KŌDAI MATH. SEM. REP.
19 (1967), 317—360

FIBRED SPACES WITH INVARIANT RIEMANNIAN METRIC

BY KENTARO YANO AND SHIGERU ISHIHARA

Introduction. In a previous paper [11], we developed the differential geometry of fibred spaces mainly from the point of view of affine geometry, that is, we studied the fibred spaces with invariant affine connection. We also studied the fibred spaces which can represent the manifolds with projective connection.

In the present paper, we would like to study the fibred spaces with invariant Riemannian metric. Since the Riemannian connection determined by the invariant Riemannian metric is also invariant, fibred spaces with invariant Riemannian metric are fibred spaces with invariant affine connection in the sense of our previous paper [11].

In §1, we define fibred spaces with invariant Riemannian metric and recall some of fundamental concepts in fibred spaces with invariant affine connection. We study also induced metric, induced connection, second fundamental tensor, co-Gauss equations, co-Weingarten equations, and Nijenhuis tensor of the second fundamental tensor.

We develop in §2 the tensor calculus in terms of local coordinates in a fibred space with invariant Riemannian metric and we study in §3 some important formulas useful for discussions which follow. §4 is devoted to the study of geodesics in the total space and in the base space.

We study in §5 structure equations and curvatures. Starting from co-Gauss, co-Codazzi and co-Ricci equations, we derive relations between the curvature of the total space and that of the base space and prove propositions in which a Kähler or an almost Kähler structure appears. In §6 we study some of interesting special cases.

In the last §7, we study the case in which a fibred space with invariant Riemannian metric is a fibred space with K-contact structure.

CONTENTS

§1. Fibred spaces with invariant Riemannian metric.
§2. The tensor calculus in fibred spaces with invariant Riemannian metric.
§3. Formulas.
§4. Geodesics.

Received February 23, 1967.

1) The number in brackets refers to the References at the end of the paper.

§ 5. Structure equations and curvatures.
§ 6. Special cases.
§ 7. The contact structures.

§ 1. Fibred spaces with invariant Riemannian metric.

Let $\tilde{M}$ and M be two differentiable manifolds[1], of $n+1$ dimensions and of n dimensions respectively, and assume that there exists a differentiable mapping $\pi: \tilde{M} \to M$, which is onto and of the maximum rank n. Then, for each point P of M, the inverse image $\pi^{-1}(\mathrm{P})$ of P is a 1-dimensional submanifold of $\tilde{M}$. We denote $\pi^{-1}(\mathrm{P})$ by F_{P} and call F_{P} the *fibre* over the point P of M. We suppose *every fibre* F_{P} *to be connected.* We assume moreover that there are given in $\tilde{M}$ a vector field $\tilde{C}$ which is non-zero and tangent to the fibre everywhere, and a Riemannian metric $\tilde{g}$ which is positive definite and satisfies the conditions

$$\mathcal{L}\tilde{g}=0, \qquad \tilde{g}(\tilde{C}, \tilde{C})=1, \tag{1.1}$$

$\mathcal{L}$ denoting the operator of Lie derivation with respect to the vector field $\tilde{C}$. The set $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ is called a *fibred space with invariant Riemannian metric*. The manifolds $\tilde{M}$ and M are called the *total space* and the *base space* respectively. The vector field $\tilde{C}$ and the Riemannian metric $\tilde{g}$ are called the *structure field* and the *invariant Riemannian metric*, or simply, the *invariant metric*, respectively.

If, in the total space $\tilde{M}$, we introduce a 1-form $\tilde{\eta}$ by the equation

$$\tilde{\eta}(\tilde{X})=\tilde{g}(\tilde{X}, \tilde{C}), \tag{1.2}$$

$\tilde{X}$ being an arbitrary vector field in $\tilde{M}$, then we easily obtain

$$\tilde{\eta}(\tilde{C})=1, \qquad \mathcal{L}\tilde{\eta}=0 \tag{1.3}$$

as direct consequences of (1.1) and (1.2). Thus we have a fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{\eta})$ in the sense of [11]. The distribution defined by the equation $\tilde{\eta}=0$ is called the *field of horizontal planes* in $\tilde{M}$ and the value of this distribution at a point of $\tilde{M}$ is called the *horizontal plane* at that point.

The Riemannian connection $\tilde{\nabla}$ determined by the invariant Riemannian metric $\tilde{g}$ is also invariant with respect to the infinitesimal transformation determined by the structure field $\tilde{C}$. Thus we have a fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{\eta})$ with invariant affine connection $\tilde{\nabla}$ in the sense of [11]. Therefore, to any fibred space with invariant Riemannian metric $\tilde{g}$, there corresponds naturally a fibred space with invariant affine connection $\tilde{\nabla}$. For a fibred space with invariant Riemannian metric, we use the same notations and terminologies as those introduced in [11] for fibred space with invariant affine connection.

We recall some of notations and terminologies introduced in [11] for fibred spaces $(\tilde{M}, M, \pi; \tilde{C}, \tilde{\eta})$.

1) The manifolds, the objects and the mappings we discuss are supposed to be of differentiability class C^∞. The manifolds are assumed to be connected.

1. $T(\tilde{M})$ is the tangent bundle of $\tilde{M}$.

2. $\mathcal{T}^r_s(\tilde{M})$ is the space of all tensor fields of type (r, s), i.e., of contravariant degree r and of covariant degree s, in $\tilde{M}$.

3. $$\mathcal{T}(\tilde{M})=\sum_{r,s}\mathcal{T}^r_s(\tilde{M}).$$

The notations $T(M)$, $\mathcal{T}^r_s(M)$ and $\mathcal{T}(M)$ denote respective the spaces with respect to M corresponding to $T(\tilde{M})$, $\mathcal{T}^r_s(\tilde{M})$ and $\mathcal{T}(\tilde{M})$ respectively.

A linear endomorphism $\tilde{T}\rightarrow\tilde{T}^H$ of $\mathcal{T}(\tilde{M})$ is defined by the following properties:

(H. 1) $$\tilde{f}^H=\tilde{f} \qquad \text{for } \tilde{f}\in\mathcal{T}^0_0(\tilde{M}).$$

(H. 2) $$\begin{aligned}\tilde{X}^H&=\tilde{X}-\tilde{\eta}(\tilde{X})\tilde{C}\\ &=\tilde{X}-\tilde{g}(\tilde{X}, \tilde{C})\tilde{C}\end{aligned} \qquad \text{for } \tilde{X}\in\mathcal{T}^1_0(\tilde{M}).$$

(H. 3) $$\tilde{\omega}^H=\tilde{\omega}-\tilde{\omega}(\tilde{C})\tilde{\eta} \qquad \text{for } \tilde{\omega}\in\mathcal{T}^0_1(\tilde{M}).$$

(H. 4) $$(\tilde{S}\otimes\tilde{T})^H=(\tilde{S}^H)\otimes(\tilde{T}^H) \qquad \text{for } \tilde{S}, \tilde{T}\in\mathcal{T}(\tilde{M}).$$

The tensor field $\tilde{T}^H$ is called the *horizontal part* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{T}(\tilde{M})$. If a tensor field $\tilde{T}$ in $\tilde{M}$ satisfies the condition $\tilde{T}=\tilde{T}^H$, then we call $\tilde{T}$ a *horizontal tensor field* in $\tilde{M}$. On putting

(1. 4) $$\tilde{T}^V=\tilde{T}-\tilde{T}^H,$$

we call $\tilde{T}^V$ the *non-horizontal part* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{T}(\tilde{M})$. Especially, we have

$$\tilde{X}^V=\tilde{g}(\tilde{C}, \tilde{X})\tilde{C} \qquad \text{for } \tilde{X}\in\mathcal{T}^1_0(\tilde{M}),$$

$$\tilde{\omega}^V=\tilde{\omega}(\tilde{C})\tilde{\eta} \qquad \text{for } \tilde{\omega}\in\mathcal{T}^0_1(\tilde{M}).$$

We call $\tilde{X}^V$ and $\tilde{\omega}^V$ the *vertical parts* of $\tilde{X}$ and $\tilde{\omega}$ respectively for any element $\tilde{X}$ of $\mathcal{T}^1_0(\tilde{M})$ and any element $\tilde{\omega}$ of $\mathcal{T}^0_1(\tilde{M})$.

We now introduce the following notations:

4. $\mathcal{T}^H(\tilde{M})$ is the space of all horizontal tensor fields in $\tilde{M}$. $\mathcal{T}^H(\tilde{M})\subset\mathcal{T}(\tilde{M})$.

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be *invariant* if it satisfies the condition $\mathcal{L}\tilde{T}=0$, $\mathcal{L}$ denoting the operation of Lie derivation with respect to the structure field $\tilde{C}$. We also introduce the following notations:

5. $\mathcal{J}(\tilde{M})$ is the space of all invariant tensor fields in $\tilde{M}$.

$$\mathcal{J}(\tilde{M})\subset\mathcal{T}(\tilde{M}), \qquad \mathcal{J}^r_s(\tilde{M})=\mathcal{J}(\tilde{M})\cap\mathcal{T}^r_s(\tilde{M}).$$

6. $\mathcal{J}^H(\tilde{M})=\mathcal{J}(\tilde{M})\cap\mathcal{T}^H(\tilde{M})$, $\qquad \mathcal{J}^{Hr}_s(\tilde{M})=\mathcal{J}^H(\tilde{M})\cap\mathcal{T}^r_s(\tilde{M})$.

7. $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$ denotes the *formal tensor product*, i.e. the tensor product of the two spaces $\mathcal{J}(\tilde{M})$ and $\mathcal{J}^H(\tilde{M})$ regarded as two abstract tensor spaces over $\tilde{M}$.

We shall now recall the operations of taking lifts and projections, which were introduced in [11]. The operation of taking lifts is a linear homomorphism $T \rightarrow T^L$ of $\mathcal{T}(M)$ into $\mathcal{T}(\tilde{M})$ characterized by the following properties:

(L. 1) $$f^L = f \circ \pi \qquad \text{for} \quad f \in \mathcal{T}_0^0(M),$$

(L. 2) for any element X of $\mathcal{T}_0^1(M)$, there exists a unique element X^L of $\mathcal{T}^H{}_0^1(\tilde{M})$ such that

$$\pi X^L = X,$$

(L. 3) $$\omega^L = {}^*\pi\omega \qquad \text{for} \quad \omega \in \mathcal{T}_1^0(M),$$

(L. 4) $$(S \otimes T)^L = (S^L) \otimes (T^L) \qquad \text{for} \quad S,\ T \in \mathcal{T}(M),$$

where the differential mapping of the projection $\pi: \tilde{M} - M$ is denoted also by π and the dual mapping of the differential mapping π is denoted by $^*\pi$. The element T^L of $\mathcal{J}^H(\tilde{M})$ is called the *lift* of T for any element T of $\mathcal{T}(M)$.

The operation of taking projection is a linear homomorphism $p: \mathcal{J}(\tilde{M}) \rightarrow \mathcal{T}(M)$, which is onto and is characterized by the following properties:

(P. 1) $$(p\tilde{f})(\mathrm{P}) = \tilde{f}(\tilde{\mathrm{P}}) \qquad \text{for} \quad \tilde{f} \in \mathcal{J}_0^0(\tilde{M}),$$

where $\tilde{\mathrm{P}}$ is an arbitrary point such that $\pi(\tilde{\mathrm{P}}) = \mathrm{P}$, P being an arbitrary point of M.

(P. 2) $$(p\tilde{X})f = p(\tilde{X}(f^L)) \qquad \text{for} \quad \tilde{X} \in \mathcal{J}_0^1(\tilde{M}),$$

f being an arbitrary element of $\mathcal{T}_0^0(M)$.

(P. 3) $$(p\tilde{\omega})(X) = p(\tilde{\omega}(X^L)) \qquad \text{for} \quad \tilde{\omega} \in \mathcal{J}_1^0(\tilde{M}),$$

X being an arbitrary element of $\mathcal{T}_0^1(M)$.

(P. 4) $$p(\tilde{S} \otimes \tilde{T}) = (p\tilde{S}) \otimes (p\tilde{T}) \quad \text{for} \quad \tilde{S},\ \tilde{T} \in \mathcal{J}(\tilde{M}).$$

We call the element $p\tilde{T}$ of $\mathcal{T}(M)$ the *projection* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{J}(\tilde{M})$.

Taking account of (L. 1)~(L. 4) and (P. 1)~(P. 4), we easily find

$$p(T^L) = T \qquad \text{for} \quad T \in \mathcal{T}(M)$$

and

$$(p\tilde{T})^L = \tilde{T}^H \qquad \text{for} \quad \tilde{T} \in \mathcal{J}(\tilde{M}).$$

Thus the two spaces $\mathcal{J}^H(\tilde{M})$ and $\mathcal{T}(M)$ are isomorphic to each other and $p: \mathcal{J}^H(\tilde{M}) \rightarrow \mathcal{T}(M)$ is the isomorphism between them. Then the operation of taking lifts is the inverse of the projection p restricted to $\mathcal{J}^H(\tilde{M})$.

We have now

PROPOSITION 1.1. *For any elements $\tilde{X}, \tilde{Y}$ of $\mathcal{J}_0^1(\tilde{M})$, we have*

$$[\tilde{X}, \tilde{Y}] \in \mathcal{J}_0^1(\tilde{M}), \qquad [\tilde{C}, \tilde{X}]=0,$$

$$(1.5) \qquad [\tilde{X}, \tilde{Y}]^H=[\tilde{X}^H, \tilde{Y}^H]^H, \qquad [\tilde{X}, \tilde{Y}]^V=[\tilde{X}^H, \tilde{Y}^H]^V = -2\tilde{\Omega}(\tilde{X}, \tilde{Y})\tilde{C}$$

and

$$(1.6) \qquad p[\tilde{X}, \tilde{Y}]=[p\tilde{X}, p\tilde{Y}],$$

where $\tilde{\Omega}$ *is an invariant 2-form defined by the equation*

$$\tilde{\Omega}=d\tilde{\eta}.$$

For any two elements X, Y *of* $\mathcal{T}_0^1(M)$, *we have*

$$(1.7) \qquad [X^L, Y^L]^H=[X, Y]^L, \qquad [X^L, Y^L]^V=-2\tilde{\Omega}(X^L, Y^L)\tilde{C}$$

(cf. Yano and Ishihara [10]).

If we define in the base space M a 2-form Ω by the equation

$$(1.8) \qquad \Omega=p\tilde{\Omega}=p(d\tilde{\eta}),$$

then, as was proved in [11], Ω *is closed* (cf. (3.6)) and the cohomology class $[\Omega]$ of order 2 determined by Ω is called the *characteristic class* of the fibred space $(\tilde{M}, M, \pi; \tilde{C})$ without horizontal planes, i.e., the class $[\Omega]$ is determined independently of the choice of the structure 1-form $\tilde{\eta}$ (in our metric case, independently of the choice of the invariant Riemannian metric $\tilde{g}$).

Since $\mathcal{J}^H(\tilde{M})$ is a subspace of $\mathcal{J}(\tilde{M})$, we denote by j: $\mathcal{J}^H(\tilde{M}) \to \mathcal{J}(\tilde{M})$ the injection. We shall now introduce a linear homomorphism i: $\mathcal{J}(\tilde{M}) \# \mathcal{J}^H(\tilde{M}) \to \mathcal{J}(\tilde{M})$ by the property

$$(\mathrm{I}.1) \qquad i(\tilde{T} \# \overset{*}{S})=\tilde{T} \otimes j(\overset{*}{S}) \qquad \text{for} \quad \tilde{T} \in \mathcal{J}(\tilde{M}),\ \overset{*}{S} \in \boldsymbol{\mathcal{J}}^H(\tilde{M}).$$

The induced metric and the induced connection. Denoting by $\tilde{\nabla}$ the Riemannian connection determined by the invariant Riemannian metric $\tilde{g}$ in $\tilde{M}$, we see that $\tilde{\nabla}$ is necessarily an invariant affine connection, i.e.,

$$\mathcal{L}(\nabla_{\tilde{Y}}\tilde{X})=0 \qquad \text{for} \quad \tilde{X}, \tilde{Y} \in \mathcal{J}_0^1(\tilde{M}).$$

Therefore we can define in the base space M an affine connection ∇ by the equation

$$(1.9) \qquad \nabla_Y X=p(\tilde{\nabla}_{Y^L} X^L) \qquad \text{for} \quad X, Y \in \mathcal{T}_0^1(M)$$

(cf. Yano and Ishihara [11]). The connection ∇ thus defined is called the *induced connection* in M. It is easily verified that the *induced connection* ∇ *is torsionless*, because so is also the Riemannian connection $\tilde{\nabla}$ in $\tilde{M}$ (cf. Yano and Ishihara [11]).

