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ON AN ULTRAHYPERELLIPTIC SURFACE WHOSE PICARD'S CONSTANT IS THREE

BY MITSURU OZAWA

1. Introduction.. Let R be an open Riemann surface. Consider the set $\mathfrak{M}(R)$ of non-constant meromorphic functions on R. Let $P(f)$ be the number of values which are not taken by $f\in\mathfrak{M}(R)$. Let $P(R)$ be the supremum of $P(f)$ when f runs over $\mathfrak{M}(R)$. Then $P(R)\geqq 2$. The significant meaning of this Picard's constant $P(R)$ lies in the following fact: If $P(R)<P(S)$, then there is no non-constant analytic mapping of R into S. [3].

Let R be an ultrahyperelliptic surface defined by $y^2=g(x)$, where $g(x)$ is an entire function having an infinite number of zeros of odd order. For this class of surfaces it is known that $P(R)\leqq 4$. Further it was proved that $P(R)=4$ if and only if

$$g(x)=f(x)^2(e^{H(x)}-\gamma)(e^{H(x)}-\delta),\qquad \gamma\delta(\gamma-\delta)\neq 0,\qquad H(0)=0,$$

where $f(x)$ is a suitable meromorphic function and $H(x)$ is a non-constant entire function. [4]. If $P(R)=3$, then

$$g(x)=f(x)^2(1-2\beta_1e^H-2\beta_2e^L+\beta_1{}^2e^{2H}-2\beta_1\beta_2e^{H+L}+\beta_2{}^2e^{2L}),$$

$$\beta_1\beta_2\neq 0,\qquad H(0)=L(0)=0,$$

with two non-constant entire functions $H(x)$, $L(x)$. Inversely if $g(x)$ has the above form, $P(R)\geqq 3$.

We shall concern with the inverse problem. In this direction we published a paper [2] in which we proved the following results:

THEOREM 1. *The surface defined by*

$$\text{(A)}\qquad y^2=1-2\beta_1e^H-2\beta_2e^L+\beta_1^2e^{2H}-2\beta_1\beta_2e^{H+L}+\beta_2^2e^{2L},\qquad \beta_1\beta_2\neq 0,\qquad H(0)=L(0)=0$$

with two non-constant entire functions $H(x)$ *and* $L(x)$ *has* $P(R)=3$, *if*

$$m(r, e^L)=o(m(r, e^H)).$$

THEOREM 2. *Let* R *be an ultrahyperelliptic surface defined by* (A). *If* H *and* L *are polynomials of degree at most two, then* $P(R)=3$ *with the following four*

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exceptional cases: (i) $H=L$; (ii) $H=2L$, $\beta_2^2=16\beta_1$; (iii) $2H=L$, $\beta_1^2=16\beta_2$; (iv) $H=-L$, $16\beta_1\beta_2=1$. *In these exceptional cases* $P(R)=4$.

In the present paper we shall prove the following

THEOREM 3. *Let* R *be an ultrahyperelliptic surface defined by* (A) *with two polynomials* H *and* L *of an arbitrary degree, then the same conclusion as in theorem* 2 *holds.*

2. Lemmas. Although we can give more general formulations of some of the following Lemmas, we shall give somewhat restrictive forms of them, since it is sufficient to apply to the present cases.

Here H and L are polynomials of the same degree and have their expansions

$$H=\sum_{n=1}^{s} h_n z^n, \qquad L=\sum_{n=1}^{s} l_n z^n.$$

LEMMA 1.

$$m(r, e^H)=\frac{1}{\pi}|h_s|r^s\left(1+O\left(\frac{1}{r}\right)\right).$$

LEMMA 2. $N(r;\ a, e^H)\sim m(r, e^H)$ *for* $a\neq 0$.

LEMMA 3. *Let* ψ *be* ae^H-be^L-1 *with two non-zero constants* a, b *and* $H\neq L$. *Then*

$$N_2(r;\ 0, \psi)\geqq m(r, e^H)-2m(r, e^L)+O(\log r)$$

for $r\geqq r_0$, *where* N_2 *indicates the* N*-function of simple* a*-points of the referred function.*

Proof. Let

$$\varphi=-\frac{\psi}{be^L+1}=-\frac{ae^H}{be^L+1}+1,$$

then

$$\varphi'=-\frac{ae^H}{(be^L+1)^2}(b(H'-L')e^L+H').$$

Hence

$$N(r;\ \infty, \varphi)=N(r;\ 0, be^L+1)\sim m(r, e^L), \qquad N(r;\ 0, \varphi-1)=0.$$

Since every common root of $b(H'-L')e^L+H'=0$, $be^L+1=0$ satisfies $L'=0$,

$$\begin{aligned} N(r;\ \infty, \varphi')&=2N(r;\ 0, be^L+1)-O(\log r)\\ &=2m(r, e^L)+O(\log r). \end{aligned}$$

