3), we get

$$\nabla_a\nabla_b r=\{bH-aK-(K^2+H^2)r\}g_{ab}. \tag{5. 5}$$

On the other hand, from (2. 16), (5. 1) and (5. 3), we obtain

$$\nabla_a(bH-aK)=Hg_a-Kf_a=-\nabla_a r,$$

which implies

$$r=-(bH-aK)+C_0,$$

C_0 being a constant. Substituting this equation into (5. 5), we get

$$\nabla_a\nabla_b r=\{-(1+H^2+K^2)r+C_0\}g_{ab}. \tag{5. 6}$$

Furthermore putting

$$c=\frac{C_0}{1+H^2+K^2},$$

which is constant, we see that (5. 6) reduces to

$$\nabla_a\nabla_b r=-\{(1+H^2+K^2)(r-c)\}g_{ab},$$

or equivalently to

$$\nabla_a \nabla_b (r-c) = -\{(1+H^2+K^2)(r-c)\} g_{ab}.$$

This equation shows that $\nabla_a r = \nabla_a (r-c)$ is an infinitesimal special concircular transformation, since $1+H^2+K^2$ is positive and constant. Q.E.D.

Taking account of Theorem 5. 3 and Theorem B, we have

LEMMA 5. 4. *Let M be a $(2n-1)$-dimensional complete connected totally umbilical surface in a $(2n+1)$-dimensional normal contact Riemannian manifold $\bar{M}$. Suppose that the covariant derivative of the mean curvature vector field of M is tangent to M, and that the scalar function r defined by* (2. 12) *is non-constant. Then M is isometric with a sphere of radius $1/\sqrt{1+H^2+K^2}$ in the Euclidean space, where H^2+K^2 is the mean curvature of M.*

Let us next consider the case in which the function r is constant. In this case, taking account of (2. 16) and (5. 1) we get

$$Kf_a - Hg_a = 0. \tag{5. 7}$$

Substituting (2. 12) into (5. 7), we obtain

$$KC_\kappa \phi_\lambda{}^\kappa B_a{}^\lambda - HD_\kappa \phi_\lambda{}^\kappa B_a{}^\lambda = (KC_\kappa - HD_\kappa)\phi_\lambda{}^\kappa B_a{}^\lambda = 0,$$

equivalently

$$\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda) B_a{}^\kappa = 0. \tag{5. 8}$$

On the other hand, by means of skew symmetry of $\phi_{\lambda\kappa}$, we have

$$\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda)(KC^\kappa - HD^\kappa) = 0. \tag{5. 9}$$

Now, we have the identity

$$H_\lambda (KC^\lambda - HD^\lambda) = (HC_\lambda + KD_\lambda)(KC^\lambda - HD^\lambda) = 0. \tag{5. 10}$$

These two equations (5. 8) and (5. 9) show that $\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda)$ is orthogonal to both of $B_a{}^\kappa$ and $(KC^\kappa - HD^\kappa)$. As a direct consequence of (5. 10) we see that $H^\kappa = HC^\kappa + KD^\kappa$ is orthogonal to both of $(KC^\kappa - HD^\kappa)$ and $B_a{}^\kappa$. Therefore $\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda)$ is proportional to H_κ and hence we can put

$$\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda) = \rho H_\kappa, \tag{5. 11}$$

with a certain function ρ. We shall prove

LEMMA. *The function ρ given by* (5. 11) *is non-zero constant if the mean curvature $h^2 = H^2 + K^2$ does not vanish.*

Proof. If $\rho = 0$, (5. 11) reduces to $\phi_{\lambda\kappa}(KC^\lambda - HD^\lambda) = 0$, which implies

$$KC^\lambda - HD^\lambda = \sigma \xi^\lambda, \tag{5. 12}$$

becaus of (1. 1). Differentiating (5. 12) covariantly we have

$$0=(\nabla_a\sigma)\xi^\lambda+\sigma B_a{}^\kappa\phi_\kappa{}^\lambda.$$

Transvecting the equation above with ξ_λ and taking account of (1. 1) we get $\nabla_a\sigma=0$, which implies