The projection $g=p\tilde{g}$ is a Riemannian metric in the base space M, which is called the *induced metric* in M, i.e., (1.10) $$g(X, Y)=p(\tilde{g}(X^L, Y^L)) \qquad \text{for} \quad X, Y\in \mathcal{T}_0^1(M).$$

Taking account of (P. 1)$\sim$(P. 4) and (1. 9), we easily find

$$\nabla_Y g=p(\tilde{\nabla}_{Y^L}\tilde{g}),$$

Y being an arbitrary element of $\mathcal{T}_0^1(M)$, as a consequence of $\tilde{g}=g^L+\tilde{\eta}\otimes\tilde{\eta}$. Thus we find $\nabla g=0$ because of $\tilde{\nabla}\tilde{g}=0$ and the equation above. Therefore we have

PROPOSITION 1. 2. *In a fibred space with invariant Riemannian metric* $\tilde{g}$, *the induced connection* ∇ *is the Riemannian connection determined by the metric* $g=p\tilde{g}$ *iuduced in the base space M.*

We have now the following formulas:

(1.11) $$\nabla_Y T=p(\tilde{\nabla}_{Y^L} T^L) \qquad \text{for} \quad Y\in\mathcal{T}_0^1(M),\ T\in\mathcal{T}(M)$$

and

(1.12) $$p(\tilde{\nabla}_{\tilde{Y}}\tilde{T})=\nabla_Y T \qquad \text{for} \quad \tilde{Y}\in\mathcal{J}^H{}_0^1(\tilde{M}),\ \tilde{T}\in\mathcal{J}^H(\tilde{M}),$$

Y and T being defined respectively by $Y=p\tilde{Y}$ and $T=p\tilde{T}$.

Van der Waerden-Bortolotti covariant derivative. Given an element $\tilde{Y}$ of $\mathcal{J}_0^1(\tilde{M})$, we define a derivation $\overset{*}{\nabla}_{\tilde{Y}}$ in the formal tensor product $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$ by the following properties:

(W. 1) $$\overset{*}{\nabla}_{\tilde{Y}}\tilde{T}=\tilde{\nabla}_{\tilde{Y}}\tilde{T} \qquad \text{for} \quad \tilde{T}\in\mathcal{J}(\tilde{M}).$$

(W. 2) $$\overset{*}{\nabla}_{\tilde{Y}}\overset{*}{S}=(\nabla_{\tilde{Y}^H}\overset{*}{S})^H \qquad \text{for} \quad \overset{*}{S}\in\mathcal{J}^H(\tilde{M}).$$

(W. 3) $$\overset{*}{\nabla}_{\tilde{Y}}(\tilde{T}\#\overset{*}{S})=(\overset{*}{\nabla}_{\tilde{Y}}\tilde{T})\#\overset{*}{S}+\tilde{T}\#(\overset{*}{\nabla}_{\tilde{Y}}\overset{*}{S}) \qquad \text{for} \quad \tilde{T}\in\mathcal{J}(\tilde{M}),\ \overset{*}{S}\in\mathcal{J}^H(\tilde{M}).$$

For any element $\overset{*}{W}$ of $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$, the correspondence $\tilde{Y}\to\overset{*}{\nabla}_{\tilde{Y}}\overset{*}{W}$ defines an element $\overset{*}{\nabla}\overset{*}{W}$ of $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$, which is called the *van der Waerden-Bortolotti covariant derivative* of $\overset{*}{W}$.

The second fundamental tensors. We shall define an element h of $\mathcal{T}_2^0(M)$ by the equation

(1.13) $$h(Y, X)=p\{\tilde{\eta}(\nabla_{Y^L}X^L)\},$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$, and an element H of $\mathcal{T}_1^1(M)$ by the equation

(1.14) $$HX=-p(\tilde{\nabla}_{\tilde{C}}X^L),$$

X being an arbitrary element of $\mathcal{T}_0^1(M)$. The equation (1. 14) is equivalent to the equation

(1. 15) $$HX=-p(\tilde{\nabla}_{X^L}\tilde{C})$$

because of $\mathcal{L}X^L=[C, X^L]=0$. We call h and H the *second fundamental tensors* in the base space M. On putting

(1. 16) $$\tilde{h}=h^L, \qquad \tilde{H}=H^L,$$

we call $\tilde{h}$ and $\tilde{H}$ the *second fundamental tensors* in the total space $\tilde{M}$.

Taking account of (1. 10) and (1. 14), we find

(1. 17) $$g(Y, HX)=-g(X, HY),$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$, because we have

$$-p\{\tilde{g}(Y^L, \tilde{\nabla}_{\tilde{C}}X^L)\}=p\{\tilde{g}(\tilde{\nabla}_{\tilde{C}}Y^L, X^L)\}$$

as a consequence of $\tilde{\nabla}_{\tilde{C}}\{\tilde{g}(X^L, Y^L)\}=0$. On the other hand, if we take account of (1. 13) and (1. 15), we obtain

(1. 18) $$g(Y, HX)=h(X, Y),$$

because we have

$$\begin{aligned}-p\{\tilde{g}(Y^L, \tilde{\nabla}_{X^L}\tilde{C})\{&=p\{\tilde{g}(\tilde{\nabla}_{X^L}Y^L, \tilde{C})\}\\&=p\{\tilde{\eta}(\tilde{\nabla}_{X^L}Y^L)\}\end{aligned}$$

as a consequence of $\tilde{g}(\tilde{C}, Y^L)=\tilde{\eta}(Y^L)=0$. Thus from (1. 17) and (1. 18) we have

PROPOSITION 1. 3. *The second fundamental tensors h and H in M have the following properties*:

(1. 19) $$h(X, Y)+h(Y, X)=0,$$

(1. 20) $$h(X, Y)=g(HX, Y),$$

X and Y *being arbitrary elements of* $\mathcal{T}_0^1M)$.

Taking account of Proposition 1. 3 and of the definitions (1. 13), (1. 14), (1. 15), we have

PROPOSITION 1. 4. *The second fundamental tensors* $\tilde{h}$ and $\tilde{H}$ *in the total space* $\tilde{M}$ *have the following properties*:

(1. 21) $$\tilde{h}(\tilde{X}, \tilde{Y})+\tilde{h}(\tilde{Y}, \tilde{X})=0,$$

(1. 22) $$\tilde{h}(\tilde{X}, \tilde{Y})=\tilde{g}(\tilde{H}\tilde{X}, \tilde{Y}),$$

$\tilde{X}$ *and* $\tilde{Y}$ *being arbitrary elements of* $\mathcal{T}_0^1(\tilde{M})$, *and* $\tilde{h}$ *and* $\tilde{H}$ *belong respectively to* $\mathcal{J}^{H_2^0}(\tilde{M})$ *and* $\mathcal{J}^{H_1^1}(\tilde{M})$. *The following formulas hold*:

$$(\tilde{\nabla}_{\tilde{Y}}\tilde{X})^V=\tilde{h}(\tilde{Y}, \tilde{X})\tilde{C}, \tag{1.23}$$

$$\tilde{\nabla}_{\tilde{C}}\tilde{X}=\tilde{\nabla}_{\tilde{X}}\tilde{C}=-\tilde{H}\tilde{X}\in \mathcal{J}^{H1}_0(\tilde{M}), \tag{1.24}$$

$$\tilde{h}(\tilde{C}, \tilde{X})=0, \qquad \tilde{H}\tilde{C}=0,$$

where $\tilde{X}$ and $\tilde{Y}$ *are arbitrary elements of* $\mathcal{J}^{H1}_0(\tilde{M})$.

As a consequence of (1.9), (1.23) and (1.24), we find the following formulas:

$$\begin{aligned} &\tilde{\nabla}_{\tilde{Y}}\tilde{X}=(\nabla_Y X)^L+\tilde{h}(\tilde{Y}, \tilde{X})\tilde{C}, \\ &\tilde{\nabla}_{\tilde{Y}}\tilde{C}=\tilde{\nabla}_{\tilde{C}}\tilde{Y}=-\tilde{H}\tilde{Y}, \qquad \tilde{\nabla}_{\tilde{C}}\tilde{C}=0 \end{aligned} \tag{1.25}$$

for any two elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{J}^{H1}_0(\tilde{M})$, where $X=p\tilde{X}$ and $Y=p\tilde{Y}$. The last equation of (1.25) is a direct consequence of the conditions $\mathcal{L}_{\tilde{C}}\tilde{g}=0$ and $\tilde{g}(\tilde{C}, \tilde{C})=1$. The first and the second equations of (1.25) are called respectively the *co-Gauss equation* and *co-Weingarten equation* of the given fibred space with invariant Riemannian metric.

Taking account of the definition of the exterior derivative, we have, for any two elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{J}^{H1}_0(\tilde{M})$,

$$\begin{aligned} 2d\tilde{\eta}(\tilde{X}, \tilde{Y}) &= \tilde{X}\tilde{\eta}(\tilde{Y})-\tilde{Y}\tilde{\eta}(\tilde{X})-\tilde{\eta}([\tilde{X}, \tilde{Y}]) \\ &= -\tilde{\eta}([\tilde{X}, \tilde{Y}]) \\ &= -\tilde{\eta}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\tilde{\nabla}_{\tilde{Y}}\tilde{X}) \end{aligned}$$

because of $\tilde{X}\tilde{\eta}(\tilde{Y})=\tilde{Y}\tilde{\eta}(\tilde{X})=0$. Thus, as a consequence of (1.23), we obtain in $\tilde{M}$

$$\tilde{h}=-d\tilde{\eta}=-\tilde{\Omega} \tag{1.26}$$

and, consequently, in M

$$h=-\Omega=-p\tilde{\Omega}=-p(d\tilde{\eta}). \tag{1.27}$$

When one of the second fundamental tensors h, H, $\tilde{h}$ and $\tilde{H}$ vanishes, all of the other second fundamental tensors vanish too. Thus, if $h=0$ holds, we find from (1.25)

$$\begin{aligned} &\tilde{\nabla}_{\tilde{Y}}\tilde{X}=(\nabla_Y X)^L, \qquad \tilde{\nabla}_{\tilde{Y}}\tilde{C}=0, \\ &\tilde{\nabla}_{\tilde{C}}\tilde{X}=0, \qquad \tilde{\nabla}_{\tilde{C}}\tilde{C}=0 \end{aligned} \tag{1.28}$$

for any elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{J}^{H1}_0(\tilde{M})$, where X and Y are defined respectively by $X=p\tilde{X}$ and $Y=p\tilde{Y}$. As is well known, when the conditions (1.28) are satisfied, the Riemannian manifold $\tilde{M}$ is locally the Pythagorean product of a Riemannian space and a straight line. In such a case, we say that the given fibred space $\tilde{M}$ with invariant Riemannian metric is *locally trivial*. Thus, taking account of (1. 26) or (1. 27), we have

PROPOSITION 1. 5. *For a fibred space $\tilde{M}$ with invariant Riemaunian metric, the following three conditions* (a), (b) *and* (c) *are equivalent to each other*:

(a) *One of the second fundamental tensors h, H, $\tilde{h}$ and $\tilde{H}$ vanishes identically.*

(b) *The field of horizontal planes, which is defined by the equation $\tilde{\eta}=0$, is integrable.*

(c) *The given fibred space $\tilde{M}$ is locally trivial.*

When one of these conditions (a), (b) *and* (c) *is satisfied the characteristic class of the corresponding fibred space without horizontal planes is zero.*

As a direct consequence of the last equation of (1. 28), we have

PROPOSITION 1. 6. *In a fibred space with invariant Riemannian metric, each fibre is a geodesic.*

When the base space M is 1-dimensional, the second fundamental tensor h vanishes identically because h is skew symmetric. Thus, as a consequence of Proposition 1. 5 we see that *any fibred space $\tilde{M}$ with invariant Riemannian metric is locally trivial and consequently the total space $\tilde{M}$ is locally flat, if $\tilde{M}$ is 2-dimensional.*

The Nijenhuis tensor of the second fundamental tensor. The Nijenhuis tensor $\tilde{N}$ of the second fundamental tensor $\tilde{H}$ in $\tilde{M}$ is by definition

$$\tilde{N}(\tilde{X},\tilde{Y})=[\tilde{H}\tilde{X},\tilde{H}\tilde{Y}]-\tilde{H}[\tilde{H}\tilde{X},\tilde{Y}]-\tilde{H}[\tilde{X},\tilde{H}\tilde{Y}]+\tilde{H}^2[\tilde{X},\tilde{Y}], \tag{1. 29}$$

$\tilde{X}$ and $\tilde{Y}$ being arbitrary elements of $\mathcal{T}_0^1(\tilde{M})$, and Nijenhuis tensor N of the second fundamental tensor H in M is by definition

$$N(X, Y)=[HX, HY]-H[HX, Y]-H[X, HY]+H^2[X, Y], \tag{1. 30}$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$. If we take X and Y arbitrarily from $\mathcal{T}_0^1(M)$, substituting $\tilde{X}=X^L$ and $\tilde{Y}=Y^L$ in both sides of (1. 29) and taking horizontal parts, we obtain the equations

$$\tilde{N}^H=N^L, \qquad p\tilde{N}=p\tilde{N}^H=N. \tag{1. 31}$$

If we take the vertical parts of both sides of (1. 29), we find

$$\begin{aligned}\{\tilde{N}(\tilde{X},\tilde{Y})\}^V&=[\tilde{H}\tilde{X},\tilde{H}\tilde{Y}]^V\\&=2\tilde{h}(\tilde{H}\tilde{X},\tilde{H}\tilde{Y})\tilde{C}\\&=2\tilde{g}(\tilde{H}^2\tilde{X},\tilde{H}\tilde{Y})\tilde{C}\end{aligned} \tag{1. 32}$$

for $\tilde{X}, \tilde{Y} \in \mathcal{T}_0^1(\tilde{M})$, as a consequence of (1. 22), (1. 26) and Proposition 1. 1, because $\tilde{H}$ belongs to $\mathcal{J}^H(\tilde{M})$. If we take arbitrarily X and Y from $\mathcal{T}_0^1(M)$ and substitute $\tilde{X}=X^L$ and $\tilde{Y}=Y^L$ in both sides of (1. 32), we obtain

$$p\{\tilde{\eta}\{\tilde{N}(X^L, Y^L)\}\}=2h(HX, HY)=2g(H^2X, HY) \tag{1. 33}$$

for $X, Y \in \mathcal{T}_0^1(M)$.

If we define an element $\tilde{S}$ of $\mathcal{J}^H{}_2^1(\tilde{M})$ by the equation

$$\tilde{S}=\tilde{N}^H, \tag{1. 34}$$

then we find $\tilde{S}=\tilde{N}-\tilde{N}^V$ and hence

$$\begin{aligned}\tilde{S}(\tilde{X}, \tilde{Y})&=\tilde{N}^H(\tilde{X}, \tilde{Y})\\ &=\tilde{N}(\tilde{X}, \tilde{Y})-[\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}]^V\\ &=\tilde{N}(\tilde{X}, \tilde{Y})-2\tilde{h}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y})\tilde{C}\end{aligned} \tag{1. 35}$$

for any elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{T}_0^1(\tilde{M})$. On the other hand, taking account of (1. 31), we obtain the relations

$$p\tilde{S}=N, \qquad \tilde{S}=N^L. \tag{1. 36}$$

We now say that a fibred space $\tilde{M}$ with invariant Riemannian metric is *normal*, if the tensor field $\tilde{S}$ defined by (1. 34) vanishes identically in $\tilde{M}$. Thus, taking account of (1. 32), (1. 34) and (1. 36), we have

PROPOSITION 1. 7. *In a fibred space $\tilde{M}$ with invariant Riemannian metric, the Nijenhuis tensor $\tilde{N}$ of the second fundamental tensor $\tilde{H}$ in $\tilde{M}$ vanishes identically if and only if one of the following conditions* (a) *and* (b) *is satisfied*:

(a) $\tilde{S}=0$ *and* $[\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}] \in \mathcal{T}^H{}_0^1(\tilde{M})$ *for* $\tilde{X}, \tilde{Y} \in \mathcal{T}_0^1(\tilde{M})$.

(b) $\tilde{S}=0$ *and* $\tilde{H}^3=0$ (*i.e.*, $H^3=0$).

PROPOSITION 1. 8. *The Nijenhuis tensor N of the second fundamental tensor H in the base space M vanishes identically if and only if the given fibred space $\tilde{M}$ is normal, i.e., if and only if $\tilde{S}=0$ in $\tilde{M}$.*

The second fundamental tensors $\tilde{H}$ and H are of the same rank, which is not greater than n, the base space M being of dimension n. If we take account of the condition (b) mentioned in Proposition 1. 7, we see that *the second fundamental tensor $\tilde{H}$ (or H) is necessarily of rank less than $n(=\dim M)$ if the Nijenhuis tensor $\tilde{N}$ of $\tilde{H}$ vanishes identically in the total space $\tilde{M}$.*

§ 2. The tensor calculus in a fibred space with invariant Riemannian metric.