Further

$$N(r;\ 0,\ \varphi')=N(r;\ 0,\ b(H'-L')e^{L}+H')-O(\log r)$$
$$=m(r,\ e^{L})+O(\log r),$$

if $H'-L'\not\equiv 0$, This is the case, since $H\not\equiv L$. Hence by the second fundamental theorem for φ

$$N(r;\ 0,\ \varphi)+O(1)\leqq T(r,\ \varphi)$$
$$\leqq N(r;\ 0,\ \varphi)+N(r;\ \infty,\ \varphi)+N(r;\ 1,\ \varphi)-N(r;\ 0,\ \varphi')$$
$$-2N(r;\ \infty,\ \varphi)+N(r;\ \infty,\ \varphi')+O(\log r\ T(r,\ \varphi))$$
$$=N(r;\ 0,\ \varphi)+O(\log r\ T(r,\ \varphi)).$$

Further

$$T(r,\ \varphi)=m(r,\ \varphi)+N(r;\ \infty,\ \varphi)$$
$$\leqq m(r,\ e^{H})+m(r,\ 1/(be^{L}+1))+m(r,\ e^{L})+O(1)$$
$$=m(r,\ e^{H})+m(r,\ e^{L})+N(r;\ \infty,\ be^{L}+1)-N(r;\ 0,\ be^{L}+1)+m(r,\ e^{L})+O(1)$$
$$=m(r,\ e^{H})+m(r,\ e^{L})+O(1)=O(r^{s}).$$

Hence

$$N(r;\ 0,\ \varphi)+O(1)\leqq T(r,\ \varphi)\leqq N(r;\ 0,\ \varphi)+O(\log r).$$

Further

$$m(r,\ e^{H})=m(r,\ (\varphi-1)(be^{L}+1))\leqq m(r,\ \varphi)+m(r,\ e^{L})+O(1).$$

Hence

$$m(r,\ e^{H})-m(r,\ e^{L})\leqq m(r,\ \varphi)+O(1)$$

and

$$m(r,\ e^{H})=m(r,\ e^{H})-m(r,\ e^{L})+N(r;\ \infty,\ \varphi)+O(\log r)$$
$$\leqq m(r,\ \varphi)+N(r;\ \infty,\ \varphi)+O(\log r)$$
$$\leqq T(r,\ \varphi)+O(\log r)\leqq N(r;\ 0,\ \varphi)+O(\log r)$$
$$=N_{2}(r;\ 0,\ \varphi)+N_{1}(r;\ 0,\ \varphi)+\bar{N}_{1}(r;\ 0,\ \varphi)+O(\log r)$$
$$\leqq N_{2}(r;\ 0,\ \varphi)+2m(r,\ e^{L})+O(\log r)$$
$$=N_{2}(r;\ 0,\ \phi)+2m(r,\ e^{L})+O(\log r).$$

This is the desired result.

The estimation of this Lemma is best possible. Consider

$$2\phi=e^{3H}-3e^{H}+2=(e^{H}+2)(e^{H}-1)^{2}.$$

Then

$$N_{2}(r;\ 0,\ \phi)=m(r,\ e^{H})+O(1),\qquad m(r,\ e^{3H})-2m(r,\ e^{H})=m(r,\ e^{H}).$$

LEMMA 4. *Let ϕ be ae^H-1 with a non-zero constant a. Then*

$$N_2(r;\ 0,\ \phi)=m(r,\ e^H)+O(\log r).$$

LEMMA 5. *Let $\phi_1=ae^H-be^L-1$, $\phi_2=ae^H+be^L+1$, $\phi_3=ae^H+be^L-1$, $\phi_4=ae^H-be^L+1$. If $H\not\equiv L$, then*

$$4|m(r,\ e^H)-m(r,\ e^L)|\leqq\sum_1^4 N_2(r;\ 0,\ \phi_j)+O(\log r).$$

Proof. Put

$$\varphi_1=-\frac{\phi_1}{be^L+1},\qquad \varphi_2=\frac{\phi_2}{be^L+1},\qquad \varphi_3=-\frac{\phi_3}{be^L-1},\qquad \varphi_4=\frac{\phi_4}{be^L-1}.$$

Then

$$\varphi_1'=-\frac{ae^H}{(be^L+1)^2}(b(H'-L')e^L+H'),\qquad \varphi_2'=\frac{ae^H}{(be^L+1)^2}(b(H'-L')e^L+H'),$$

$$\varphi_3'=-\frac{ae^H}{(be^L-1)^2}(b(H'-L')e^L-H'),\qquad \varphi_4'=\frac{ae^H}{(be^L-1)^2}(b(H'-L')e^L-H').$$

Hence

$$(N_1+\bar{N}_1)(r;\ 0,\ \varphi_1)+(N_1+\bar{N}_1)(r;\ 0,\ \varphi_2)\leqq 2m(r,\ e^L)+O(\log r)$$

and

$$(N_1+\bar{N}_1)(r;\ 0,\ \varphi_3)+(N_1+\bar{N}_1)(r;\ 0,\ \varphi_4)\leqq 2m(r,\ e^L)+O(\log r).$$

Thus

$$\begin{aligned}4m(r,\ e^H)&\leqq\sum_1^4 N(r;\ 0,\ \varphi_j)+O(\log r)\\&=\sum_1^4 N_2(r;\ 0,\ \varphi_j)+\sum_1^4(N_1+\bar{N}_1)(r;\ 0,\ \varphi_j)+O(\log r)\\&\leqq\sum_1^4 N_2(r;\ 0,\ \phi_j)+4m(r,\ e^L)+O(\log r).\end{aligned}$$

By symmetry of the given expression we have the desired result.

