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A REMARK ON PROBABILISTICAL DEFINITENESS FOR SELF-ADJOINT OPERATORS

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In his paper [4], Urbanik has introduced the concept of joint probability distribution into a system of observables and discussed the relation to the simultaneous observability, or the commutativity of the system. As a result of this argument he has proved that if the system is commutative then for all states it has a joint probability distribution. The converse implication is also valid with an additional condition that each element of the system has discrete spectrum. Immediately after the publication of this paper, Varadarajan [5] has shown independently the above converse implication without assuming any special conditions, which is given as a result of his generalized probability theory.

In the present paper we define a certain modified condition for the Urbanik's case, about a set of self-adjoint operators, i.e., say, probabilistical semi-definiteness, and show this to be necessary and sufficient for its commutativity under the restriction of their boundedness. Then it follows that probabilistical semidefiniteness and probabilistical definiteness are equivalent whenever each element of the system is bounded.

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§ 1. Let R be the real line and $\mathfrak{B}$ its Borel field. For any index set I, R^I will denote the Cartesian product set of all points $X=\Pi x_i$, $i\in I$, consisting of real numbers x_i which correspond to each $i\in I$. If I is a finite set of n elements, then R^I is a finite n-dimensional space, and it will often be denoted by R^n and its Borel field by $\mathfrak{B}^n$. Let $p_{i_1,\dots,i_n}$ be a projection which maps the points $X=\Pi x_i$ of R^I into the points whose coordinates vanish except their finite number of indices $i_1\in I,\cdots,i_n\in I$ and whose i_j-coordinates $(j=1,\cdots,n)$ remain invariant. In the following we identify this projected space with an n-dimensional space R^n. For all n-dimensional Borel sets $M^{(n)}$ and any $i_1,\cdots,i_n$-th coordinates, $\mathfrak{B}^I$ will stand for a Borel field generated by cylinder sets of the form $p^{-1}_{i_1,\dots,i_n}(M^{(n)})$.

If a probability measure Q is given on the measurable space $(R^I,\mathfrak{B}^I)$, we define the probability measure $Q_{i_1,\dots,i_n}$ on $(R^n,\mathfrak{B}^n)$ by

$$Q_{i_1,\dots,i_n}(M^{(n)})=Q(p^{-1}_{i_1,\dots,i_n}(M^{(n)}) \quad \text{for} \quad M^{(n)}\in\mathfrak{B}^n,$$

and also for any real valued measurable function $q(X)$ on R^I we define the prob-

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ability measure $Q_{q(X)}$ on $(R, \mathfrak{B})$ by

$$Q_{q(X)}(M)=Q(q^{-1}(M)) \quad \text{for} \quad M\in\mathfrak{B}.$$

Let A be a self-adjoint operator on a Hilbert space $\mathfrak{H}$ and $A=\int\lambda dE_\lambda$ its spectral resolution. Then for any unit vector ϕ in the domain of A

$$(*) \qquad P_A^\phi(M)=\int_M d(E_\lambda\phi, \phi) \quad \text{for} \quad M\in\mathfrak{B}$$

is a probability measure on $(R, \mathfrak{B})$. Throughout this paper we assume that a vector ϕ which appears in such a formula is always in the domain of a suitable operator.

§ **2.** The following proposition is an extention of the theorem of Cramér and Wold [1] and we shall omit the proof, because it is almost similar to the original one.

PROPOSITION. *Let Q and Q' be probability measures on $(R^I, \mathfrak{B}^I)$. If for any finite number of indices $i_j\in I\,(j=1,\cdots,n)$ and for any $a_j\in R\,(j=1,\cdots,n)$ $Q_{q(X)}(M)=Q'_{q(X)}(M)$ holds for arbitrary $M\in\mathfrak{B}$, where $q(X)=\sum_{j=1}^n a_jx_{i_j}$ for $X=\Pi x_i\in R^I$, then Q and Q' are identical, i.e., $Q=Q'$.*