$$\sigma\phi_\kappa{}^\lambda B_a{}^\kappa=0. \tag{5.13}$$

Now, $\phi_\kappa{}^\lambda B_a{}^\kappa$ does not vanish because $\phi_\kappa{}^\lambda$ is of rank $2n$. Suppose that σ is zero, then we have $H=K=0$ because of (5. 12). $H=K=0$ contradicts the assumption. Therefore σ is not zero. This result that $\phi_\kappa{}^\lambda B_a{}^\kappa$ and σ are both not zero contradicts (5. 13). So, ρ is non-zero. Next, if we differentiate the both sides of (5. 11) covariantly, we have

$$\nabla_a(\phi_{\lambda\kappa}(KC^\lambda-HD^\lambda))=(aK-bH)G_{\kappa\nu}B_a{}^\nu, \tag{5.14}$$

$$\nabla_a(\rho H_\kappa)=(\nabla_a\rho)H_\kappa-\rho(H^2+K^2)B_{a\kappa}, \tag{5.15}$$

because of (1. 3), (2. 15), (5. 1) and (5. 3). Comparing (5. 14) with (5. 13), we have $\nabla_a\rho=0$, which implies that ρ is constant. Thus the proof is completed.

Transvecting (5. 11) with ξ^κ, we get

$$H_\kappa\xi^\kappa=0$$

because the left hand side becomes zero by (1. 1) and ρ is non-zero constant. Substituting (5. 2) in the equation above, we have

$$(HC_\kappa+KD_\kappa)\xi^\kappa=Ha+Kb=0. \tag{5.16}$$

On the other hand, since r is constant, (5. 5) implies

$$bH-aK=(H^2+K^2)r. \tag{5.17}$$

Summing up, we get relations (5. 16) and (5. 17), when the function r is constant and the mean curvature h does not vanish.

We now prove

LEMMA 5. 5. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional normal contact Riemannian manifold $\bar{M}$. Suppose that the covariant derivative of the mean curvature vector field H^λ of M is tangent to M. If the mean curvature h of M does not vanish and the function r defined by* (2. 12) *is constant, then the vector field u_a defined by* (2. 13) *is a Killing vector field of constant length and satisfies the following equation*:

$$C^2\nabla_a\nabla_b u_c=u_b g_{ac}-u_c g_{ab}. \tag{5.18}$$

Proof. First of all, we shall prove that the vector u^a has constant length. Making use of (2.13) and (2. 14), we have

(5. 19) $$u^a u_a = 1-(a^2+b^2).$$

On the other hand, from (5. 16) and (1. 17) we get

$$a=-Kr, \qquad b=Hr,$$

if $h^2=H^2+K^2\neq 0$. Accordingly we have $a^2+b^2=(H^2+K^2)r^2$. Therefore, the length of u^a which is given by (5. 19) is constant, because r and H^2+K^2 are constant by virtue of Theorem 5. 2.

Substituting (5. 1) and (5. 16) into the fifth equation of (2. 16), we have

$$\nabla_a u_b + \nabla_b u_a = 0$$

by virtue of skew symmetry of f_{ab}. This equation shows that u^a is a Killing vector field.

Finally, differentiating the fifth equation of (2. 16) covariantly and taking account of (5. 1), (5. 3) and (5. 16), we have

(5. 20) $$\nabla_a \nabla_b u_c = (Hf_b + Kg_b)g_{ac} - (Hf_c + Kg_c)g_{ab} + u_b g_{ac} - u_c g_{ab}.$$

On the other hand, by differentiating (5. 16) covariantly we have

$$Hf_c + Kg_c - (H^2+K^2)u_c = 0$$

by virtue of (2. 16) and (5. 3).

Substituting the equation above into (5. 20), we get

$$\nabla_a \nabla_b u_c = (1+H^2+K^2)u_b g_{ac} - (1+H^2+K^2)u_c g_{ab},$$

that is,

$$\frac{1}{1+H^2+K^2}\nabla_a \nabla_b u_c = u_b g_{ac} - u_c g_{ab}.$$

Therefore u_a satisfies the equation (5. 18). Thus the proof is completed.