LEMMA 6. *Let ϕ_j be the same as in Lemma 5. Further assume $h_s=e^{\pi i/3}l_s$. Then*

$$4m(r,\ e^H)\leqq\sum_1^4 N_2(r;\ 0,\ \phi_j)+O(\log r).$$

Proof. Consider multiple zeros of ϕ_1. These are common zeros of two equations $\phi_1\equiv ae^H-be^L-1=0$ and $\phi_1'\equiv aH'e^H-bL'e^L=0$. Then every common root whose modulus is sufficiently large satisfies

$$\begin{cases} ae^H = \dfrac{L'}{L'-H'} = \dfrac{e^{\pi i/3} h_s s z^{s-1}\left(1+O\left(\dfrac{1}{z}\right)\right)}{(e^{\pi i/3}-1) h_s s z^{s-1}\left(1+O\left(\dfrac{1}{z}\right)\right)} = e^{-\pi i/3}\left(1+O\left(\dfrac{1}{z}\right)\right), \\ be^L = \dfrac{H'}{L'-H'} = \dfrac{h_s s z^{s-1}\left(1+O\left(\dfrac{1}{z}\right)\right)}{(e^{\pi i/3}-1) h_s s z^{s-1}\left(1+O\left(\dfrac{1}{z}\right)\right)} = e^{-2\pi i/3}\left(1+O\left(\dfrac{1}{z}\right)\right). \end{cases}$$

Thus we have

$$h_s z^s\left(1+O\left(\frac{1}{z}\right)\right) = 2n\pi i - \frac{\pi}{3} i - \log a + O\left(\frac{1}{z}\right),$$

$$e^{\pi i/3} h_s z^s\left(1+O\left(\frac{1}{z}\right)\right) = 2m\pi i - \frac{2}{3}\pi i - \log b + O\left(\frac{1}{z}\right)$$

and hence

$$z = A n^{1/s}\left(1+O\left(\frac{1}{z}\right)\right),$$

$$z = B m^{1/s}\left(1+O\left(\frac{1}{z}\right)\right), \qquad AB \neq 0.$$

Therefore $z\to\infty$ implies $n\to\infty$, $m\to\infty$ and vice versa. However

$$e^{\pi i/3} = \frac{m}{n}\left(1+O\left(\frac{1}{z}\right)\right)\left(1+O\left(\frac{1}{m}\right)\right)\left(1+O\left(\frac{1}{n}\right)\right).$$

This implies that $m/n \to \exp(\pi i/3)$ as $z\to\infty$. This is a contradiction, since m/n is a real rational number. Hence there is no common zero of $\phi_1=0$ and $\phi_1'=0$ having a sufficiently large modulus. This shows that

$$(N_1+\bar{N}_1)(r;\ 0,\ \phi_1) = O(\log r).$$

Similarly we have the same fact for any ϕ_j. Further there is no common zero of $\phi_j=0$, $\phi_k=0$ for $j\neq k$. As in Lemma 5 we have

$$4m(r, e^H) \leqq \sum_1^4 N_2(r;\ 0,\ \phi_j) + \sum_1^4 (N_1+\bar{N}_1)(r;\ 0,\ \phi_j) + O(\log r)$$

$$= \sum_1^4 N_2(r;\ 0,\ \phi_j) + O(\log r).$$

This is the desired result.

3. Proof of theorem 3. First of all it is sufficient to consider the case that the degree of $H(x)$ is coincident with that of $L(x)$. In fact if the degree of $H(x)$ is greater than that of $L(x)$, then $m(r, e^L) = o(m(r, e^H))$. By theorem 1 we have

$P(R)=3$ in this case.

Assume that $P(R)=4$. Then there are a non-constant entire function $K(x)$ and two constants γ, δ $(\gamma\delta(\gamma-\delta)\neq 0)$ and a meromorphic function $f(x)$ such that

$$G\equiv 1-2\beta_1 e^H-2\beta_2 e^L+\beta_1^2 e^{2H}-2\beta_1\beta_2 e^{H+L}+\beta_2^2 e^{2L}$$
$$=f^2(e^K-\gamma)(e^K-\delta), \qquad K(0)=0.$$

By Lemma 5, which was proved in [4], we have

$$N_2(r;\ 0,\ e^K-\gamma)\sim N_2(r;\ 0,\ e^K-\delta)\sim m(r,\ e^K)$$

outside a set of finite logarithmic measure. Since all the simple zeros of $(e^K-\gamma)\cdot(e^K-\delta)$ are the zeros of G, we have

$$2m(r,\ e^K)\sim N_2(r;\ 0,\ (e^K-\gamma)(e^K-\delta))\leqq N(r;\ 0,\ G)\leqq m(r,\ G)$$

outside a set of finite logarithmic measure. If K is a transcendental entire function or a polynomial of degree greater than that of H, then we have

$$\rho_G=\overline{\lim_{r\to\infty}}\frac{\log m(r,\ G)}{\log r}\geqq\overline{\lim_{r\to\infty}}\frac{\log m(r,\ e^K)}{\log r}$$
$$>\overline{\lim_{r\to\infty}}\frac{\log m(r,\ e^H)}{\log r}=\text{the degree of } H.$$

However by its form $\rho_G\leqq$the degree of H, which is a contradiction. Hence the degree of K must be less than or equal to that of H. Let p denote the degree of K and let s denote the degree of H. Then $s\geqq p$.