Let $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ be a fibred space with invariant Riemannian metric $\tilde{g}$. Since the projection π: $\tilde{M}\rightarrow M$ is differentiable and of the maximum rank everywhere, there exists, for any point $\tilde{\mathrm{P}}$ of $\tilde{M}$, a coordinate neighborhood $\tilde{U}$ containing $\tilde{\mathrm{P}}$ in $\tilde{M}$ such that $U=\pi(\tilde{U})$ is a coordinate neighborhood of the point $\mathrm{P}=\pi(\tilde{\mathrm{P}})$ in M and the intersection $F_{\mathrm{Q}}\cap\tilde{U}$ is expressed in $\tilde{U}$ by equations

$$y^{1}=a^{1},\ y^{2}=a^{2},\ \cdots,\ y^{n}=a^{n},$$

$a^{1}, a^{2}, \cdots, a^{n}$ being constant, with respect to certain local coordinates $(y^{1}, y^{2}, \cdots, y^{n}, y^{n+1})$ defined in $\tilde{U}$, where Q is an arbitrary point in U. We call such a neighborhood $\tilde{U}$ a *cylindrical neighborhood* of $\tilde{M}$. Since we restrict ourselves only to cylindrical neighborhoods in $\tilde{M}$, we call them simply *neighborhoods* of $\tilde{M}$. Given a neighborhood $\tilde{U}$ in $\tilde{M}$, the set $(\tilde{U}, U, \pi; \tilde{C}, \tilde{g})$ is a fibred space with invariant Riemannian metric $\tilde{g}$, where $U=\pi(\tilde{U})$, π is the restriction of the projection π: $\tilde{M}\rightarrow M$ to $\tilde{U}$, and $\tilde{C}$ and $\tilde{g}$ are respectively the restrictions of the structure field $\tilde{C}$ and the invariant Riemannian metric $\tilde{g}$ to $\tilde{U}$. In the sequel, we shall identify the operations of taking horizontal parts, lifts, projections, etc. in the fibred space $(\tilde{U}, U, \pi; \tilde{C}, \tilde{g})$ with the corresponding operations in the given fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ respectively.

Let $(x^{h})=(x^{1}, x^{2}, \cdots, x^{n+1})$ be coordinates defined in $\tilde{U}$ and $(\xi^{a})=(\xi^{1}, \xi^{2}, \cdots, \xi^{n})$ coordinates defined in $U=\pi(\tilde{U})$, where $\tilde{U}$ is a neighborhood in $\tilde{M}$.[1] We denote by E^{h} and g_{ji} the components of the structure field $\tilde{C}$ and the invariant Riemannian metric $\tilde{g}$ with respect to coordinates (x^{h}) defined in $\tilde{U}$.[2] Then the structure 1-form $\tilde{\eta}$ defined by (1. 2) has components of the form

$$E_{i}=g_{ih}E^{h},\quad \text{i.e.,}\quad \tilde{\eta}=E_{i}dx^{i}. \tag{2. 1}$$

Taking a point P with coordinates (ξ^{a}) arbitrarily in U, we may assume that $F_{\mathrm{P}}\cap\tilde{U}$ is expressed by n equations

$$\xi^{a}=\xi^{a}(x^{h}) \tag{2. 2}$$

in $\tilde{U}$, where n functions $\xi^{a}(x^{h})$ are differentiable in $\tilde{U}$ and their Jacobian matrix $(\partial\xi^{a}/\partial x^{h})$ is of rank n. Putting

$$E_{i}{}^{a}=\partial_{i}\xi^{a} \tag{2. 3}$$

1) The indices h, i, j, k, l, m, s, t run over the range $\{1, 2, \cdots, n+1\}$ and the indices a, b, c, d, e, f run over the range $\{1, 2, \cdots, n\}$. The so-called Einstein's summation convention is used with respect to these two systems of indices.

2) The components of a tensor field $\tilde{T}$ in $\tilde{M}$ with respect to coordinates (x^{h}) defined in $\tilde{U}$, or, the components of $\tilde{T}$ in $\tilde{U}$ mean the components of $\tilde{T}$ with respect to the natural frame $\{\partial/\partial x^{i}\}$.

where ∂_i denotes the operator

$$\partial_i = \frac{\partial}{\partial x^i},$$

we know that n local covector fields $\tilde{\zeta}^a$ with components $E_i{}^a$ are linearly independent in $\tilde{U}$. Since the structure field $\tilde{C}$ is tangent to fibres, we find

$$E^i E_i = g_{ji} E^j E^i = 1, \qquad E^i E_i{}^a = 0, \tag{2. 4}$$

the first equation being a direct consequence of (1. 1) and (2. 1). Then the $n+1$ local covector fields $\tilde{\zeta}^a$ and $\tilde{\eta}$ are linearly independant in $\tilde{U}$.

Taking account of (2. 4), we see that the inverse of the matrix $(E_i{}^a, E_i)$ has the form

$$(E_i{}^a, E_i)^{-1} = \begin{pmatrix} E^h{}_b \\ E^h \end{pmatrix}, \tag{2. 5}$$

where $E^h{}_b$, for each fixed index b, are components of a local vector field $\tilde{B}_b$ in $\tilde{U}$. Then the $n+1$ local vector fields $\tilde{B}_b$ and $\tilde{C}$ are linearly independent in $\tilde{U}$. The equation (2. 5) is equivalent to the conditions

$$\begin{aligned} &E^i{}_b E_i{}^a = \delta_b^a, \qquad && E^i{}_b E_i = 0, \\ &E^i E_i{}^a = 0, && E^i E_i = 1, \end{aligned} \tag{2. 6}$$

that is,

$$\begin{aligned} &\tilde{\zeta}^a(\tilde{B}_b) = \delta_b^a, \qquad && \tilde{\eta}(\tilde{B}_b) = 0, \\ &\tilde{\zeta}^a(\tilde{C}) = 0, && \tilde{\eta}(\tilde{C}) = 1, \end{aligned} \tag{2. 6$'$}$$

or, to the condition

$$E_i{}^a E^h{}_a + E_i E^h = \delta_i^h. \tag{2. 7}$$

The first and the second equations of (2. 6) or of (2. 6)$'$ show that n local vector fields $\tilde{B}_b$ span the horizontal plane, which is defined by the equation $\tilde{\eta} = E_i dx^i = 0$, at each point of $\tilde{U}$.

Applying the Lie derivative $\mathcal{L}$ with respect to the structure field $\tilde{C}$, we find

$$\begin{aligned} &\mathcal{L} E^h{}_b = 0, \qquad && \mathcal{L} E^h = 0, \\ &\mathcal{L} E_i{}^a = 0, && \mathcal{L} E_i = 0 \end{aligned} \tag{2. 8}$$

(cf. (2. 8) in Yano and Ishihara [11]).

Horizontal parts. Let there be given a tensor field $\tilde{T}$, say, of type (1. 1) in the total space $\tilde{M}$. Then $\tilde{T}$ has components of the form

$$\tilde{T}_i{}^h = T_b{}^a E_i{}^b E^h{}_a + T_b{}^0 E_i{}^b E^h + T_0{}^a E_i E^h{}_a + T_0{}^0 E_i E^h \tag{2.9}$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, where $T_b{}^a$, $T_b{}^0$, $T_0{}^a$ and $T_0{}^0$ are all functions in $\tilde{U}$. Then, taking account of (H. 1)~(H. 4), we easily see that the horizontal part $\tilde{T}^H$ of $\tilde{T}$ has in $\tilde{U}$ components

$$\tilde{T}^H{}_i{}^h = T_b{}^a E_i{}^b E^h{}_a.$$

Invariant functions. Let $\tilde{f}$ be an invariant function in the total space $\tilde{M}$. We have by definition $\mathcal{L}\tilde{f}=0$, which implies that $\tilde{f}$ is expressed as

$$\tilde{f} = f(\xi^a(x^h)) \tag{2.10}$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, where $\xi^a(x^h)$ are the functions appearing in the equation (2. 2) defining fibres. Taking account of (2. 10), we find the formula

$$\partial_i \tilde{f} = E_i{}^a \partial_a f \tag{2.11}$$

(cf. (2. 11) in Yano and Ishihara [11]), where ∂_a means the operator defined by

$$\partial_a = \frac{\partial}{\partial \xi^a}.$$

The function $\tilde{f}$ is invariant and consequently its projection $f=p\tilde{f}$ is a function in the base space M and, conversely, the lift f^L of f coincides with $\tilde{f}$. Thus, in the sequel, *we shall identify any invariant function $\tilde{f}$ with its projection $f=p\tilde{f}$ and denote the invariant function $\tilde{f}$ by the same symbol f as its projection.*

Invariant tensor fields, projections and lifts. Let there be given a tensor field $\tilde{T}$, say, of type (1. 1) in the total space $\tilde{M}$. Then $\tilde{T}$ *is invariant if and only if it has in each neighborhood $\tilde{U}$ components of the form* (2. 9) *with invariant functions $T_b{}^a$, $T_b{}^0$, $T_0{}^a$ and $T_0{}^0$ in $\tilde{U}$.* Thus, taking account of (P. 1)~(P. 4), we easily see that for an invariant tensor field $\tilde{T}$, say, of type (1. 1), *its projection $T=p\tilde{T}$ has components $T_b{}^a$ with respect to coordinates (ξ^a) defined in $U=\pi(\tilde{U})$.* [1)]

Let there be given a tensor field T, say, of type (1, 1) in the base space M and let $T_b{}^a$ be the components of T in U. Then, taking account of (L. 1)~(L. 4), we easily see that *the lift T^L of T has components of the form*

$$\tilde{T}_i{}^h = T_b{}^a E_i{}^b E^h{}_a \tag{2.12}$$

1) The components of a tensor field T in M with respect to coordinates (ξ^a) defined in U, or, the components of T in U mean the components of T with respect to the natural frame $\{\partial/\partial\xi^b\}$ at each point of U.

with respect to coordinates (x^h) defined in $\tilde{U}$, $\tilde{U}$ being a neighborhood of $\tilde{M}$ such that $U=\pi(\tilde{U})$.

Invariant Riemannian metric. Let g_{ji} be the components of the invariant Riemannian metric $\tilde{g}$ in each neighborhood $\tilde{U}$ of $\tilde{M}$. Then the Riemannian metric $g=p\tilde{g}$ induced in the base space M has components

$$g_{cb}=E^j{}_cE^i{}_bg_{ji} \tag{2.13}$$

in $U=\pi(\tilde{U})$. Thus, taking account of (2.6) and (2.7), we have the formula

$$g_{ji}=g_{cb}E_j{}^cE_i{}^b+E_jE_i \tag{2.14}$$

by virtue of $g_{ji}E^jE^i=E^iE_i=1$.

If we define g^{ih} by the equation

$$(g^{ih})=(g_{ji})^{-1}, \quad \text{i.e.} \quad g_{ji}g^{ih}=\delta_j^h, \tag{2.15}$$

then g^{ih} are components of an element $\tilde{G}$ of $\mathcal{J}_0^2(\tilde{M})$ in $\tilde{U}$. The projection $G=p\tilde{G}$ has components g^{ba} in U such that

$$(g^{ba})=(g_{cb})^{-1}, \quad \text{i.e.,} \quad g_{cb}g^{ba}=\delta_c^a. \tag{2.16}$$

Thus we obtain the following formulas:

$$\begin{aligned} g^{ba}&=E_i{}^bE_h{}^ag^{ih}, \\ g^{ih}&=g^{ba}E^i{}_bE^h{}_a+E^iE^h. \end{aligned} \tag{2.17}$$

Moreover, taking account of (2.6), we easily find the following formulas:

$$\begin{aligned} E_i{}^a&=g_{ih}g^{ba}E^h{}_b, \qquad & E_i&=g_{ih}E^h, \\ E^h{}_b&=g^{hi}g_{ab}E_i{}^a, \qquad & E^h&=g^{hi}E_i. \end{aligned} \tag{2.18}$$

The Riemannian connection. The Riemannian connection $\tilde{\nabla}$ determined by the given invariant Riemannian metric $\tilde{g}$ is also invariant and has the Christoffel's symbols $\{{}_j{}^h{}_i\}$ constructed from g_{ji} as its coefficients in each neighborthood $\tilde{U}$ of the total space $\tilde{M}$. For any vector field $\tilde{X}$ in $\tilde{M}$, its covariant derivative $\tilde{\nabla}\tilde{X}$ has components of the form

$$\tilde{\nabla}_j\tilde{X}^h=\partial_j\tilde{X}^h+\{{}_j{}^h{}_i\}\tilde{X}^i$$

in $\tilde{U}$, $\tilde{X}^h$ being the components of $\tilde{X}$ in $\tilde{U}$.

If we take account of (1.25), we obtain

$$\tag{2.19} \begin{aligned} &\tilde{\nabla}_j E^h{}_b = \Gamma_c{}^a{}_b E_j{}^c E^h{}_a + h_{cb} E_j{}^c E^h - h_b{}^a E_j E^h{}_a, \\ &\tilde{\nabla}_j E^h = -h_c{}^a E_j{}^c E^h{}_a \end{aligned}$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, where h_{cb} and $h_b{}^a$ are invariant functions in $\tilde{U}$ and $\Gamma_c{}^b{}_a$ are also invariant functions in $\tilde{U}$ such that

$$\Gamma_c{}^a{}_b = \Gamma_b{}^a{}_c,$$

because the local vector fields $\tilde{B}_b$ with components $E^h{}_b$ are horizontal and invariant in $\tilde{U}$ (cf. (2.15) in Yano and Ishihara [11]). Comparing (1.25) and (2.19), we know that *the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ have in $\tilde{U}$ components of the form*

$$\tilde{h}_{ji} = h_{cb} E_j{}^c E_i{}^b \qquad \text{and} \qquad \tilde{h}_i{}^h = h_b{}^a E_i{}^b E^h{}_a$$

respectively and hence *the second fundamental tensors h and H have in $U = \pi(\tilde{U})$ components of the form*

$$h_{cb} \qquad \text{and} \qquad h_b{}^a$$

respectively. Therefore, Proposition 1.3 is equivalent to the fact that the conditions

$$\tag{2.20} \tilde{h}_{ji} + \tilde{h}_{ij} = 0, \qquad h_{cb} + h_{bc} = 0,$$

$$\tag{2.21} \tilde{h}_{ji} = \tilde{h}_j{}^h g_{ih}, \qquad h_{cb} = h_c{}^a g_{ba}$$

hold. The first and the second equations of (2.19) are called the *co-Gauss equations* and *co-Weingarten equations* respectively.

Let $\tilde{X}$ be an element of $\mathcal{J}^1_0(\tilde{M})$. Then $\tilde{X}$ has components of the form

$$\tilde{X}^h = X^a E^h{}_a + X^0 E^h$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, where X^a and X^0 are invariant functions in $\tilde{U}$. Taking account of (2.19), we find that the covariant derivative $\tilde{\nabla}\tilde{X}$ has components of the form

$$\tag{2.22} \tilde{\nabla}_j \tilde{X}^h = \{\nabla_c X^a - X^0 h_c{}^a\} E_j{}^c E^h{}_a + \{h_{ca} X^a + \partial_c X^0\} E_j{}^c E^h - \{h_b{}^a X^b\} E_j E^h{}_a,$$

where we have put

$$\nabla_c X^a = \partial_c X^a + \Gamma_c{}^a{}_b X^b.$$

When $\tilde{X}$ belongs to $\mathcal{J}^{H1}_0(\tilde{M})$, the formula (2.22) reduces to

$$\tag{2.23} \tilde{\nabla}_j \tilde{X}^h = (\nabla_c X^a) E_j{}^c E^h{}_a + (h_{ca} X^a) E_j{}^c E^h - (h_b{}^a X^b) E_j E^h{}_a$$

because of $X^0 = 0$.