Taking account of Lemma 5. 5 and Theorem A, we have

LEMMA 5. 6. *Let M be a $(2n-1)$-dimensional totally umbilical surface in a $(2n+1)$-dimensional normal contact Riemannian manifold $\bar{M}$. Suppose that the covariant derivative of the mean curvature vector H^λ of M is tangent to M. If the mean curvature h of M does not vanish and the function r defined by (2. 12) is constant, then the induced Riemannian metric g_{ab} of M is homothetic to the associated Riemannian metric of the normal contact Riemannian manifold.*

Combining Lemma 5. 4 and Lemma 5. 6, we have

THEOREM 5. 7. *Let M be a $(2n-1)$-dimensional complete connected totally umbilical surface in a $(2n+1)$-dimensional normal contact Riemannian manifold $\bar{M}$. Suppose that the covariant derivative of the mean curvature vector field of M is* *tangent to M, and that the mean curvature* $h^2=H^2+K^2$ *of M does not vanish. Then either of the following two cases occurs*:

(1) *M is isometric with a sphere of a radius* $1/\sqrt{1+H^2+K^2}$ *in the Euclidean* 2*n-space.*

(2) *M is homothetic to a* (2*n*−1)*-dimensional normal contact Riemannian manifold.*

§ 6. Totally umbilical hypersurfaces in certain normal contact Riemannian manifolds.

Let $\bar{M}$ be a $(2n+1)$-dimensional normal contact Riemannian manifold whose Ricci tensor has a special form

$$R_{\lambda\mu}=\alpha G_{\lambda\mu}+\beta\xi_\lambda\xi_\mu, \tag{6.1}$$

where α and β are necessarily constant (cf. [7]).

Let M be a totally umbilical hypersurface in $\bar{M}$.

The Codazzi equation of the hypersurface M is given by

$$\nabla_j H_{ih}-\nabla_i H_{jh}=B_j{}^\nu B_i{}^\mu B_h{}^\lambda C^\kappa R_{\nu\mu\lambda\kappa}. \tag{6.2}$$

By transvecting g^{ih} to (6. 2), we have

$$\nabla_j H^h{}_h-\nabla^h H_{jh}=B_j{}^\nu(G^{\mu\lambda}-C^\mu C^\lambda)C^\kappa R_{\nu\mu\lambda\kappa}=B_j{}^\nu C^\kappa R_{\nu\kappa}.$$

Substituting (6. 1) into the equation above, we have

$$(n-1)\nabla_j H=B_j{}^\nu C^\kappa(\alpha G_{\nu\kappa}+\beta\xi_\nu\xi_\kappa)=\beta a u_j, \tag{6.3}$$

since the hypersurface M is totally umbilical.

Differentiating (6. 3) covariantly, we have

$$(n-1)\nabla_i\nabla_j H=\beta\{(f_i-u_iH)u_j+a(f_{ij}+aHg_{ij})\}. \tag{6.4}$$

Because the left hand side of (6. 4) is symmetric with respect to i and j, we get

$$\beta(f_ju_i-f_iu_j+2af_{ji})=0. \tag{6.5}$$

Transvecting u^j to (6. 5), we have

$$\beta(3a^2-1)f_i=0, \tag{6.6}$$

by virtue of (2. 6). Transvecting f^i to (6. 6), we have

$$\beta(3a^2-1)(1-a^2)=0, \tag{6.7}$$

which implies

$$a^2=\frac{1}{3} \qquad \text{or} \qquad a^2=1, \tag{6.8}$$

if $\beta \neq 0$. We assume now that the constant β is not zero. Then the function a^2 is necessarily equal to a constant, i.e. equal to 1/3 or 1, since a is continuous. When $a^2=1/3$, we have easily $H=0$ because of the last equation of (2. 8) and (2. 6). When $a^2=1$, we have by virtue of the third equation of (2. 6) $u_j=0$, which means together with (6. 3) that H is constant. However, the fact that a and H are both constant contradicts Lemma 4. 1. Therefore we can conclude that β is zero. Thus we get

THEOREM 6. 1. *If, in a normal contact Riemannian manifold whose Ricci tensor has the form of* (6. 1), *there exists a totally umbilical hypersurface, the space is necessarily an Einstein space.*

COROLLARY 6. 2. *Let* $\bar{M}$ *be a normal contact Riemannian manifold whose Ricci tensor has the form of* (6. 1). *If* $\bar{M}$ *is not an Einstein space, then there is no totally umbilical hypersurface.*

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