Now we shall prove the inversely directed inequality $s\leqq p$ and hence $s=p$. If $H\equiv L$, then

$$G(z)\equiv 1-2(\beta_1+\beta_2)e^H+(\beta_1-\beta_2)^2 e^{2H}.$$

This implies that $P(R)=4$. Hence we may omit this case. Since $H\not\equiv L$ and

$$G=G_1G_2G_3G_4,\quad G_1=1-\sqrt{\beta_1}e^{H/2}-\sqrt{\beta_2}e^{L/2},\quad G_2=1-\sqrt{\beta_1}e^{H/2}+\sqrt{\beta_2}e^{L/2},$$
$$G_3=1+\sqrt{\beta_1}e^{H/2}-\sqrt{\beta_2}e^{L/2},\quad G_4=1+\sqrt{\beta_1}e^{H/2}+\sqrt{\beta_2}e^{L/2},$$

we can make use of Lemma 5 for G and we have

$$\frac{2}{\pi}\,||h_s|-|l_s||r^s\left(1+O\left(\frac{1}{r}\right)\right)\leqq\sum_1^4 N_2(r;\ 0,\ G_j)+O(\log r)$$
$$=N_2(r;\ 0,\ G)+O(\log r)$$
$$=N_2(r;\ 0,\ (e^K-\gamma)(e^K-\delta))+O(\log r)$$
$$\leqq 2m(r,\ e^K)+O(\log r)=O(r^p).$$

If $|h_s| \neq |l_s|$, then $s \leqq p$. Further consider Ge^{-2L} and Ge^{-2H}. Then applying Lemma 5 and assuming $|h_s - l_s| \neq |l_s|$ we have $s \leqq p$ similarly. Therefore we may assume that $|h_s| = |l_s| = |h_s - l_s|$, that is, $h_s = l_s \exp(\pi i/3)$ or $l_s = h_s \exp(\pi i/3)$. Now by Lemma 6 we have

$$\frac{2}{\pi}|h_s| r^s\left(1+O\left(\frac{1}{r}\right)\right) \leqq 2m(r, e^H)$$

$$\leqq N_2(r;\ 0, G) + O(\log r) \leqq 2m(r, e^K) + O(\log r) = O(r^p).$$

Hence $s \leqq p$. Therefore we have $s = p$.

By a suitable transformation $x \to \alpha x$ we may assume that

$$K(x) = x^p + \alpha_{p-1}x^{p-1} + \cdots + \alpha_1 x.$$

Put

$$H(x) = \sum_1^p h_n x^n, \qquad L(x) = \sum_1^p l_n x^n.$$

Let $z_n^{(j)}$ be a zero of $e^K - \gamma$ such that for $j = 0, 1, \cdots, p-1$

$$z_n^{(j)} \begin{cases} = (2n\pi)^{1/p} i^{1/p} \exp(2j\pi i/p)\,(1 + O(1/n^{1/p})) & \text{if} \quad n > 0, \\ = (-2n\pi)^{1/p}(-i)^{1/p} \exp(2j\pi i/p)\,(1 + O(1/n^{1/p})) & \text{if} \quad n < 0. \end{cases}$$

All of those roots with fixed j are called roots of (j)-direction. Since $K = 2\pi i n + \log \gamma$ has p solutions for each n and p solutions have almost the same modulus, each (j)-direction has an infinite number of simple zeros of $e^K - \gamma$ for each sufficiently large positive n and negative n, that is, $m(r, e^K)/2p$ simple zeros for positive n and for negative n, respectively. Evidently

$$K(z_n^{(j)}) = z_n^{(j)p} + \alpha_{p-1} z_n^{(j)p-1} + \cdots + \alpha_1 z_n^{(j)} = \log \gamma + 2n\pi i.$$

Hence

$$H(z_n^{(j)}) = h_p(\log \gamma + 2n\pi i) + \sum_{m=2}^{p} (h_{m-1} - \alpha_{m-1} h_p) z_n^{(j)m-1}$$

and

$$L(z_n^{(j)}) = l_p(\log \gamma + 2n\pi i) + \sum_{m=2}^{p} (l_{m-1} - \alpha_{m-1} l_p) z_n^{(j)m-1}.$$

Let

$$X_0 = e^{2\pi i h_p}, \qquad Y_0 = e^{2\pi i l_p}, \qquad A = \beta_1 e^{h_p \log \gamma}, \qquad B = \beta_2 e^{l_p \log \gamma}$$

and

$$U_n^{(j)} = \exp\left(\sum_{m=1}^{p-1} (h_m - \alpha_m h_p) z_n^{(j)m}\right), \quad V_n^{(j)} = \exp\left(\sum_{m=1}^{p-1} (l_m - \alpha_m l_p) z_n^{(j)m}\right),$$

$$F_n^{(j)} = 1 - 2AX_0^n U_n^{(j)} - 2BY_0^n V_n^{(j)} + A^2 X_0^{2n} U_n^{(j)2} - 2ABX_0^n Y_0^n U_j^{(j)} V_n^{(j)} + B^2 Y_0^{2n} V_n^{(j)2}.$$

Evidently $F_n^{(j)}=G(z_n^{(j)})=0$ for any simple zero $z_n^{(j)}$ of $e^K-\gamma$.