Let X be an element of $\mathcal{T}_0^1(M)$ and put $\tilde{X}=X^L$. Then $\tilde{X}$ has the components $\tilde{X}=X^a E^h{}_a$, X^a being the components of X in $U=\pi(\tilde{U})$. Thus, taking account of (2. 23), we see that the projection $p(\tilde{\nabla}\tilde{X})$ of $\tilde{\nabla}\tilde{X}$ has components

$$\nabla_c X^a=\partial_c X^a+\Gamma_c{}^a{}_b X^b.$$

Therefore the connection ∇ induced in the base space M has coefficients $\Gamma_c{}^a{}_b$ in U, since $p(\tilde{\nabla}\tilde{X})$ is nothing but ∇X by means of the definition (1. 9). Consequently, $\Gamma_c{}^a{}_b$ should coincide with the Chistoffel's symbols $\{{}_c{}^a{}_b\}$ constructed from g_{cb}, because the induced connection ∇ coincides with the Riemannian connection determined by the induced metric g. Thus the *co-Gauss equations* and the *co-Weingrten equations* (2. 19) reduce respectively to

$$\tilde{\nabla}_j E^h{}_b=\{{}_c{}^a{}_b\}E_j{}^c E^h{}_a+h_{cb}E_j{}^c E^h-h_b{}^a E_j E^h{}_a,$$

(2. 24)

$$\tilde{\nabla}_j E^h=-h_c{}^a E_j{}^c E^h{}_a.$$

Van der Waerden-Bortolotti covariant derivatives. Let there be given an element of the formal tensor product $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$, say, $\overset{*}{T}$ belonging to $\mathcal{J}_1^1(\tilde{M})\#\mathcal{J}^H{}_1^1(\tilde{M})$. Then $\overset{*}{T}$ is expressed as follows:

$$\overset{*}{T}=\overset{*}{T}_k{}^j{}_b{}^a\tilde{e}^k\#\tilde{e}_j\#\tilde{\zeta}^b\#\tilde{B}_a$$

in each cylindrical neighborhood $\tilde{U}$ of $\tilde{M}$, $\overset{*}{T}_k{}^j{}_b{}^a$ being invariant functions in $\tilde{U}$, where $\{\tilde{e}_j\}=\{\partial/\partial x^j\}$ is the natural frame of coordinates (x^h) defined in $\tilde{U}$, $\{\tilde{e}^k\}$ the dual base to $\{\tilde{e}_j\}$, $\tilde{B}_a$ local vector fields having components $E^h{}_a$ and $\tilde{\zeta}^b$ local covector fields having components $E_i{}^b$, all in $\tilde{U}$. We call $\overset{*}{T}_k{}^j{}_b{}^a$ the *components* of $\overset{*}{T}$ with respect to coordinates (x^h) and coordinates (ξ^a) defined in $\tilde{U}$ and in $U=\pi(\tilde{U})$, or simply the *components* of $\overset{*}{T}$ in $(\tilde{U}, U)$. Let i: $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})\to\mathcal{J}(\tilde{M})$ be the linear homomorphism defined by (I. 1)$\sim$(I. 3) in §1. Then the image $\tilde{T}=i(\overset{*}{T})$ has in $\tilde{U}$ components of the form

$$\tilde{T}_k{}^j{}_i{}^h=\overset{*}{T}_k{}^j{}_b{}^a E_i{}^b E^h{}_a. \tag{2. 25}$$

Conversely, we have

$$\overset{*}{T}_k{}^j{}_b{}^a=\tilde{T}_k{}^j{}_i{}^h E^i{}_b E_h{}^a. \tag{2. 26}$$

Let $\overset{*}{T}$ be the element of $\mathcal{J}(\tilde{M})\#\mathcal{J}^H(\tilde{M})$ considered above. Then the *van der Waerden-Bortolotti covariant derivative* $\overset{*}{\nabla}\overset{*}{T}$ of $\overset{*}{T}$ has components of the form

$$\overset{*}{\nabla}_l\overset{*}{T}_k{}^j{}_b{}^a=\partial_l\overset{*}{T}_k{}^j{}_b{}^a+\{{}_l{}^j{}_m\}\overset{*}{T}_k{}^m{}_b{}^a-\{{}_l{}^m{}_k\}\overset{*}{T}_m{}^j{}_b{}^a+E_l{}^d(\{{}_d{}^a{}_e\}\overset{*}{T}_k{}^j{}_b{}^e-\{{}_d{}^e{}_b\}\overset{*}{T}_k{}^j{}_e{}^a) \tag{2. 27}$$

by virtue of (W. 1)$\sim$(W. 4) given in §1 (cf. (2. 34) in Yano and Ishihara [11]). We put conventionally in $\tilde{U}$

$$
\begin{aligned}
\overset{*}{\nabla}_j E^h{}_b &= \partial_j E^h{}_b + \{{}_j{}^h{}_i\} E^i{}_b - \{{}_c{}^a{}_b\} E_j{}^c E^h{}_a \\
&= \tilde{\nabla}_j E^h{}_b - \{{}_c{}^a{}_b\} E_j{}^c E^h{}_a,
\end{aligned}
\tag{2.28}
$$

$$
\begin{aligned}
\overset{*}{\nabla}_j E_i{}^a &= \partial_j E_i{}^a - \{{}_j{}^h{}_i\} E_h{}^a + \{{}_c{}^a{}_b\} E_j{}^c E_i{}^b \\
&= \tilde{\nabla}_j E_i{}^a + \{{}_c{}^a{}_b\} E_j{}^c E_i{}^b,
\end{aligned}
\tag{2.29}
$$

which are the components of $(\tilde{\nabla}\tilde{B}_a)^V$ and $(\tilde{\nabla}\tilde{\zeta}^b)^V$ respectively. If we take account of the formula giving $\tilde{\nabla}_l \tilde{T}_{k}{}^j{}_i{}^h$, where $\tilde{T}_k{}^j{}_i{}^h$ are the components of $\tilde{T}=i(\overset{*}{T})$, then we have the formula

$$
\begin{aligned}
\tilde{\nabla}_l T_k{}^j{}_i{}^h &= \tilde{\nabla}_l(\overset{*}{T}_k{}^j{}_b{}^a E_i{}^b E^h{}_a) \\
&= (\overset{*}{\nabla}_l \overset{*}{T}_k{}^j{}_b{}^a) E_i{}^b E^h{}_a + \overset{*}{T}_k{}^j{}_b{}^a (\overset{*}{\nabla}_l E_i{}^b) E^h{}_a + \overset{*}{T}_k{}^j{}_b{}^a E_i{}^b (\overset{*}{\nabla}_l E^h{}_a)
\end{aligned}
\tag{2.30}
$$

because of (2. 27), (2. 28) and (2. 29).

The *co-Gauss equations* and the *co-Weingarten equations* (2. 24) reduce respectively to the equations

$$
\begin{aligned}
\overset{*}{\nabla}_j E^h{}_b &= h_{cb} E_j{}^c E^h - h_b{}^a E_j E^h{}_a, \\
\tilde{\nabla}_j E^h &= -h_c{}^a E_j{}^c E^h{}_a \\
&= -\tilde{h}_j{}^h,
\end{aligned}
\tag{2.31}
$$

which are equivalent to the equations

$$
\begin{aligned}
\overset{*}{\nabla}_j E_i{}^a &= h_c{}^a E_j{}^c E_i + h_b{}^a E_j E_i{}^b, \\
\tilde{\nabla}_j E_i &= -h_{cb} E_j{}^c E_i{}^b \\
&= -\tilde{h}_{ji}.
\end{aligned}
\tag{2.32}
$$

§ 3. Formulas.

Ricci formulas. As is well known, we have the Ricci formula

$$
\tilde{\nabla}_k \tilde{\nabla}_j \tilde{X}^h - \tilde{\nabla}_j \tilde{\nabla}_k \tilde{X}^h = \tilde{K}_{kji}{}^h \tilde{X}^i
\tag{3.1}
$$

for any element $\tilde{X}$ of $\mathcal{T}^1_0(\tilde{M})$, $\tilde{X}^h$ being the components of $\tilde{X}$, where $\tilde{K}_{kji}{}^h$ denote the components of the curvature tensor $\tilde{K}$ of the invariant Riemannian metric $\tilde{g}$ given in $\tilde{M}$ and are defined by

$$
\tilde{K}_{kji}{}^h = \partial_k \{{}_j{}^h{}_i\} - \partial_j \{{}_k{}^h{}_i\} + \{{}_k{}^h{}_l\}\{{}_j{}^l{}_i\} - \{{}_j{}^h{}_l\}\{{}_k{}^l{}_i\}.
$$

For any element $\overset{*}{T}$ of $\mathcal{I}^1_0(\tilde{M}) \# \mathcal{I}^{H}{}^0_1(\tilde{M})$, we have the formula

$$\overset{*}{\nabla}_k\overset{*}{\nabla}_j T^h{}_b-\overset{*}{\nabla}_j\overset{*}{\nabla}_k T^h{}_b=\tilde{K}_{kji}{}^h T^i{}_b-E_k{}^d E_j{}^c K_{dcb}{}^a T^h{}_a \tag{3.2}$$

by virtue of (2. 27), $T^h{}_b$ being the components of $\overset{*}{T}$, where $K_{dcb}{}^a$ denote the components of the curvature tensor K of the Riemannian metric g induced in the base space M and are defined by

$$K_{dcb}{}^a=\partial_d\{{}_c{}^a{}_b\}-\partial_c\{{}_d{}^a{}_b\}+\{{}_d{}^a{}_e\}\{{}_c{}^e{}_b\}-\{{}_c{}^a{}_e\}\{{}_d{}^e{}_b\}$$

(cf. (2. 37) in Yano and Ishihara [11]). The formula (3. 2) is *the Ricci formula for the van der Waerden-Bortolotti covariant differentiation.*

Some formulas. If we take account of (2. 28), (2. 29) and (2. 30), we easily find

$$\begin{aligned}\tilde{\nabla}_j\tilde{h}_i{}^h&=(\nabla_c h_b{}^a)E_j{}^c E_i{}^b E^h{}_a+(h_c{}^e h_e{}^a)E_j{}^c E_i E^h{}_a+(h_c{}^e h_{be})E_j{}^c E_i{}^b E^h,\\ \tilde{\nabla}_j\tilde{h}_{ih}&=(\nabla_c h_{ba})E_j{}^c E_i{}^b E_h{}^a-(h_c{}^e h_{ae})E_j{}^c E_i E_h{}^a+(h_c{}^e h_{be})E_j{}^c E_i{}^b E_h,\end{aligned} \tag{3.3}$$

the second equation of which implies

$$\tilde{\nabla}_j\tilde{h}_{ih}+\tilde{\nabla}_i\tilde{h}_{hj}+\tilde{\nabla}_h\tilde{h}_{ji}=(\nabla_c h_{ba}+\nabla_b h_{ac}+\nabla_a h_{cb})E_j{}^c E_i{}^b E_h{}^a \tag{3.4}$$

because of $h_c{}^e h_{be}=h_b{}^e h_{ce}$.

The structure 1-form $\tilde{\eta}$ has the local expression $\tilde{\eta}=E_i dx^i$ in each neighborhood $\tilde{U}$ of $\tilde{M}$. Taking account of (2. 29), we obtain

$$\tilde{\Omega}=d\tilde{\eta}=-\tilde{h}_{ji}dx^j\wedge dx^i \tag{3.5}$$

which is equivalent to (1. 26). Since $dd\tilde{\eta}=0$, if we take account of (3. 3), we easily obtain the following identities:

$$\begin{aligned}&\tilde{\nabla}_j\tilde{h}_{ih}+\tilde{\nabla}_i\tilde{h}_{hj}+\tilde{\nabla}_h\tilde{h}_{ji}=0,\\ &\nabla_c h_{ba}+\nabla_b h_{ac}+\nabla_a h_{cb}=0,\end{aligned} \tag{3.6}$$

which show that *the* 2*-form* $\Omega=p(d\tilde{\eta})$ *defined by* (1. 8) *or by* (1. 26) *is closed.*

We have now the following formulas:

$$[\tilde{B}_c, \tilde{B}_b]=2h_{cb}\tilde{C}, \qquad [\tilde{C}, \tilde{B}_b]=0 \tag{3.7}$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$ as consequences of (2. 24).

Nijenhuis tensors. Denote by $\tilde{N}$ the Nijenhuis tensor of the second fundamental tensor $\tilde{H}$ in $\tilde{M}$. Then, as is well known, $\tilde{N}$ has components of the form

(3. 8) $$\tilde{N}_{ji}{}^h=\tilde{h}_j{}^t\tilde{\nabla}_t\tilde{h}_i{}^h-h_i{}^t\tilde{\nabla}_t\tilde{h}_j{}^h-(\tilde{\nabla}_j\tilde{h}_i{}^t-\tilde{\nabla}_i\tilde{h}_j{}^t)\tilde{h}_t{}^h$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, $\tilde{h}_i{}^h$ being the components of $\tilde{H}$. Substituting the first equation of (3. 3) in (3. 8), we easily obtain the formula

(3. 9) $$\tilde{N}_{ji}{}^h=N_{cb}{}^aE_j{}^cE_i{}^bE^h{}_a+2(h_c{}^eh_b{}^dh_{ed})E_j{}^cE_i{}^bE^h,$$

$h_b{}^a$ and h_{cb} being the components of the second fundamental tensors H and h in M respectively, where $N_{cb}{}^a$ denote the components of the Nijenhuis tensor N of H and are given by the equation

(3. 10) $$N_{cb}{}^a=h_c{}^e\nabla_eh_b{}^a-h_b{}^e\nabla_eh_c{}^a-(\nabla_ch_b{}^e-\nabla_bh_c{}^e)h_e{}^a.$$

The equation (3. 9) is equivalent to the equations

(3. 11) $$p\tilde{N}=N,\quad p\{\tilde{\eta}(\tilde{N}(X^L,\ Y^L))\}=2h(HX, HY),$$

X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$. The equation (3. 11) are direct consequences of (1. 31) and (1. 33).

The tensor field $\tilde{S}=\tilde{N}^H$ defined by (1. 34) has components of the form

(3. 12) $$\begin{aligned}\tilde{S}_{ji}{}^h&=N_{cb}{}^aE_j{}^cE_i{}^bE^h{}_a\\&=\tilde{N}_{ji}{}^h-2(h_c{}^eh_b{}^dh_{ed})E_j{}^cE_i{}^bE^h\\&=\tilde{N}_{ji}{}^h-2\tilde{h}_j{}^t\tilde{h}_i{}^s\tilde{h}_{ts}E^h\end{aligned}$$

becouse of (3. 9).

Remark. If we suppose that $\tilde{H}$ satisfies the condition

$$\tilde{H}^2=-I+\tilde{\eta}\otimes\tilde{C},\ \text{i.e.,}$$

$$\tilde{h}_i{}^t\tilde{h}_t{}^h=-\delta_i^h+E_iE^h,$$

then we easily find

$$\begin{aligned}2\tilde{h}_j{}^t\tilde{h}_i{}^s\tilde{h}_{ts}&=2\tilde{h}_{ji}\\&=-(\tilde{\nabla}_jE_i-\tilde{\nabla}_iE_j)\end{aligned}$$

by virtue of (2. 32). Thus, in this case, we have the following expression of $\tilde{S}_{ji}{}^h$:

$$\tilde{S}_{ji}{}^h=\tilde{N}_{ji}{}^h+(\tilde{\nabla}_jE_i-\tilde{\nabla}_iE_j)E^h.$$

This expression of $\tilde{S}_{ji}{}^h$ was introduced by Sasaki and Hatakeyama, when $(\tilde{H}, \tilde{C}, \tilde{\eta})$ defines an almost contact structure (cf. Sasaki [4], Sasaki and Hatakeyama [5], [6]).

Coming back to our case, if we assume that the Nijenhuis tensor N of H

vanishes identically in the base space M, then we have

$$h_c{}^e\nabla_e h_{ba}-h_b{}^e\nabla_e h_{ca}-(\nabla_c h_b{}^e-\nabla_b h_c{}^e)h_{ea}=0, \tag{3.13}$$

which is equivalent to $N_{cba}=0$, where $N_{cba}=N_{cb}{}^e g_{ea}$. On the other hand, we have from (3.6)

$$\nabla_d h_{cb}+\nabla_c h_{bd}+\nabla_b h_{dc}=0. \tag{3.14}$$

Transvecting $g^{de}h_{ea}$ to (3.14), we find

$$(\nabla_c h_b{}^e-\nabla_b h_c{}^e)h_{ea}=h_a{}^e\nabla_e h_{cb}. \tag{3.15}$$

If we substitute (3.15) in (3.13), we obtain

$$h_c{}^e\nabla_e h_{ba}+h_b{}^e\nabla_e h_{ac}-h_a{}^e\nabla_e h_{cb}=0 \tag{3.16}$$

and, changing cyclically the indices a, b and c in (3.16),

$$h_b{}^e\nabla_e h_{ac}+h_a{}^e\nabla_e h_{cb}-h_c{}^e\nabla_e h_{ba}=0. \tag{3.17}$$

Thus, if we add these two equations (3.16) and (3.17), then we get

$$h_b{}^e\nabla_e h_{ac}=0, \quad \text{i.e.,} \quad h_b{}^e\nabla_e h_c{}^a=0. \tag{3.18}$$

Conversely, if we assume that the equation (3.18) holds in M, then, taking account of (3.14) or (3.15), we see that the equation (3.13) holds, i.e., that the Nijenhuis tensor N of H vanishes identically in M. Therefore, taking account of (3.3), (3.14) and Proposition 1.8, we have

PROPOSITION 3.1. *In a fibred space $\tilde{M}$ with invariant Riemannian metric, the following five conditions* (a), (b), (c), (d) *and* (e) *are equivalent to each other*:

(a) *The Nijenhuis tensor N of H vanishes identically in the base space M.*

(b) *The given fibred space is normal, i.e., $\tilde{S}=0$ in $\tilde{M}$.*

(c) *The Nijenhuis tensor $\tilde{N}$ of $\tilde{H}$ has components of the form*

$$\begin{aligned}\tilde{N}_{ji}{}^h&=2\tilde{h}_j{}^t\tilde{h}_i{}^s\tilde{h}_{ts}E^h\\&=2(h_c{}^e h_b{}^d h_{ed})E_j{}^c E_i{}^b E^h\end{aligned}$$

in the total space $\tilde{M}$.