First of all we shall prove that $|X_0|=|Y_0|=1$. Assume that $|X_0|<1$, $|Y_0|<1$. Since $U_n^{(j)}=\exp O(n^{(p-1)/p})$, $V_n^{(j)}=\exp O(n^{(p-1)/p})$, we have $X_0^n U_n^{(j)}\to 0$, $Y_0^n V_n^{(j)}\to 0$ as $n\to\infty$. Hence $F_n^{(j)}\to 1$ as $n\to\infty$, which contradicts $F_n^{(j)}=0$. Assume $|X_0|>1$, $|Y_0|>1$. Then $X_0^n U_n^{(j)}\to 0$, $Y_0^n V_n^{(j)}\to 0$ as $n\to-\infty$. Hence $F_n^{(j)}\to 1$ as $n\to-\infty$, which contradicts $F_n^{(j)}=0$. Assume $|X_0|>1$, $|Y_0|\leqq 1$. Then $X_0^n U_n^{(j)}=\exp O(n)$ and $Y_0^n V_n^{(j)}=\exp O(n^{(p-1)/p})$ as $n\to\infty$. Hence

$$F_n^{(j)}=1-2(AX_0^n U_n^{(j)}+BY_0^n V_n^{(j)})+(AX_0^n U_n^{(j)}-BY_0^n V_n^{(j)})^2\to\infty$$

as $n\to\infty$, which is a contradiction. Similarly we can conclude that $|X_0|\leqq 1$, $|Y_0|>1$ does not occur. Thus we have the desired result: $|X_0|=|Y_0|=1$.

Next we shall prove that $h_m-\alpha_m h_p=l_m-\alpha_m l_p=0$ for every m $(1\leqq m\leqq p-1)$. Let m_0 and m_1 be the highest indices for which $h_m-\alpha_m h_p\neq 0$, $l_m-\alpha_m l_p\neq 0$, respectively. Assume $m_0>m_1$. If $p\neq 2m_0$, then there is an index j such that

$$|\arg\{(h_{m_0}-\alpha_{m_0}h_p)z_n^{(j)m_0}\}|$$
$$=\left|\arg(h_{m_0}-\alpha_{m_0}h_p)+\frac{\pi}{2}\frac{m_0}{p}+\frac{2\pi m_0}{p}j+o(1)\right|<\frac{\pi}{2}\qquad(\text{mod } 2\pi).$$

Hence

$$\exp\Re\{(h_{m_0}-\alpha_{m_0}h_p)z_n^{(j)m_0}\}=\exp O(n^{m_0/p})$$

for any sufficiently large n. Hence $U_n^{(j)}=\exp O(n^{m_0/p})$. However $V_n^{(j)}$ is at most $\exp O(n^{m_1/p})$. Hence $F_n^{(j)}\to\infty$ as $n\to\infty$. If $p=2m_0$ and if there is an index j satisfying the above condition on the argument, then we have the same contradiction. If $p=2m_0$ and if for every j

$$\left|\arg(h_{m_0}-\alpha_{m_0}h_p)+\frac{\pi}{4}+\pi j\right|=\frac{\pi}{2}\qquad(\text{mod } 2\pi),$$

then

$$\arg(h_{m_0}-\alpha_{m_0}h_p)=\frac{\pi}{4}\qquad(\text{mod } \pi).$$

Consider $-n$ with the same index j. Then

$$\left|\arg(h_{m_0}-\alpha_{m_0}h_p)-\frac{\pi}{4}+j\pi+o(1)\right|=o(1)\qquad(\text{mod } \pi),$$

which implies that

$$\arg(h_{m_0}-\alpha_{m_0}h_p)z_n^{(j)m_0}\to 0\qquad(\text{mod } \pi)$$

as $n\to-\infty$. Hence taking an index j^* (j or $j+1$) such that

$$\exp(\Re\{(h_{m_0}-\alpha_{m_0}h_p)z_n^{(j^*)m_0}\})=\exp O((-n)^{1/2})$$

as $n\to-\infty$, $U_n^{(j^*)}\to\infty$ as $n\to-\infty$. On the other hand $V_n^{(j^*)}=\exp O((-n)^{m_1/p})$ as n

$\to -\infty$, which implies that $F_n^{(j^*)}\to\infty$. This is untenable. When $m_0<m_1$, the same contradiction appears. Hence we have $m_0=m_1$.

If $p\neq 2m_0$ and $p\geqq 3$, then among three successive indices j (mod p) there are two indices such that

$$L_{m_0,p}^{(j)}\equiv \arg(l_{m_0}-\alpha_{m_0}l_p)+\frac{\pi}{2}\frac{m_0}{p}+\frac{2\pi m_0}{p}j\neq\frac{\pi}{2} \qquad (\mathrm{mod}\ \pi).$$

Similarly we have the same fact for

$$H_{m_0,p}^{(j)}\equiv \arg(h_{m_0}-\alpha_{m_0}h_p)+\frac{\pi}{2}\frac{m_0}{p}+\frac{2\pi m_0}{p}j.$$

Hence we can select an index j such that $H_{m_0,p}^{(j)}$ and $L_{m_0,p}^{(j)}$ are not $\pi/2$ (mod π). If both of $H_{m_0,p}^{(j)}$, $L_{m_0,p}^{(j)}$ lie in an open interval $(\pi/2, 3\pi/2)$ (mod 2π), then $U_n^{(j)}\to 0$, $V_n^{(j)}\to 0$ as $n\to\infty$. Hence $F_n^{(j)}\to 1$ as $n\to\infty$. If $L_{m_0,p}^{(j)}$ does not lie and $H_{m_0,p}^{(j)}$ lie in that interval, then $V_n^{(j)}\to\infty$, $U_n^{(j)}\to 0$ as $n\to\infty$. This implies that $F_n^{(j)}\to\infty$ as $n\to\infty$. If both of $L_{m_0,p}^{(j)}$, $H_{m_0,p}^{(j)}$ do not lie in that interval, then there is another index j^* ($j+[p/2]-1$ or $j+[p/2]$ or $j+[p/2]+1$) such that $L_{m_0,p}^{(j^*)}$ and $H_{m_0,p}^{(j^*)}$ lie in that interval. This leads to a contradiction similarly.

If $p=2m_0$, then either $H_{m_0,p}^{(j)}=\pi/2$ (mod π) or not and either $L_{m_0,p}^{(j)}=\pi/2$ (mod π) or not. If $H_{m_0,p}^{(j)}\neq\pi/2$, $L_{m_0,p}^{(j)}\neq\pi/2$ (mod π) leads to a contradiction similarly. If $H_{m_0,p}^{(j)}=\pi/2$ (mod π), $L_{m_0,p}^{(j)}\neq 0$ (mod π), then consider $-n$ with the same index j. Then

$$\arg(h_{m_0}-\alpha_{m_0}h_p)z_n^{(j)m_0}=o(1)$$

and

$$\arg(l_{m_0}-\alpha_{m_0}l_p)z_n^{(j)m_0}\neq\frac{\pi}{2} \qquad (\mathrm{mod}\ \pi),$$

which leads to a contradiction. If $H_{m_0,p}^{(j)}=\pi/2$ (mod π), $L_{m_0,p}^{(j)}=0$ (mod π), then $\arg(h_{m_0}-\alpha_{m_0}h_p)=\pi/4$ (mod π), $\arg(l_{m_0}-\alpha_{m_0}l_p)=-\pi/4$ (mod π). In this case take an index j such that $L_{m_0,p}^{(j)}=0$ (mod 2π). Then

$$V_n^{(j)}=\exp\Re\{(l_{m_0}-\alpha_{m_0}l_p)(2\pi)^{1/2}n^{1/2}i^{1/2}e^{\pi ji}\}\left(1+O\left(\frac{1}{n^{1/p}}\right)\right)$$

is greater than $U_n^{(j)}$ in the order so that $V_n^{(j)}\pm U_n^{(j)}\to\infty$ as $n\to\infty$. Hence we have a contradiction.

Thus we have the desired result: $h_m-\alpha_m h_p=l_m-\alpha_m l_p=0$ for every m. This fact implies that $H=h_pK$ and $L=l_pK$ and $U_n^{(j)}=V_n^{(j)}=1$. From now on we shall omit the subscript p from h_p and l_p. Now we have

$$1-2AX_0^n-2BY_0^n+A^2X_0^{2n}-2ABX_0^nY_0^n+B^2Y_0^{2n}=0$$

for every $n\geqq n_0$ and $-n\leqq -n_0$. Then we have $X_0=Y_0=1$ [3]. This implies that h and l are integers. Further we have

$$G=F(e^{K/2})=f^2(e^{K/2}-\sqrt{\gamma})(e^{K/2}+\sqrt{\gamma})(e^{K/2}-\sqrt{\delta})(e^{K/2}+\sqrt{\delta}),$$

$$F(\chi)=1-2\beta_1\chi^{2h}-2\beta_2\chi^{2l}+\beta_1^2\chi^{4h}-2\beta_1\beta_2\chi^{2(h+l)}+\beta_2^2\chi^{4l}.$$