(d) $$h_c{}^e\nabla_e h_b{}^a=0, \ \textit{i.e.,}\ \nabla_{HX}H=0$$

for any element X of $\mathcal{T}_0^1(M)$.

(e) $$\begin{aligned}\tilde{h}_j{}^t\tilde{\nabla}_t\tilde{h}_i{}^h&=\tilde{h}_j{}^t\tilde{h}_t{}^s\tilde{h}_s{}^h E_i+\tilde{h}_j{}^t\tilde{h}_t{}^s\tilde{h}_{is}E^h\\&=(h_c{}^e h_e{}^d h_d{}^a)E_j{}^c E_i E^h{}_a+(h_c{}^e h_e{}^d h_{bd})E_j{}^c E_i{}^b E^h\end{aligned}$$

in the total space $\tilde{M}$.

If we assume in Proposition 3. 1 that the second fundamental tensor H is non-singular everywhere in the base space M, then, taking account of (3. 3), we have

PROPOSITION 3. 2. *A fibred space $\tilde{M}$ with invariant Riemannian metric such that the second fundamental tensor H is non-singular is normal, i.e.,* $\tilde{S}=0$, *if and only if one of the following equivalent conditions* (a) *and* (b) *is satisfied*:

(a) $$\nabla H=0, \text{ i.e., } \nabla_c h_b{}^a=0$$

in the base space M.

(b) $$\tilde{\nabla}_j \tilde{h}_i{}^h=\tilde{h}_j{}^s\tilde{h}_s{}^h E_i+\tilde{h}_j{}^s\tilde{h}_{is}E^h$$

$$=(h_c{}^e h_e{}^a)E_j{}^c E_i E^h{}_a+(h_c{}^e h_{be})E_j{}^c E_i{}^b E^h$$

in the total space $\tilde{M}$.

Taking account of Propositions 1. 7 and 3. 1, we have

PROPOSITION 3. 3. *In a fibred space $\tilde{M}$ with invariant Riemannian metric, the following three conditions* (a), (b) *and* (c) *are equivalent to each other*:

(a) $$\tilde{N}=0$$

in the total space $\tilde{M}$.

(b) $$h_c{}^e\nabla_e h_b{}^a=0, \qquad H^3=0$$

in the base space M.

(c) $$\tilde{h}_j{}^s\tilde{\nabla}_s\tilde{h}_i{}^h=0, \qquad \tilde{H}^3=0$$

in the total space $\tilde{M}$.

§ 4. Geodesics.

Let there be given, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, a curve $\tilde{C}$ expressed by equations $x^h=x^h(t)$ in a neighborhood $\tilde{U}$ of $\tilde{M}$, t being a parameter. Denoting by C the image $\pi(\tilde{C})$ of $\tilde{C}$ by the projection π: $\tilde{M}\to M$, we may assume that C is expressed by equations $\xi^a=\xi^a(t)$ in $U=\pi(\tilde{U})$, where the functions $\xi^a(t)$ in the right-hand side are defined by $\xi^a(t)=\xi^a(x^h(t))$, the functions $\xi^a(x^h)$ being those appearing in (2. 2). Then we find along C

$$\frac{d\xi^a}{dt}=E_i{}^a\frac{dx^i}{dt}$$

and hence, differentiating covariantly both sides, we obtain

$$\frac{\delta^2 \xi^a}{dt^2} = E_i{}^a \frac{\delta^2 x^i}{dt^2} + \frac{d\xi^b}{dt} E^j{}_b(\overset{*}{\nabla}_j E_i{}^a) \frac{dx^i}{dt}, \tag{4.1}$$

where we have put

$$\frac{\delta^2 x^h}{dt^2} = \frac{d^2 x^h}{dt^2} + \{_j{}^h{}_i\} \frac{dx^j}{dt} \frac{dx^i}{dt}, \quad \frac{\delta^2 \xi^a}{dt^2} = \frac{d^2 \xi^a}{dt^2} + \{_c{}^a{}_b\} \frac{d\xi^c}{dt} \frac{d\xi^b}{dt}.$$

Substituting (2.32) in (4.1), we find along C

$$\frac{\delta^2 \xi^a}{dt^2} = E_i{}^a \frac{\delta^2 x^i}{dt^2} + \left(E_i \frac{dx^i}{dt}\right) h_b{}^a \frac{d\xi^b}{dt}. \tag{4.2}$$

A curve $\tilde{C}$ in $\tilde{M}$ is said to be *horizontal* if its tangent vector is horizontal at each point of $\tilde{C}$. Thus we have from (4.2)

PROPOSITION 4.1. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the projection C of a geodesic $\tilde{C}$ given in $\tilde{M}$ is also a geodesic in the base space M with respect to the induced metric g, if and only if one of the following two conditions* (a) *and* (b) *holds:*

(a) *$\tilde{C}$ is a horizontal geodesic in $\tilde{M}$.*

(b) *The tangent vector $\tilde{v}$ of $\tilde{C}$ satisfies*

$$\tilde{H}\tilde{v} = \tilde{a}\tilde{v} \qquad or \qquad Hv = av$$

along $\tilde{C}$, v being the projection $p\tilde{v}$ of $\tilde{v}$ and tangent to $\tilde{C}$, where $\tilde{a}$ and a are functions along $\tilde{C}$ and C respectively (cf. Proposition 4.4).

Taking account of Proposition 1.5 and 4.1, we have

PROPOSITION 4.2. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the projection of a geodesic given arbitrarily in $\tilde{M}$ is also a geodesic in the base space M with respect to the induced metric g, if and only if, h, H, $\tilde{h}$ and $\tilde{H}$ vanish identically, or, equivalently, if and only if the given fibred space $\tilde{M}$ is locally trivial.*

When a curve $\tilde{C}$ is horizontal in $\tilde{M}$ and a curve C is the projection $\pi(\tilde{C})$ of $\tilde{C}$ in the base space M, the curve $\tilde{C}$ is called a *horizontal lift* of C. Let C be a curve in M and be expressed by equations $\xi^a = \xi^a(t)$ in a neighborhood U of M. If its horizontal lift $\tilde{C}$ is expressed by equations $x^h = x^h(t)$ in a neighborhood $\tilde{U}$ of $\tilde{M}$ such that $U = \pi(\tilde{U})$, then the functions $x^h(t)$ should satisfy the differential equations

$$\frac{dx^h}{dt} = E^h{}_a \frac{d\xi^a}{dt}$$

along $\tilde{C}$ in $\tilde{U}$. Differentiating covariantly both sides, we find along the lift $\tilde{C}$

$$\frac{\delta^2 x^h}{dt^2} = E^h{}_a \frac{\xi^2 \xi^a}{dt^2} + E^j{}_b(\overset{*}{\nabla}_j E^h{}_a) \frac{d\xi^b}{dt} \frac{d\xi^a}{dt}$$

which reduces to

(4. 3) $$\frac{\delta^2 x^h}{dt^2} = E^h{}_a \frac{\delta^2 \xi^a}{dt^2}$$

because of (2. 31), since $h_{cb} + h_{bc} = 0$ holds. Thus we have

PROPOSITION 4. 3. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, any horizontal lift $\tilde{C}$ of a geodesic C given arbitrarily in the base space M is also a geodesic in $\tilde{M}$, and the projection π: $\tilde{M} \to M$ preserves the affine parameters on $\tilde{C}$ and on C.*

PROPOSITION 4. 4. *If, for a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, there are given an arbitrary point $\tilde{P}$ in $\tilde{M}$ and an arbitrary horizental vector $\tilde{V}$ at $\tilde{P}$, then there exists always a unique horizontal geodesic passing through the point $\tilde{P}$ and being tangent to $\tilde{V}$ at $\tilde{P}$.*

PROPOSITION 4. 5. *If a fibred space $\tilde{M}$ with invariant Riemannian metric is complete with respect to the given invariant metric $\tilde{g}$, then the base space M is also complete with respect to the induced metric g.*

We assume that, along any horizontal geodesic in a fibred space $\tilde{M}$ with invariant Riemannian metric, any affine parameter takes an arbitrarily given real value. If this is the case, the fibred space $\tilde{M}$ is said to be *horizontally complete*. We now have

PROPOSITION 4. 6. *In a fibred space $\tilde{M}$ with invariant Riemannian metric, the base space M is complete with respect to the induced metric if the given fibred space $\tilde{M}$ is horizontally complete.*

§ 5. Structure equations and curvatures.

Let there be given a fibred space with invariant Riemannian metric $\tilde{g}$. Then, as were obtained in § 4 of [11], the following *structure equations* can be established:

(5. 1) $$\tilde{K}_{dcb}{}^a = K_{dcb}{}^a - (h_d{}^a h_{cb} - h_c{}^a h_{db}) + 2h_{dc} h_b{}^a,$$

(5. 2) $$\tilde{K}_{dcb}{}^0 = \nabla_d h_{cb} - \nabla_c h_{db},$$

(5. 3) $$\tilde{K}_{doo}{}^a = -h_d{}^e h_e{}^a$$

by virtue of the Ricci formulas (3. 1) and (3. 2), where we have put

$$(5.4)\qquad \begin{aligned} \tilde{K}_{dcb}{}^{a} &= \tilde{K}_{kji}{}^{h} E^{k}{}_{d} E^{j}{}_{c} E^{i}{}_{b} E_{h}{}^{a}, \\ \tilde{K}_{dcb}{}^{o} &= \tilde{K}_{kji}{}^{h} E^{k}{}_{d} E^{j}{}_{c} E^{i}{}_{b} E_{h}, \\ \tilde{K}_{doo}{}^{a} &= \tilde{K}_{kji}{}^{h} E^{k}{}_{d} E^{j} E^{i} E_{h}{}^{a}, \end{aligned}$$

$\tilde{K}_{kji}{}^{h}$ being the components of the curvature tensor $\tilde{K}$ of the invariant Riemannian metric $\tilde{g}$. The equation (5. 1), (5. 2), and (5. 3) are called the *co-Gauss equation*, the *co-Codazzi equation* and the *co-Ricci equation*, respectively.

The curvature tensor $\tilde{K}$ of the invariant Riemannian metric $\tilde{g}$ is obviously invariant. The condition $p\tilde{K}=K$ is equivalent to the condition

$$(h_{d}{}^{a} h_{cb} - h_{c}{}^{a} h_{db}) - 2h_{dc} h_{b}{}^{a} = 0$$

by virtue of (5. 1), (5. 2) and (5. 3), where K is the curvature tensor of the induced metric g in M. Transvecting g^{cb} to both sides of the equation above and contracting with respect to the indices a and d, we get the equation

$$(5.5)\qquad g^{ab} g^{cb} h_{ac} h_{db} = 0$$

by virtue of $h_{cb}=-h_{bc}$. The equation (5. 5) implies $h_{cb}=0$. Thus, taking account of Proposition 1. 5, we have

PROPOSITION 5. 1. *For a fibred space with invariant Riemannian metric $\tilde{g}$, the projection $p\tilde{K}$ of the curvature tensor $\tilde{K}$ of the invariant Riemannian metric $\tilde{g}$ coincides with the curvature tensor K of the Riemannian metric g induced in the base space M, if and only if the given fibred space is locally trivial.*

Denote by $\tilde{\gamma}(\tilde{X}, \tilde{Y})$ the sectional curvature with respect to the section determined by two vectors $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$ and by $\gamma(X, Y)$ the sectional curvature with respect to the section determined by two vectors X and Y in the base space M. Then we have

$$\tilde{\gamma}(\tilde{X}, \tilde{Y}) = \frac{\tilde{K}_{kjih} \tilde{X}^{h} \tilde{Y}^{j} \tilde{X}^{i} \tilde{Y}^{h}}{(g_{kh} g_{ji} - g_{jh} g_{ki}) \tilde{X}^{k} \tilde{Y}^{j} \tilde{X}^{i} \tilde{Y}^{h}}$$

and the corresponding formula for $\gamma(X, Y)$, where $\tilde{X}^{h}$ and $\tilde{Y}^{h}$ are the components of $\tilde{X}$ and $\tilde{Y}$ respectively and

$$\tilde{K}_{kjih} = \tilde{K}_{kji}{}^{t} g_{th}.$$

Thus, taking account of (5. 1), we find

$$(5.6)\qquad \gamma(X, Y) - p\{\tilde{\gamma}(X^{L}, Y^{L})\} = 3\{h(X, Y)\}^{2} |X \wedge Y|^{2} \geqq 0$$

for any two vector fields X and Y in M, where we have put

$$|X\wedge Y|^2=(g_{ki}g_{jh}-g_{ji}g_{kh})X^k Y^j X^i Y^h, \tag{5.7}$$

X^h and Y^h being the componnets of X and Y respectively. Therefore, taking account of Proposition 1. 5 and (5. 6), we have

PROPOSITION 5. 2. *In any fibred space $\tilde{M}$ with invariant Riemannian metric, the sectional curvatures satisfy the inequality*

$$\gamma(X, Y)\geqq p\{\tilde{\gamma}(X^L, Y^L)\}$$

for any elements X and Y of $\mathcal{T}_0^1(M)$, M being the base space. The equality $\gamma(X, Y)=p\{\tilde{\gamma}(X^L, Y^L)\}$ holds for any elements X and Y of $\mathcal{T}_0^1(M)$ if and only if the given fibred space is locally trivial

As a consequence of (5. 3), we obtain

$$\begin{aligned}\tilde{\gamma}(\tilde{X}, \tilde{C})&=-\tilde{g}(\tilde{X}, \tilde{H}^2\tilde{X})|\tilde{X}\wedge\tilde{C}|^2\\&=\tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{X})|\tilde{X}\wedge\tilde{C}|^2\geqq 0\end{aligned} \tag{5.8}$$

for any element $\tilde{X}$ of $\mathcal{T}_0^1(\tilde{M})$, where $|\tilde{X}\wedge\tilde{Y}|^2$ is defined in $\tilde{M}$ for two vector fields $\tilde{X}$ and $\tilde{Y}$ by an equation similar to (5. 7). Thus, taking account of Proposition 1. 5 and (5. 8), we have

PROPOSITION 5. 3. *For any fibred space $\tilde{M}$ with invariant Riemannian metric, the sectional curvature with respect to any section containing the vector $\tilde{C}$ is non-negative at each point of $\tilde{M}$. The sectional curvature with respect to any section containing the vector $\tilde{C}$ vanishes at each point of $\tilde{M}$, if and only if the given fibred space is locally trivial. The sectional curvature with respect to any section containing the vector $\tilde{C}$ is non-zero at each point of $\tilde{M}$, if and only if the second fundamental tensor H is of the maximum rank n in the base space M, where M is n-dimensional.* (*If this is the case, $n=\dim M$ is necessarily even.*)

If we assume now that the sectional curvature $\tilde{\gamma}(\tilde{X}, \tilde{C})$ is a non-zero constant A for any element $\tilde{X}$ of $\mathcal{T}^{H}{}_0^1(\tilde{M})$, then by virtue of (5. 8) we get

$$A=c^2>0,$$

$$h_b{}^e h_e{}^a=-c^2\delta_b^a, \qquad \text{i.e.,} \qquad H^2=-c^2 I,$$

which implies

$$f_b{}^e f_e{}^a=-\delta_b^a, \qquad \text{i.e.,} \qquad f^2=-I, \tag{5.9}$$

$f_b{}^a$ being defined by

$$f_b{}^a = \frac{1}{c} h_b{}^a, \qquad \text{i.e.,} \quad f = \frac{1}{c} H. \tag{5.10}$$

On putting

$$f_{cb} = f_c{}^e g_{eb},$$

we have

$$\begin{aligned} &f_{cb} + f_{bc} = 0, \\ &\nabla_c f_{ba} + \nabla_b f_{ac} + \nabla_a f_{cb} = 0 \end{aligned} \tag{5.11}$$

as consequences of (3.6) and $h_{cb} + h_{bc} = 0$. A set (g, f) of a Riemannian metric g and a tensor field f of type (1, 1) is called an *almost Kählerian structure*, if (g, f) satisfies the conditions (5.9) and (5.11). Thus we have

PROPOSITION 5.4. *If, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the sectional curvature with respect to any section coutaining the vector $\tilde{C}$ is a non-zero constant A at each point $\tilde{P}$ of $\tilde{M}$, A being independent of $\tilde{P}$, then the constant A should be positive and the set (g, f) defines an almost Kählerian structure in the base space M, g being the induced metric in M and f being defined by $f = (1/c)H$, where H is the second fundamental tensor in M and $A = c^2$ $(c \neq 0)$. If this is the case, the base space M is necessarily even-dimensional.*

Conversely if the set (g, f), f being defined by $f = (1/c)H$ with a constant c, is an almost Kählerian structure in M, then the sectional curvature with respect to any section containing the vector $\tilde{C}$ is constant and equal to c^2 at each point of $\tilde{M}$.