Since $e^{K/2}-\chi_0$, $\chi_0\neq 0$ has only a finite number of multiple zeros and an infinite number of simple zeros and $e^{K/2}$ has no zero, $\sqrt{\gamma}$, $-\sqrt{\gamma}$, $\sqrt{\delta}$ and $-\sqrt{\delta}$ must be the simple zeros of $F(\chi)$. In the first place we assume that $0<h<l$. Evidently we have

$$F(\chi)=F_1(\chi)F_2(\chi)F_3(\chi)F_4(\chi),$$

$$F_1=1-\sqrt{\beta_1}\chi^h-\sqrt{\beta_2}\chi^l, \qquad F_2=1-\sqrt{\beta_1}\chi^h+\sqrt{\beta_1}\chi^l,$$

$$F_3=1+\sqrt{\beta_1}\chi^h-\sqrt{\beta_2}\chi^l, \qquad F_4=1+\sqrt{\beta_1}\chi^h+\sqrt{\beta_2}\chi^l.$$

Since no two of F_1, F_2, F_3, F_4 have common zero, we may seek for all the multiple zeros of each function F_j. Since there is no triple zero in each factor F_j, every multiple zero is a double zero. From the equations

$$\begin{cases} F_1(\chi)=0, \\ F_1'(\chi)=0; \end{cases} \quad \begin{cases} F_2(\chi)=0, \\ F_2'(\chi)=0; \end{cases} \quad \begin{cases} F_3(\chi)=0, \\ F_3'(\chi)=0; \end{cases} \quad \begin{cases} F_4(\chi)=0, \\ F_4'(\chi)=0; \end{cases}$$

we have

$$\begin{cases} \chi^h=X, \\ \chi^l=Y; \end{cases} \quad \begin{cases} \chi^h=X, \\ \chi^l=-Y; \end{cases} \quad \begin{cases} \chi^h=-X, \\ \chi^l=Y; \end{cases} \quad \begin{cases} \chi^h=-X, \\ \chi^l=-Y; \end{cases}$$

respectively, where $\chi=l/(l-h)\sqrt{\beta_1}$, and $Y=h/(h-l)\sqrt{\beta_2}$. Thus every double zero is a common point between h-th roots of X and l-th roots of Y or that of X and of $-Y$ or that of $-X$ and of Y or that of $-X$ and of $-Y$, respectively. Let $E(u, p)$ be the set of $|u|^{1/p}\exp\{(\arg u)/p+2n\pi i/p\}$, $n=0, 1, \cdots, p-1$. If $E(X, h)\frown E(Y, l)\neq\phi$, then there are d common points of $E(X, h)$ and $E(Y, l)$, where d is the greatest common measure of h and l.

If there is no double zero in $F(\chi)$, then we have $4l=4$, that is,

$$0<h<l=1,$$

which is untenable. Hence $E(X, h)\frown E(Y, l)\neq\phi$.

If $E(-X, h)\frown E(Y, l)=\phi$, $E(X, h)\frown E(-Y, l)=\phi$ and $E(-X, h)\frown E(-Y, l)=\varphi$, then we have $4l-2d=4$, that is, $0<2h<2l=2+d\leqq 2+h$. Hence $h=d=1$. Thus $2l=3$, which is untenable.

If $E(-X, h)\frown E(Y, l)\neq\phi$ but $E(X, h)\frown E(-Y, l)=\phi$ and $E(-X, h)\frown E(-Y, l)=\phi$, then $E(-X, h)\frown E(Y, l)$ contains just d points and hence $4l-4d=4$. Therefore $h<l=1+d\leqq 1+h$. Thus we have $d=1$, $h=1$, $l=2$. Then $\beta_1^2=16\beta_2$ holds.

If further $E(X, h)\frown E(-Y, l)\neq\phi$, then $E(-X, h)\frown E(-Y, l)\neq\phi$ and these two sets contain just d points, respectively. Thus we have $2d\leqq h$ and $4l-8d=4$. Then

$0<h<l=1+2d\leqq 1+h$, which implies $d=1$, $h=2$ and $l=3$. This is untenable, since $E(-X, 2)\frown E(-Y, 3)=\phi$.

Next assume that $h<0<l$. Then, putting $h=-k$, we get

$$F(\chi)\frac{\chi^{4k}}{\beta_1^2}=1-2\frac{\beta_2}{\beta_1}\chi^{2(l+h)}-2\frac{1}{\beta_1}\chi^{2k}+\frac{\beta_2^2}{\beta_1^2}\chi^{4(l+k)}-2\frac{\beta_2}{\beta_1^2}\chi^{2l+4k}+\frac{1}{\beta_1^2}\chi^{4k}.$$

Since $0<k<l+k$, we can apply the above result. Then we have $l+k=2k=2$. This implies that $l=k=1$ and hence $h=-1$, $l=1$ and $16\beta_1\beta_2=1$. If $l<h<0$, we put $h=-k$ and $l=-m$. Then we have

$$F(\chi)\frac{\chi^{4m}}{\beta_2^2}=1-2\frac{1}{\beta_2}\chi^{2m}-2\frac{\beta_1}{\beta_2}\chi^{2(m-k)}+\frac{1}{\beta_2^2}\chi^{4m}-2\frac{\beta_1}{\beta_2^2}\chi^{4m-2k}+\frac{\beta_1^2}{\beta_2^1}\chi^{4(m-k)}.$$

Since $0<m-k<n$, we can apply the above fact. Then we have $m-k=1$, $m=2$ and $\beta_1^2=16\beta_2$, that is, $h=-1$, $l=-2$ and $\beta_1^2=16\beta_2$.