Let $\tilde{K}_{ji}$ be the components of the Ricci tensor $\tilde{R}$ of the invariant Riemannian metric $\tilde{g}$ in the total space $\tilde{M}$. Then we have by definition

$$\tilde{K}_{ji} = \tilde{K}_{tji}{}^t,$$

$\tilde{K}_{ji}$ being symmetric in j and i. If we put

$$\begin{aligned} &\tilde{K}_{cb} = \tilde{K}_{ji} E^j{}_c E^i{}_b, \\ &\tilde{K}_{co} = K_{ji} E^j{}_c E^i, \qquad \tilde{K}_{oo} = K_{ji} E^j E^i, \end{aligned} \tag{5.12}$$

and, if we take account of (5.1), (5.2) and (5.3), then we have

$$\begin{aligned} &\tilde{K}_{cb} = K_{cb} + 2h_c{}^e h_{eb}, \\ &\tilde{K}_{co} = -\nabla_e h_c{}^e, \\ &\tilde{K}_{oo} = g^{dc} g^{ba} h_{db} h_{ca} \geqq 0, \end{aligned} \tag{5.13}$$

where K_{cb} denote the components of the Ricci tensor R of the induced metric g in the base space M and is given by

$$K_{cb}=K_{ecb}{}^{e}.$$

Thus, taking account of Proposition 1.5 and the last equation of (5.13), we have

PROPOSITION 5.5. *For any fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the inequality*

$$\tilde{R}(\tilde{C},\tilde{C})\geqq 0$$

holds everywhere in $\tilde{M}$, where $\tilde{R}$ denotes the Ricci tensor of $\tilde{g}$. The equality $\tilde{R}(\tilde{C},\tilde{C})=0$ holds everywhere in $\tilde{M}$, if and only if the given fibred space is locally trivial.

As a consequence of the first equation of (5.13), we have

$$R(X,X)=p\{\tilde{R}(X^L,X^L)\}=2g(HX,HX)\geqq 0 \tag{5.14}$$

for any element X of $\mathcal{T}_0^1(M)$. Thus, taking account of Proposition 1.5 and (5.14), we have

PROPOSITION 5.6. *In any fibred space with invariant Riemannian metric $\tilde{g}$, the inequality*

$$R(X,X)\geqq p\{\tilde{R}(X^L,X^L)\}$$

holds for any element X of $\mathcal{T}_0^1(M)$, M being the base space, where $\tilde{R}$ and R denote the Ricci tensors of the invariant metric $\tilde{g}$ and the metric g induced in M respectively. The equality $R(X,X)=p\{\tilde{R}(X^L,X^L)\}$ holds for any elements X of $\mathcal{T}_0^1(M)$, if and only if the given fibred space is locally trivial

Taking account of the second equation of (5.13), we obtain

$$\tilde{R}(\tilde{C},\tilde{X})=0 \qquad \text{for} \quad \tilde{X}\in\mathcal{T}^H(\tilde{M}) \tag{5.15}$$

if and only if

$$\nabla_e h_b{}^e=0 \tag{5.16}$$

holds. The condition (5.15) is equivalent to the fact that the vector $\tilde{C}$ is a proper vector of the Ricci tensor $\tilde{R}$. On the other hand, if we take account of (3.6), we see that the condition (5.16) is equivalent to the condition that the second fundamental tensor h is harmonic in the base space M. Thus we have

PROPOSITION 5.7. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$,* *the second fundamental tensor h is harmonic in the base space M if and only if the structure field $\tilde{C}$ is a proper vector of the Ricci tensor $\tilde{R}$ of the invariant metric $\tilde{g}$ everywhere in $\tilde{M}$.*

Denote by $\tilde{k}$ and k the curvature scalars of the invariant Riemannian metric $\tilde{g}$ and the induced metric g respectively. Then we get by definition

$$\tilde{k}=\tilde{K}_{ji}g^{ji}, \qquad k=K_{cb}g^{cb}. \tag{5.17}$$

Thus, as a consequence of (5.13), we obtain

$$k-p\tilde{k}=g^{dc}g^{ba}h_{db}h_{ca}\geqq 0 \tag{5.18}$$

at each point of the base space M. Therefore, taking account of inequality Proporition 1.5 and (5.18), we have

PROPOSITION 5.8. *For any fibred space with invariant Riemannian metric $\tilde{g}$, the inequality*

$$k\geqq p\tilde{k}$$

holds everywhere in the base space M, where $\tilde{k}$ and k are the curvature scalars of the invariant Riemannian metric $\tilde{g}$ and the induced metric g respectively. The equality $k=p\tilde{k}$ holds everywhere in M, if and only if the given fibred space is locally trivial.

Consider two vector fields $\tilde{X}$ and $\tilde{Y}$ in the total space $\tilde{M}$. We denote by $\tilde{K}(\tilde{X},\tilde{Y})$ a tensor field of type (1, 1) defined by the equation

$$\tilde{K}(\tilde{X},\tilde{Y})\tilde{Z}=(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}-\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z})-\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}$$

for any element $\tilde{Z}$ of $\mathcal{T}^1_0(\tilde{M})$. The tensor field $\tilde{K}(\tilde{X},\tilde{Y})$ is called the *curvature transformation* with respect to $\tilde{X}$ and $\tilde{Y}$, when it is regarded as an endomorphism of the tangent bundle $T(\tilde{M})$. We easily see that $\tilde{K}(\tilde{X},\tilde{Y})$ has components of the form

$$\tilde{T}_i{}^h=\tilde{X}^k\tilde{Y}^j\tilde{K}_{kji}{}^h,$$

$\tilde{X}^h$ and $\tilde{Y}^h$ being the components of $\tilde{X}$ and $\tilde{Y}$ respectively. For any element $\tilde{Z}$ of $\mathcal{T}^1_0(\tilde{M})$, the vector field $\tilde{K}(\tilde{X},\tilde{Y})\tilde{Z}$ has components of the form

$$V^h=\tilde{X}^h\tilde{Y}^j\tilde{Z}^i\tilde{K}_{kji}{}^h,$$

$\tilde{Z}^h$ being the components of $\tilde{Z}$.

If we take account of (3.6) and (5.2), we easily see that the condition

$$\nabla_c h_{ba}=0, \qquad \text{i.e.} \qquad \nabla h=0$$

is equivalent to the condition that the vector field $\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z}$ is horizontal for any three horizontal vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$, or equivalently to the condition that the curvature transformation $\tilde{K}(\tilde{X}, \tilde{Y})$ preserves the horizontal plane invariant at each point of $\tilde{M}$, $\tilde{X}$ and $\tilde{Y}$ being arbitrary horizontal vector fields. Thus, taking account of (3. 3), we have

PROPOSITION 5. 9. *For a fibred space $\tilde{M}$ with invariant Riemannian metric, the following two conditions* (a) *and* (b) *are equivalent to each other*:

(a) *The second fundamental tensor H is covariantly constant in the base space M, that is,*

$$\nabla_c h_b{}^a=0, \quad i.e., \quad \nabla H=0 \quad in \quad M,$$

or equivalently

$$\tilde{\nabla}_j \tilde{h}_i{}^h=\tilde{h}_j{}^s\tilde{h}_s{}^h E_i+\tilde{h}_j{}^s\tilde{h}_{si}E^h \quad in \quad \tilde{M}.$$

(b) *The curvature transformation with respect to any two horizontal vector fields preserves the horizontal plane invariant at each point of $\tilde{M}$.*

Taking account of Propositions 3. 2 and 5. 9, we have

PROPOSITION 5. 10. *A fibred space $\tilde{M}$ with invariant Riemannian metric is normal, i.e., $\tilde{S}=\tilde{N}^H$ defined by* (1. 34) *vanishes identically in $\tilde{M}$, the second fundamental tensor H in the base space M being assumed to be non-singular everywhere, if and only if one of the conditions* (a) *and* (b) *mentioned in Proposition* 5. 9 *is satisfied.*

As a corollary to Propositions 5. 4 and 5. 9, we have

PROPOSITION 5. 11. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the set $(g, (1/c)H)$ is a Kählerian structure in the base space M, where c is a non-zero constant, g and H being the induced metric and the second fundamental tensor in M respectively, if and only if the following two couditions* (a) *and* (b) *are satisfied*:

(a) *The sectional curvature with respect to any section containing the vector $\tilde{C}$ is a non-zero constant A at each point $\tilde{P}$ of $\tilde{M}$, A being independent of $\tilde{P}$.*

(b) *The curvature transformation $\tilde{K}(\tilde{X}, \tilde{Y})$ with respect to any two horizontal vector fields $\tilde{X}$ and $\tilde{Y}$ preserves the horizontal plane at each point of $\tilde{M}$.*

In a normal fibred space with invariant Riemannian metric $\tilde{g}$, the set $(g, (1/c)H)$ is a Kählerian structure in M, if and only if the condition (a) *is satisfied.*

In Proposition 5. 11, the Kählerian structure $(g, (1/c)H)$ is said to be *induced in the base space M.*

§ 6. Special cases.

Locally flat fibred spaces. We now suppose that, in a fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ with invariant Riemannian metric $\tilde{g}$, the invariant metric $\tilde{g}$ is locally flat, i.e., that $\tilde{K}_{kji}{}^{h}=0$ everywhere in $\tilde{M}$. Then the equation (5. 3) reduces to

$$h_d{}^e h_e{}^a=0,$$

which implies (5. 5). Thus we have $h_{cb}=0$.

Substituting $\tilde{K}_{dcb}{}^{a}=\tilde{K}_{kji}{}^{h}E^k{}_dE^j{}_cE^i{}_bE_h{}^a=0$ and $h_{cb}=0$ in (5. 1), we find $K_{dcb}{}^{a}=0$. Therefore, taking account of Proposition 5. 1, we have

PROPOSITION 6. 1. *For a fibred space with invariant Riemannian metric $\tilde{g}$, the invariant metric $\tilde{g}$ is locally flat, if and only if the given fibred space is locally trivial and the metric g induced in the base space M is locally flat.*

Einstein fibred spaces. We assume that, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is an Einstein metric, i.e.,

$$\tilde{K}_{ji}=\frac{\tilde{k}}{n+1}g_{ji} \tag{6. 1}$$

where $\tilde{K}_{ji}$ and $\tilde{k}$ are the Ricci tensor and the curvature scalar of $\tilde{g}$ respectively, $\tilde{M}$ being $(n+1)$-dimensional. Then, combining (5. 13) and (6. 1), we find

$$\nabla_e h_b{}^e=0, \qquad \tilde{k}=(n+1)g^{kj}g^{ih}\tilde{h}_{ki}\tilde{h}_{jh}\geqq 0. \tag{6. 2}$$

The equation $\nabla_e h_b{}^e=0$ together with (3. 6) shows that the second fundamental tensor h in M is harmonic. Thus, taking account of Proposition 1. 5, (3. 3), (3. 6) and (6. 2), we have

PROPOSITION 6. 2. *If, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is an Einstein metric, then the following two conditions* (a) *and* (b) *hold*:

(a) *The scalar curvature $\tilde{k}$ of $\tilde{g}$ is non-negative in $\tilde{M}$.*

(b) *The second fundamental tensor h in the base space M is a harmonic tensor, or equivalently, the equation*

$$\tilde{\nabla}_s\tilde{h}_i{}^s=(\tilde{h}_s{}^t\tilde{h}_t{}^s)E_i$$

holds in $\tilde{M}$ (cf. Proposition 5. 7).

When the invariant Riemannian metric $\tilde{g}$ is an Einstetn metric, the curvature scalar $\tilde{k}$ vanishes identically in $\tilde{M}$ if and only if the given fibred space is locally trivial and the induced metric g is an Einstein metric with vanishing curvature scalar in the base space M.

We assume now that, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the invariant metric $\tilde{g}$ and the metric g induced in the base space M are both Einstein metrics, i.e.,

$$\tilde{K}_{ji}=\frac{\tilde{k}}{n+1}g_{ji}, \qquad K_{cb}=\frac{k}{n}g_{cb}, \tag{6.3}$$

$\tilde{k}$ and k being the curvature scalars of $\tilde{g}$ and g respectively, where $\dim\tilde{M}=n+1$ and $\dim M=n$. Then, combining (5.13) and (6.3), we get the following equations:

$$\begin{gathered} h_b{}^e h_e{}^a=-\frac{1}{2}\left(\frac{k}{n}-\frac{p\tilde{k}}{n+1}\right)\delta_b^a, \\ \nabla_e h_b{}^c=0, \qquad \frac{p\tilde{k}}{n+1}g^{dc}g^{ba}h_{db}h_{ca}\geqq 0. \end{gathered} \tag{6.4}$$

On the other hand, taking account of Proposition 5.8 we have $k\geqq p\tilde{k}$, which implies

$$\frac{1}{2}\left(\frac{k}{n}-\frac{p\tilde{k}}{n+1}\right)=c^2>0, \tag{6.5}$$

if $\tilde{k}\neq 0$. We assume that $\tilde{k}$ and k are constant (as is well known, $\tilde{k}$ and k are constant if $n\geqq 3$). If we put

$$f_b{}^a=\frac{1}{c}h_b{}^a, \tag{6.6}$$

then, as consequences of (6.4) and $h_{cb}+h_{bc}=0$, we find

$$\begin{gathered} f_b{}^e f_e{}^a=-\delta_b^a, \qquad f_{cb}+f_{bc}=0, \qquad \nabla_e f_c{}^e=0, \\ \tilde{k}=-n(n+1)c^2, \qquad k=n(n+2)c^2, \end{gathered} \tag{6.7}$$

where c is a constant and f_{cb} is defined by

$$f_{cb}=f_c{}^e g_{eb}. \tag{6.8}$$

Moreover, as a direct consequence of the identity (3.6), we get

$$\nabla_c f_{ba}+\nabla_b f_{ac}+\nabla_a f_{cb}=0. \tag{6.9}$$

Summing up (6.7), (6.8) and (6.9), we know that the set (g, f) defines an almost Kählerian structure in the base space M, where f is the tensor field of type (1, 1) with components $f_b{}^a$ and f_{cb} is a harmonic tensor field. Thus we have

PROPOSITION 6.3. *Suppose that, in a fibred space $\tilde{M}$ with invariant Riemannian* *metric $\tilde{g}$, the metric $\tilde{g}$ and the metric g induced in the base space M are both Einstein metrics with constant scalar carvature. Then $\tilde{M}$ and M are necessarily odd- and even-dimensional respectively, and, the set (g, f) is an almost Kählerian structure in M such that $f_{cb}=f_c{}^e g_{eb}$ is a harmonic tensor field, $f_b{}^a$ being the components of the tensor field f of type $(1, 1)$ defined by $f=-(1/c)H$, where H is the second fundamental tensor in M and c is a positive constant defined by*

$$\tilde{k}=n(n+1)c^2, \qquad k=n(n+2)c^2,$$

$\tilde{k}$ and k being the curvature scalars of $\tilde{g}$ and g respectively.

Conformally flat fibred spaces. We now assume that, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is conformally flat. Then we have by definition

$$(6.10) \qquad \tilde{K}_{kji}{}^h=\frac{1}{n-1}(\delta_k^h\tilde{K}_{ji}-\delta_j^h\tilde{K}_{ki}+\tilde{K}_{kt}g^{th}g_{ji}-\tilde{K}_{jt}g^{th}g_{ki})-\frac{\tilde{k}}{n(n-1)}(\delta_k^h g_{ji}-\delta_j^h g_{ki}),$$

if $\dim\tilde{M}=n+1>3$, where $\tilde{K}_{ji}$ and $\tilde{k}$ are respectively the Ricci tensor and the curvature scalar of $\tilde{g}$. Thus, taking account of (5.2) and (6.10), we find

$$(6.11) \qquad -\frac{1}{n-1}(g_{db}\tilde{K}_{co}-g_{cb}\tilde{K}_{do})=\nabla_d h_{cb}-\nabla_c h_{db},$$

and, transvecting g^{db}, we get

$$\tilde{K}_{co}=-\nabla_e h_c{}^e.$$

Comparing this equation with the second equation of (5.13), we obtain $\tilde{K}_{co}=0$, which implies together with (6.11)

$$\nabla_d h_{cb}-\nabla_c h_{db}=0.$$

Thus, taking account of (3.6), we have

$$\nabla_d h_{cb}=0, \quad \text{i.e.,} \quad \nabla h=0.$$

Consequently, if we take account of Proposition 3.1, we have

PROPOSITION 6.4. *If, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is conformally flat, then the second fundamental tensor H in the base space is covariant constant, i.e., $\nabla H=0$, and hence the given fibred space is normal, i.e., $\tilde{S}=0$.*

Fibred spaces of constant curvature. Suppose that, in a fibred space with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is of constant curvature. Then we have by definition

$$\tilde{K}_{kji}{}^{h} = \frac{\tilde{K}}{n(n+1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki}), \tag{6.12}$$

where $\tilde{k}$ is the curvature scalar of $\tilde{g}$ and a non-zero constant. Substituting (6.12) in (5.1), (5.2) and (5.3), we obtain respectively the equations

$$K_{dcb}{}^{a} = \frac{\tilde{k}}{n(n+1)}(\delta_d^a g_{cb} - \delta_c^a g_{db}) + (h_d{}^a h_{cb} - h_c{}^a h_{db} - 2h_{dc} h_b{}^a), \tag{6.13}$$

$$\nabla_d h_{cb} - \nabla_c h_{db} = 0, \tag{6.14}$$

$$h_d{}^e h_e{}^a = -\frac{\tilde{k}}{n(n+1)}\delta_d^a, \quad \text{i.e.,} \quad H^2 = -\frac{\tilde{k}}{n(n+1)}I. \tag{6.15}$$

If we contract with respect to indices a and d in (6.15), we find

$$\tilde{k} = -(n+1)h_d{}^e h_e{}^d > 0$$

by virtue of $\tilde{k} \neq 0$ and $h_{cb} = -h_{bc}$. If we put

$$c^2 = \frac{\tilde{k}}{n(n+1)} \qquad (c > 0) \tag{6.16}$$

and

$$f_b{}^a = \frac{1}{c} h_b{}^a, \quad \text{i.e.,} \quad f = \frac{1}{c}H, \tag{6.17}$$

then we find

$$f_b{}^e f_e{}^a = -\delta_b^a, \quad \text{i.e.,} \quad f^2 = -I \tag{6.18}$$

as a consequence of (6.15). On the other hand, taking account of Proposition 6.4, we obtain

$$\nabla_c f_b{}^a = 0, \quad \text{i.e.} \quad \nabla f = 0. \tag{6.19}$$

If we substitute (6.17) in the right-hand side of (6.13), we get

$$K_{dcb}{}^{a} = \frac{k}{n(n+1)}\{(\delta_d^a g_{cb} - \delta_c^a g_{db}) + (f_d{}^a f_{cb} - f_c{}^a f_{db} - 2f_{dc} f_b{}^a)\}, \tag{6.20}$$

where we have put

$$k = n(n+2)c^2 \tag{6.21}$$

and

$$f_{ba}=f_b{}^e g_{ea},$$

which satisfies the equation

$$f_{cb}+f_{bc}=0 \tag{6.22}$$

because of $h_{cb}+h_{bc}=0$. We note here that $n=\dim M$ is necessarily even because of (6.18). Thus, summing up (6.16), (6.18), (6.19), (6.20), (6.21) and (6.22), we have

PROPOSITION 6.5. *Suppose that, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ is of non-zero constant curvature $\tilde{k}$. Then the total space $\tilde{M}$ and the base space M are necessarily odd- and even-dimensional respectively, the curvature scalar $\tilde{k}$ is necessarily positive, and, the set (g, f) in the base space M is a Kählerian structure of positive constant holomorphic sectional curvature k, g being the metric induced in the base space M and f being the tensor field of type $(1, 1)$ defined by $f=(1/c)H$, where H is the second fundamental tensor in M and c is a positive constant defined by*

$$\tilde{k}=n(n+1)c^2, \qquad k=n(n+2)c^2 \qquad (n=\dim M)$$

(cf. Kurita [2], Tashiro and Tachibana [9]).

As a corollary to Proposition 6.5, we have

PROPOSITION 6.6. *Suppose that, in a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the metric $\tilde{g}$ in $\tilde{M}$ is of non-zero constant curvature $\tilde{k}$ and the metric g induced in the base space M is also of constant curvature k. Then $\tilde{M}$ and M are necessarily 3- and 2-dimensional respectively and the two curvatures $\tilde{k}$ and k should be positive.*

In connection with Proposition 6.5, we shall discuss the fibring of the odd-dimensional unit sphere S^{n+1} defined by an equation

$$(y_1)^2+(y_2)^2+\cdots+(y_{n+2})^2=1$$

with respect to rectangular coordinates $(y_1, y_2, \cdots, y_{n+2})$ defined in an even-dimensional Euclidean space E^{n+2}, where n is even and sometimes denoted by $n=2m$. Let us assume that there exists a fibred space $(S^{n+1}, M, \pi; \tilde{C}, \tilde{g})$ with invariant Riemannian metric $\tilde{g}$, where $\tilde{g}$ is the natural Riemannian metric induced in S^{n+1} and obviously of constant curvature 1. Since the structure field $\tilde{C}$ is a Killing vector field in S^{n+1}, there exists a vector field V in E^{n+2} such that V has components of the form

$$V_A=\sum_{B=1}^{n+2} a_{AB}y_B \qquad (A=1, 2, \cdots, n+2) \tag{6.23}$$

and the restriction of V to S^{n+1} coincides with $\tilde{C}$, where the matrix (a_{AB}) is constant and skew-symmetric. On the other hand, the structure field $\tilde{C}$ is Killing vector field of unit length. Then, as is well known, we get

$$\tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{C}, \tilde{C})=0 \tag{6.24}$$

for any vector field $\tilde{Y}$ in S^{n+1}.

If we take rectangular coordinates $(y_1, y_2, \cdots, y_{n+2})$ suitably in E^{n+2}, as a consequence of (6.24), we may assume that the components V_A $(A=1, 2, \cdots, n+2)$ given by (6.23) have the following form:

$$V_1=-y_2,\ V_2=y_1,\ V_3=-y_4,\ V_4=y_3, \cdots,\ V_{n+1}=-y_{n+2},\ V_{n+2}=y_{n+1}. \tag{6.25}$$

Thus, if we introduce in E^{n+2} complex coordinates $(Z_1, Z_2, \cdots, Z_{m+1})$ by

$$Z_1=y_1+\sqrt{-1}y_2,\ Z_2=y_3+\sqrt{-1}y_4, \cdots,\ Z_{m+1}=y_{n+1}+\sqrt{-1}y_{n+2} \qquad (n=2m),$$

then the components given by (6.25) reduces to

$$U_1=\sqrt{-1}Z_1,\ U_2=\sqrt{-1}Z_2, \cdots,\ U_{m+1}=\sqrt{-1}Z_{m+1}$$

with respect to $(Z_1, Z_2, \cdots, Z_{m+1})$. Therefore, the vector field V generates a 1-parameter group Φ of rotations $\tau(\theta)$ having the representation

$$\tau(\theta)=\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 & \cdots & 0 \\ 0 & e^{\sqrt{-1}\theta} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{\sqrt{-1}\theta} \end{pmatrix} \tag{6.26}$$

with respect to $(Z_1, Z_2, \cdots, Z_{m+1})$ in E^{n+2}. Then *each fibre in S^{n+1} is an orbit of the group Φ*. Consequently, the projection π: $S^{n+1}\to M$ coincides with the natural mapping

$$\pi:\ S^{n+1}\to CP(m) \qquad (n=2m),$$

where $CP(m)$ denotes the complex projective space of complex dimension m. Thus we have

PROPOSITION 6.7. *Let S^{n+1} be an odd-dimensional sphere (i.e., $n=2m$) with the natural Riemannian metric $\tilde{g}$ of positive curvature* 1. *If there exists a fibred space $(S^{n+1}, M, \pi;\ \tilde{C}, \tilde{g})$ with invariant Riemannian metric $\tilde{g}$, then its projection coincides with the natural projection π: $S^{n+1}\to CP(m)$, the base space M coinciding with the complex projective space $CP(m)$, and the metric g induced in M defines the natural Kählerian structure of positive constant holomorphic sectional curvature together* *with the second fundamental tensor H of the given fibred space.*

If we assume now that, for a normal fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the second fundamental tensor H is of the maximum rank n in the base space M, where $n=\dim M$ is necessarily even, then we obtain $\nabla H=0$, i.e., $\nabla_c h_b{}^a=0$ in M as a consequence of Proposition 3. 2. Thus, taking account of (5. 13), we find the following equations:

$$\begin{aligned}&\tilde{K}_{ji}=(K_{cb}+2h_c{}^e h_{eb})E_j{}^c E_i{}^b+\tilde{K}_{oo}E_j E_i,\\ (6.\,27)\qquad &\\ &\tilde{\nabla}_k\tilde{K}_{ji}=(\nabla_d K_{cb})E_k{}^d E_j{}^c E_i{}^b+P_{dc}E_k{}^d E_j{}^c E_i+P_{db}E_k{}^d E_j E_i{}^b+(P_{cb}+P_{bc})E_k E_j{}^c E_i{}^b,\end{aligned}$$

P_{cb} being defined by

$$P_{cb}=h_c{}^e K_{eb}-2h_c{}^e h_b{}^d h_{ed}-\tilde{K}_{00}h_{cb},$$

where $\tilde{K}_{oo}$ reduces to a positive constant. Therefore, taking account of the second equation of (6. 27), we see that the condition $\tilde{\nabla}_k\tilde{K}_{ji}=0$ is equivalent to the conditions

$$\nabla_d K_{cb}=0 \quad\text{and}\quad P_{cb}=0,$$

or equivalently to the condition

$$(6.\,28)\qquad K_{cb}=\tilde{K}_{oo}g_{cb}+2h_c{}^e h_{eb} \qquad \text{in}\quad M,$$

which is equivalent again to the condition

$$\tilde{K}_{ji}=\tilde{K}_{00}g_{ji}+4(h_c{}^e h_{eb})E_j{}^c E_i{}^b \qquad \text{in}\quad \tilde{M}.$$

Thus, taking account of (6. 27) and (6. 28), we have

PROPOSITION 6. 8. *Let $\tilde{M}$ be a normal fibred space with invariant Riemannian metrie $\tilde{g}$ and suppose that the second fundamental tensor H is of the maximum rank n in the base space M, where $n=\dim M$ is necessarily even. Then the equality*

$$p(\tilde{\nabla}\tilde{R})=\nabla R$$

holds, $\tilde{R}$ and R being the Ricci tensors of the invariant metric $\tilde{g}$ and the metric g induced in M respectively. The Ricci tensor $\tilde{R}$ is covariantly constant, i.e., $\tilde{\nabla}\tilde{R}=0$ in $\tilde{M}$, if and only if the Ricci tensor R has components of the form

$$K_{cb}=\tilde{K}_{oo}g_{cb}+2h_c{}^e h_{eb}$$

in M, where $\tilde{K}_{oo}=g^{dc}g^{ba}h_{db}h_{ca}$ is a positive constant.

We assume that the second fundamental tensor H is of the maximum rank n in the base space M, for a normal fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, where $n=\dim M$ is necessarily even. Then, taking account of (5. 1), (5. 2) and (5. 3), we easily see that the curvature tensor $\tilde{K}$ of the invariant Riemannian metric $\tilde{g}$ has components of the form

$$\begin{aligned}\tilde{K}_{kjih}=&Q_{dcba}E_k{}^dE_j{}^cE_i{}^bE_h{}^a\\ &+(h_c{}^eh_{ea})E_kE_j{}^cE_iE_h{}^a-(h_c{}^eh_{eb})E_kE_j{}^cE_i{}^bE_h\\ &-(h_d{}^eh_{ea})E_k{}^dE_jE_iE_h{}^a+(h_d{}^eh_{eb})E_k{}^dE_jE_i{}^bE_h\end{aligned}\tag{6. 29}$$

because of $\nabla H=0$, where we have put

$$\begin{gathered}Q_{dcba}=K_{dcba}-(h_{da}h_{cb}-h_{ca}h_{db}-2h_{dc}h_{ba}),\\ \tilde{K}_{kjih}=\tilde{K}_{kji}{}^tg_{th},\qquad K_{dcba}=K_{dcb}{}^eg_{ea}.\end{gathered}\tag{6. 30}$$

If we differentiate covariantly both sides of (6. 29), and, if we take account of (2. 32) and the identity $K_{dcb}{}^eh_e{}^a=K_{dce}{}^ah_b{}^e$, which is a direct consequence of $\nabla_ch_b{}^a=0$, then we obtain the equation

$$\begin{aligned}\tilde{\nabla}_l\tilde{K}_{kjih}=&(\nabla_eK_{dcba})E_l{}^eE_k{}^dE_j{}^cE_i{}^bE_h{}^a\\ &+h_e{}^f(Q_{fcba}-Q_{ca}g_{fb}+Q_{cb}g_{fa})E_l{}^eE_kE_j{}^cE_i{}^bE_h{}^a\\ &+h_e{}^f(Q_{dfba}-Q_{db}g_{fa}+Q_{da}g_{fb})E_l{}^eE_k{}^dE_jE_i{}^bE_h{}^a\\ &+h_e{}^f(Q_{dcfa}-Q_{ca}g_{fd}+Q_{da}g_{fc})E_l{}^eE_k{}^dE_j{}^cE_iE_h{}^a\\ &+h_e{}^f(Q_{dcbf}-Q_{db}g_{fc}+Q_{cb}g_{fd})E_l{}^eE_k{}^dE_j{}^cE_i{}^bE_h,\end{aligned}\tag{6. 31}$$

Q_{cb} being defined by

$$Q_{cb}=h_c{}^eh_{eb}.\tag{6. 32}$$

By virtue of (6. 31) and (6. 32), the condition $\tilde{\nabla}_l\tilde{K}_{kjih}=0$ is equivalent to the conditions

$$\begin{gathered}\nabla_eK_{dcba}=0,\\ Q_{dcba}-Q_{ca}g_{db}+Q_{cb}g_{da}=0.\end{gathered}\tag{6. 33}$$

On the other hand, we find

$$Q_{cb}=h_c{}^eh_{eb}=-c^2g_{cb}\tag{6. 34}$$

because of $\nabla_dQ_{cb}=0$ and $Q_{cb}=Q_{bc}$, since $\tilde{M}$ is an irreducible Riemannian manifold (cf. Proposition 6. 10).

If we substitute (6. 34) in the second equation of (6. 33), we have

$$K_{dcba}=c^2(g_{da}g_{cb}-g_{ca}g_{db}+f_{da}f_{cb}-f_{ca}f_{db}-2f_{dc}f_{ca}), \tag{6. 35}$$

$f_b{}^a$ and f_{cb} being defined respectively by

$$f_b{}^a=\frac{1}{c}h_b{}^a, \qquad f_{cb}=f_c{}^e g_{eb}.$$

Substituting (6. 33) and (6. 35) in (6. 29), we find

$$\tilde{K}_{kjih}=c^2(g_{kh}g_{ji}-g_{jh}g_{ki}),$$

which means that the invariant Riemannian metric $\tilde{g}$ is of positive constant curvature in $\tilde{M}$. Summing up, we have

PROPOSITION 6. 9. *Let $\tilde{M}$ be a normal fibred space with invariant Riemannian metric $\tilde{g}$ and assume that the second fundamental tensor H is of the maximum rank n in the base space M, where $n=\dim M$ is necessarily even. Then the equality*

$$p(\tilde{\nabla}\tilde{K})=\nabla K$$

holds, $\tilde{K}$ and K being the curvature tensors of the invariant metric $\tilde{g}$ and the induced metric g respectively. The invariant Riemannian metric $\tilde{g}$ is locally symmetric, i.e., $\tilde{\nabla}\tilde{K}=0$ in $\tilde{M}$, if and only if the invariant metric $\tilde{g}$ is of positive constant curvatnre.

Take a horizontal vector $\tilde{X}$ at a point $\tilde{\mathrm{P}}$ of a normal fibred space $\tilde{M}$ with invariant Riemannian metric, whose second fundamental tensor H is of the maximum rank n in the base space M, $n=\dim M$ being necessarily even. Then the curvature transformation $\tilde{K}(\tilde{X}, \tilde{C})$ has components of the form

$$\begin{aligned} A(\tilde{X})_i{}^h &= \tilde{X}^k E^j \tilde{K}_{kji}{}^h \\ &= \bar{X}_i E^h - E_i \bar{X}^h \end{aligned} \tag{6. 36}$$

by virtue of (6. 29), where $\bar{X}^h$ are the components of $H^2\tilde{X}$ and $\bar{X}_i=g_{is}\bar{X}^s$. Taking another horizontal vector $\tilde{Y}$ at the point $\tilde{\mathrm{P}}$, we know that the bracket product $[\tilde{K}(\tilde{X}, \tilde{C}), \tilde{Y}(\tilde{Y}, \tilde{C})]$ has components of the form

$$\begin{aligned} B(\tilde{X}, \tilde{Y})_i{}^h &= A_i{}^s(\tilde{X})A_s{}^h(\tilde{Y})-A_i{}^s(\tilde{Y})A_s{}^h(\tilde{X}) \\ &= \bar{X}_i\bar{Y}^h-\bar{Y}_i\bar{X}^h \end{aligned} \tag{6. 37}$$

as a consequence of (6. 36), where $\bar{Y}^h$ are the components of $H^2\tilde{Y}$ and $\bar{Y}_i=g_{is}\bar{Y}^s$. Therefore, linear combinations of matrices $(A(\tilde{X})_i{}^h)$ and $(B(\tilde{X}, \tilde{Y})_i{}^h)$, $\tilde{X}$ and $\tilde{Y}$ being arbitrary horizontal vectors at the point P, span the Lie algebra of the group $SO(n+1)$ of all rotations, because the tensor field H is of the maximum rank n. Since in our case the manifold $\tilde{M}$ is orientable, we thus have

PROPOSITION 6. 10. *If, for a normal fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the second fundamental tensor H is of the maximum rank n in the base space M, where $n=\dim M$ is necessarily even, then the homogeneous holonomy group of the Riemannian manifold $\tilde{M}$ coincides with the group $SO(n+1)$ of all rotations.*

§7. Contact structure.