If $l=h$, then we have

$$G(z)=1-2(\beta_1+\beta_2)e^H+(\beta_1-\beta_2)^2e^{2H}.$$

If $\beta_1\neq\beta_2$, then

$$G(z)=MN\left(e^H-\frac{1}{M}\right)\left(e^H-\frac{1}{N}\right),\qquad M=(\sqrt{\beta_1}+\sqrt{\beta_2})^2,\ N=(\sqrt{\beta_1}-\sqrt{\beta_2})^2.$$

Hence $P(R)=4$. If $\beta_1=\beta_2$, then $G(z)=1-4\beta_1e^H$ and hence $P(R)=4$.

Summing up these results we have theorem 3.

3. We shall apply this theorem 3 to an analytic mapping. Let S be an ultrahyperelliptic surface of finite order with $P(S)=4$, which is defined by

$$y^2=g(x),\ g(x)=(e^{K(x)}-\gamma)(e^{K(x)}-\delta),\ \gamma\delta(\gamma-\delta)\neq 0,\ K(0)=0.$$

Here "of finite order" means that $K(x)$ is a polynomial of x. Let R be an ultrahyperelliptic surface of finite order with $P(R)=3$, which is defined by

$$y^2=G(x),\ G=1-2\beta_1e^H-2\beta_2e^L+\beta_1^2e^{2H}-2\beta_1\beta_2e^{H+L}+\beta_2^2e^{2L}$$

with two polynomials H and L, $H(0)=L(0)=0$ and with two non-zero constants β_1, β_2.

Assume there exists an analytic mapping φ from S into R. Then by a theorem in [5] there exist an entire function h and a meromorphic function f such that $f^2g=G\circ H$. Then by a theorem in [1] $h(z)$ must be a polynomial of z. Hence $H\circ h$ and $L\circ h$ must be polynomials of z. Therefore by the proof of our theorem 3 we have either $H\circ h-H\circ h(0)=L\circ h-L\circ h(0)$ or $H\circ h-H\circ h(0)=2(L\circ h-L\circ h(0))$, $\beta_2^2e^{2L\circ h(0)}=16\beta_1e^{H\circ h(0)}$ or $2(H\circ h-H\circ h(0))=L\circ h-L\circ h(0)$, $\beta_1^2e^{2H\circ h(0)}=16\beta_2e^{L\circ h(0)}$ or $H\circ h-H\circ h(0)=-L\circ h+L\circ h(0)$, $16\beta_1\beta_2e^{H\circ h(0)+L\circ h(0)}=1$.

Put

$$H(z)=\sum_{n=1}^{p} h_n z^n, \qquad L(z)=\sum_{n=1}^{p} l_n z^n, \qquad h(z)=\sum_{n=0}^{\nu} a_n z^n.$$

(I) $H\circ h-H\circ h(0)=L\circ h-L\circ h(0)$. Then we have $h_n=l_n$, $n=1, \cdots, p$. Hence $H=L$, which implies that $P(R)=4$. This is a contradiction.

(II) $H\circ h-H\circ h(0)=2(L\circ h-L\circ h(0))$, $\beta_2^2 e^{2L\circ h(0)}=16\beta_1 e^{H\circ h(0)}$. Then we have $h_n=2l_n$, $n=1, \cdots, p$ and $H\circ h(0)=2L\circ h(0)$. Hence $H=2L$ and $\beta_2^2=16\beta_1$, which implies that $P(R)=4$. This is again a contradiction.

(III) $2(H\circ h-H\circ h(0))=L\circ h-L\circ h(0)$, $\beta_1^2 e^{2H\circ h(0)}=16\beta_2 e^{L\circ h(0)}$. This case leads to a contradiction similarly.

(IV) $H\circ h-H\circ h(0)=-(L\circ h-L\circ h(0))$, $16\beta_1\beta_2 e^{H\circ h(0)+L\circ h(0)}=1$. Then we have $h_n=-l_n$, $n=1, \cdots, p$ and hence $H=-L$ and $H\circ h(0)=-L\circ h(0)$. This implies that $16\beta_1\beta_2=1$ and hence $P(R)=4$, which is a contradiction.

THEOREM 4. *Let R and S be two ultrahyperelliptic surfaces with $P(R)=3$ and $P(S)=4$ and of finite order defined above. Then there is no analytic mapping from S into R.*

This is a partial solution of our problem in [5].

REFERENCES

[1] HIROMI, G., AND H. MUTŌ, On the existence of analytic mappings, I. Kōdai Math. Sem. Rep. **19** (1967), 236–244.

[2] HIROMI, G., AND M. OZAWA, On the existence of analytic mappings between two ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **17** (1965), 281–306.

[3] OZAWA, M. On complex analytic mappings. Kōdai Math. Sem. Rep. **17** (1965), 93–102.

[4] OZAWA, M. On ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **17** (1965), 103–108.

[5] OZAWA, M. On the existence of analytic mappings. Kōdai Math. Sem. Rep. **17** (1965), 191–197.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.