In a differentiable manifold $\tilde{M}$ of odd dimensions, a *contact metric structures* is a set $(\tilde{g}, \tilde{\eta}, \tilde{\phi})$ of a Riemannian metric $\tilde{g}$, a 1-form $\tilde{\eta}$ and a tensor field $\tilde{\phi}$ of tpye (1, 1) such that

$$\tilde{\phi}^2=-I+\tilde{\eta}\otimes\tilde{\xi}, \qquad \tilde{\eta}(\tilde{\xi})=1,$$

$$\tilde{\phi}\tilde{\xi}=0, \qquad \tilde{\eta}(\tilde{\phi}\tilde{X})=0$$

for any vector field $\tilde{X}$ in $\tilde{M}$, I denoting the identity tensor field of type (1, 1) in $\tilde{M}$, where the tensor field $\tilde{\phi}$ of type (1, 1) and the vector field $\tilde{\xi}$ are defined in $\tilde{M}$ respectively by the equations

$$\tilde{g}(\tilde{\xi}, \tilde{X})=\tilde{\eta}(\tilde{X}),$$

$$\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y})=d\tilde{\eta}(\tilde{X}, \tilde{Y}),$$

$\tilde{X}$ and $\tilde{Y}$ being arbitrary vector fields in $\tilde{M}$ (cf. Sasaki [4]). When the vector field $\tilde{\xi}$ is a Killing vector field with respect to $\tilde{g}$, i.e., when

$$\mathcal{L}_{\tilde{\xi}}\tilde{g}=0,$$

the contact structure $(\tilde{g}, \tilde{\eta}, \tilde{\phi})$ is called a *K-contact structure* (cf. Hatakeyama, Ogawa and Tanno [1]).

Coming back to fibred spaces with invariant Riemannian metric, if we take account of Proposition 5. 4, we have

PROPOSITION 7. 1. *In a fibred space $\tilde{M}$ with invariant Riemannian metric $\tilde{g}$, the set $(\tilde{g}, \tilde{\eta}, (1/c)\tilde{H})$ is a K-contact structure, $\tilde{H}$ denoting the second fundamental tensor in $\tilde{M}$ and c being a non-zero constant, if and only if the sectional curvature with respect to any section containing the vector $\tilde{C}$ is a constant A everywhere in $\tilde{M}$, where A is necessarily equal to c^2* (cf. Hatakeyama, Ogawa and Tanno [1]).

When a fibred space with invariant Riemannian metric satisfies the conditions mentioned in Proposition 7. 1, the fibred space is called a *fibred space with K-contact structure*. Then Proposition 5. 4 is equivalent to the fact that *a fibred* *space with invariant Riemannian metric is a fibred space with K-contact structure if and only if the set (g, f) is an almost Kählerian structure in the base space M, where g is the induced metric and f is defined by $f=(1/c)H$, H being the second fundamental tensor in M and c being a non-zero constant.*

Taking account of (2. 32), Propositions 3. 2 and 5. 9, we have

PROPOSITION 7. 2. *For a fibred space $\tilde{M}$ with K-contact structure, the following six conditions* (a)$\sim$(f) *are equivalent to each other*:

(a) *M is normal, i.e., $\tilde{S}=0$ in $\tilde{M}$.*

(b) *The set (g, f) is a Kählerian structure in the base space M, where g is the induced metric and f is defined by $f=(1/c)H$, H being the second fundamental tensor in M and c being a non-zero constant.*

(c) $N=0$ *in M.*

(d) $\nabla_c h_{ba}=0$, *i.e.*, $\nabla h=0$ *in M.*

(e) $\tilde{\nabla}_k \tilde{h}_{ji}=c^2(g_{ki}E_j-g_{kj}E_i)$ *in $\tilde{M}$.*

(f) $\tilde{\nabla}_k \tilde{\nabla}_j E_i=c^2(-g_{ki}E_j+g_{kj}E_i)$ *in $\tilde{M}$.*

In a normal fibred space $\tilde{M}$ with K-contact structure, the equation (5. 13) reduces to

$$\tilde{K}_{cb}=K_{cb}-2c^2 g_{cb}, \qquad \tilde{K}_{co}=0, \qquad \tilde{K}_{oo}=nc^2, \tag{7. 1}$$

or equivalently to

$$\tilde{K}_{ji}=(K_{cb}-2c^2 g_{cb})E_j{}^c E_i{}^b+nc^2 E_j E_i, \tag{7. 2}$$

c being a certain non-zero constant, where $n=\dim M$. Thus, taking account of (7. 1) or (7. 2), we have

PROPOSITION 7. 3. *In a normal fibred space $\tilde{M}$ with K-contact structure, the metric g induced in the base space M is an Einstein metric, if and only if the condition*

$$\tilde{K}_{ji}=Ag_{ji}+BE_jE_i, \qquad A+B=nc^2 \tag{7. 3}$$

holds with a non-zero constant c, where A and B are certain invariant functions in $\tilde{M}$.

Differentiating covariantly both sides of (7. 3), we find

$$\tilde{\nabla}_k \tilde{K}_{ji}=(\partial_k A)g_{ji}+(\partial_k B)E_jE_i-B(\tilde{h}_{kj}E_i+\tilde{h}_{ki}E_j)$$

because of (2. 32). Transvecting g^{ki}, we get

$$g^{ts}\tilde{\nabla}_t \tilde{K}_{js}=(n+1)\partial_j A \tag{7. 4}$$

because of $\partial_k B=E_k{}^d \partial_d B$. On the other hand, transvecting g^{ji} to both sides of (7. 3) and differentiating covariantly, we obtain

$$\tilde{\nabla}_j(g^{ts}\tilde{K}_{ts})=(n+1)\partial_j A+\partial_j B. \tag{7.5}$$

If we substitute (7. 4) and (7. 5) in the identity

$$\tilde{\nabla}_j(g^{ts}\tilde{K}_{ts})=2(\tilde{\nabla}_t\tilde{K}_{js})g^{ts},$$

which is a direct consequence of the Ricci identity, then we have

$$(n+1)\partial_j A-\partial_j B=0. \tag{7.6}$$

On the other hand, we have from the second equation of (7. 3)

$$\partial_j A+\partial_j B=0. \tag{7.7}$$

Combining (7. 5) and (7. 7), we find $\partial_j A=\partial_j B=0$, i.e., the fact that A and B are constant. Thus we have

PROPOSITION 7. 4. *If, in a normal fibred space with K-contact structure, the Ricci tensor $\tilde{R}$ has the form*

$$\tilde{R}=A\tilde{g}+B\tilde{\eta}\otimes\tilde{\eta},$$

then A and B are necessarily constant (cf. Okumura [3]).

Let $\tilde{M}$ be a normal fibred space with K-contact structure. Then a Kählerian structure (g,f) is induced in the base space M as a consequence of Proposition 7. 2. Thus, taking account of (5. 1), (5. 2) and (5. 3), we have

$$\begin{aligned}\tilde{K}_{kji}{}^h=K_{kji}{}^h&-c^2(\tilde{f}_k{}^h\tilde{f}_{ji}-\tilde{f}_j{}^h\tilde{f}_{ki}-2\tilde{f}_{kj}\tilde{f}_i{}^h)\\&-c^2(E_kE^hg_{ji}-E_jE^hg_{ki}+(\delta_k^hE_j-\delta_j^hE_k)E_i)\end{aligned} \tag{7.8}$$

with a non-zero constant c, where

$$\tilde{f}_i{}^h=f_b{}^aE_i{}^bE^h{}_a$$

and

$$K_{kji}{}^h=K_{dcb}{}^aE_k{}^dE_j{}^cE_i{}^bE^h{}_a$$

are the components of the lifts K^L and f^L respectively and $\tilde{f}_{ji}=\tilde{f}_j{}^hg_{hi}$, $\tilde{K}$ and K being the curvature tensors of the invariant Riemannian metric $\tilde{g}$ and the induced metric g respectively.

We now define in the base space M a tensor field Z of type (1, 3) with the following components:

$$Z_{dcb}{}^a=K_{dcb}{}^a-c^2(\delta_d^ag_{cb}-\delta_c^ag_{db}+f_d{}^af_{cb}-f_c{}^af_{db}-2f_{dc}f_b{}^a). \tag{7.9}$$

Then the equation (7. 8) reduces to

$$\tilde{K}_{kji}{}^h = c^2(\delta_k^h g_{ji} - \delta_j^h g_{ki}) + Z_{kji}{}^h, \tag{7.10}$$

where we have put

$$Z_{kji}{}^h = Z_{dcb}{}^a E_k{}^d E_j{}^c E_i{}^b E^h{}_a$$

which are the components of the lift Z^L.

When the induced Kählerian structure (g, f) is of constant holomorphic sectional curvature in the base space M, we have $Z_{dcb}{}^a = 0$, which implies together with (7.10)

$$\tilde{K}_{kji}{}^h = c^2(\delta_k^h g_{ji} - \delta_j^h g_{ki}).$$

Therefore we have

PROPOSITION 7.5. *A normal fibred space $\tilde{M}$ with K-contact structure is of positive constant curvature if and only if the induced Kählerian structure is of constant holomorphic sectional curvature in the base space M.*

Differentiating covariantly both sides of (7.2), we have

$$\tilde{\nabla}_k \tilde{K}_{ji} = (\nabla_d K_{cb}) E_k{}^d E_j{}^c E_i{}^b + (K_{cb} - 2c^2 g_{cb})\{(\overset{*}{\nabla}_k E_j{}^c) E_i{}^b + E_j{}^c (\overset{*}{\nabla}_k E_i{}^b)\}$$
$$+ nc^2\{(\tilde{\nabla}_k E_j) E_i + E_j (\tilde{\nabla}_k E_i)\},$$

which reduces to

$$\tilde{\nabla}_k \tilde{K}_{ji} = (\nabla_d K_{cb}) E_k{}^d E_j{}^c E_i{}^b + c\tilde{f}_k{}^s \tilde{Z}_{si} E_j + c\tilde{f}_k{}^s \tilde{Z}_{sj} E_i + cE_k(\tilde{f}_j{}^s \tilde{Z}_{si} + \tilde{f}_i{}^s \tilde{Z}_{sj}) \tag{7.11}$$

by virtue of (2.32), (7.2) and $h_b{}^a = cf_b{}^a$, $h_{cb} = cf_{cb}$, where we have put

$$\tilde{Z}_{ji} = (R_{cb} - (n+2)c^2 g_{cb}) E_j{}^c E_i{}^b. \tag{7.12}$$

Thus, taking account of (7.11) and (7.12), we have

PROPOSITION 7.6. *Let $\tilde{M}$ be a normal fibred space with K-contact structure. Then the equation*

$$p(\tilde{\nabla}\tilde{R}) = \nabla R$$

holds, $\tilde{R}$ and R being the Ricci tensors of the invariant Riemannian metric $\tilde{g}$ and the induced metric g respectively.

The Ricci tensor $\tilde{R}$ is covariantly constant in $\tilde{M}$, i.e., $\tilde{\nabla}\tilde{R} = 0$, if and only if the invariant metric $\tilde{g}$ is an Einstein metric in $\tilde{M}$. If this is the case, the induced metric g is also an Einstein metric in the base space M (cf. Okumura [3], Tanno [7]).

If we differentiate covariantly both sides of (7.10) and put $\tilde{K}_{kjih} = \tilde{K}_{kji}{}^s g_{sh}$, then we have

$$\tilde{\nabla}_l \tilde{K}_{kjih} = (\nabla_e K_{dcba}) E_l{}^e E_k{}^d E_j{}^c E_i{}^b E_h{}^a + K_{dcba}\{(\overset{*}{\nabla}_l E_k{}^d) E_j{}^c E_i{}^b E_h{}^a + E_k{}^d (\overset{*}{\nabla}_l E_j{}^c) E_i{}^b E_h{}^a + E_k{}^d E_j{}^c (\overset{*}{\nabla}_l E_i{}^b) E_h{}^a + E_k{}^d E_j{}^c E_i{}^b (\overset{*}{\nabla}_l E_h{}^a)\},$$

which reduces to

$$\begin{aligned} \tilde{\nabla}_l \tilde{K}_{kjih} &= (\nabla_e K_{dcba}) E_l{}^e E_k{}^d E_j{}^c E_i{}^b E_h{}^a \\ &+ c E_l{}^e \{(f_e{}^f Z_{fcba}) E_k E_j{}^c E_i{}^b E_h{}^a + (f_e{}^f Z_{dfca}) E_k{}^d E_j E_i{}^b E_h{}^a \\ &+ (f_e{}^f Z_{dcfa}) E_k{}^e E_j{}^c E_i E_h{}^a + (f_e{}^f Z_{dcbf}) E_k{}^d E_j{}^c E_i{}^b E_h\} \end{aligned} \tag{7.13}$$

by virtue of (2. 32) and the identity

$$Z_{dcbe} f_a{}^e + Z_{dcea} f_b{}^e = 0,$$

where we have put $Z_{dcba} = Z_{dcb}{}^e g_{ea}$. Taking account of (7. 9) and (7. 13), we find $\tilde{\nabla}_l \tilde{K}_{kjih} = 0$ if and only if $Z_{dcba} = 0$ holds. Therefore, if we take account of (7. 10), we have

PROPOSITION 7. 7. *Let $\tilde{M}$ be a normal fibred space with K-contact structure. Then the equation*

$$p(\tilde{\nabla}\tilde{K}) = \nabla K$$

holds, $\tilde{K}$ and K being the curvature tensors of the invariant metric $\tilde{g}$ and the induced metric g respectively.

The invariant Riemannian metric $\tilde{g}$ is locally symmetric in $\tilde{M}$, i.e., $\tilde{\nabla}\tilde{K} = 0$, if and only if the invariant metric $\tilde{g}$ is of constant cnrvature (cf. Okumura [3]).

We have Propositions 7. 6 and 7. 7 as corollaries to Propositions 6. 8 and 6. 9 respectively. We have the following Proposition 7. 8 as a direct corollary to Proposition 6. 10.

PROPOSITION 7. 8. *The homogeneous holonomy group of a normal fibred space $\tilde{M}$ with K-contact structure coincides with the group* SO$(n+1)$ *of all rotations, where* $n+1 = \dim \tilde{M}$. (Tashiro [8]).

REFERENCES

[1] HATAKEYAMA, Y., Y. OGAWA AND S. TANNO, Some properties of manifolds with contact metric structure. Tôhoku Math. J. **15** (1963), 42–48.
[2] KURITA, M., On normal contact metric manifolds. J. Math. Soc. Japan **15** (1963), 304–318.
[3] OKUMURA, M., Some remarks on space with a certain contact structure. Tôhoku Math. J. **14** (1962), 135–145.
[4] SASAKI, S., On differentiable manifolds with certain structures which are closely

related to almost contact structure I. Tôhoku Math. J. **12** (1960), 459–476.
[5] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II. Tôhoku Math. J. **13** (1961), 281–294.
[6] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures. J. Math. Soc. Japan **14** (1962), 249–271.
[7] TANNO, S., Some transformations on manifolds with almost contact and contact structure II. Tôhoku Math. J. **15** (1963), 322–331.
[8] TASHIRO, Y., On contact structure of hypersurfaces II. Tôhoku Math. J. **15** (1963), 167–175.
[9] TASHIRO, Y., AND S. TACHIBANA, On Fubinian and C-Fubinian manifolds. Kōdai Math. Sem. Rep. **15** (1963), 176–183.
[10] YANO, K., AND S. ISHIHARA, Fibred spaces and projectable tensor fields. Perspectives in Geometry and Relativity. (1966), 468–481.
[11] YANO, K., AND S. ISHIHARA, Differential geometry of fibred spaces. Kōdai Math. Sem. Rep. **19** (1967), 257–288.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.