According to Urbanik [4], we shall reformulate the joint probability distribution and the probabilistical definiteness for a set $\mathfrak{M}$ of self-adjoint operators on $\mathfrak{H}$.[1] The set $\mathfrak{M}$ is said to have a joint probability distribution at a unit vector $\phi\in\mathfrak{H}$, if there exists a probability measure Q^ϕ on $(R^{\mathfrak{M}}, \mathfrak{B}^{\mathfrak{M}})$ depending on ϕ such that for any finite number of $A_1, \cdots, A_n$ of $\mathfrak{M}$ and for any $a_j\in R\,(j=1, \cdots, n)$

$$(**) \qquad P^\phi{}_{\sum_n a_jA_j}(M)=Q^\phi{}_{\sum_n a_jx_{A_j}}(M) \quad \text{for} \quad X=\Pi x_T\in R^{\mathfrak{M}} \text{ and } M\in\mathfrak{B},$$

where $\sum_n$ means $\sum_{j=1}^n$, $\mathfrak{M}$ is used as an index set and the left hand side is defined by (*). And $\mathfrak{M}$ is said to be *probabilistically definite,* if the formula (**) holds for all unit vector $\phi\in\mathfrak{H}$. Now, we shall introduce a modified concept of the above.

DEFINITION. The set $\mathfrak{M}$ of self-adjoint operators is *probabilistically semi-definite,* if for all unit vectors $\phi\in\mathfrak{H}$ there exists a probability measure Q^ϕ on $R^{\mathfrak{M}}$ such that for any pair A, B of $\mathfrak{M}$ and $M\in\mathfrak{B}$

$$P^\phi_{A^2\pm B}(M)=Q^\phi_{(x_A)^2\pm x_B}(M), \qquad P^\phi_{A\pm B}(M)=Q^\phi_{x_A\pm x_B}(M), \qquad P^\phi_B(M)=Q^\phi_{x_B}(M),$$

where $X=\Pi x_T$, $T\in\mathfrak{M}$.

THEOREM 1. *Let $\mathfrak{M}$ be a set of self-adjoint operators.*

(1) *The set $\mathfrak{M}$ is probabilistically definite, if and only if for any unit vector*

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1) The original definition by Urbanik was introduced only for a finite collection of self-adjoint operators. However it is easily described, under his way, for a infinite collection of such operators. This is analogous to the term "to have a joint distribution in the weak sense" defined by Varadarajan. The definition is sufficient to define only for any pair of self-adjoint operators.

$\psi\in\mathfrak{H}$ there exist a probability space $(\Omega, \mathfrak{B}, \mu^{\psi})$ and a set $\{\hat{x}_T\colon T\in\mathfrak{M}\}$ of random variables on Ω associated with $\mathfrak{M}$ bijectively[2] *such that*

$$\left(p\left(\sum_{j=1}^{n} a_j A_j\right)\psi, \psi\right)=\int_{\Omega} p\left(\sum_{j=1}^{n} a_j \hat{x}_{A_j}(\omega)\right)d\mu^{\psi}(\omega) \qquad a_j\in R, \qquad A_j\in\mathfrak{M}$$

for every real polynomial p.

(2) *The set $\mathfrak{M}$ is probabilistically semi-definite, if and only if for any unit vector $\psi\in\mathfrak{H}$ there exist a probability space $(\Omega, \mathfrak{B}, \mu^{\psi})$ and a set $\{\hat{x}_T\colon T\in\mathfrak{M}\}$ of random variables on Ω associated with $\mathfrak{M}$ bijectively such that*

$$(p(A^2\pm B)\psi, \psi)=\int_{\Omega} p(\hat{x}_A(\omega)^2\pm\hat{x}_B(\omega))d\mu^{\psi}(\omega),$$

$$(p(A\pm B)\psi, \psi)=\int_{\Omega} p(\hat{x}_A(\omega)\pm\hat{x}_B(\omega))d\mu^{\psi}(\omega),$$

$$(p(B)\psi, \psi)=\int_{\Omega} p(\hat{x}_B(\omega))d\mu^{\psi}(\omega)$$

for $A, B\in\mathfrak{M}$ and for every real polynomial p.

Proof of (1). Suppose $\mathfrak{M}$ is probabilistically definite. Denote $A=\sum_{j=1}^{n} a_j A_j$, $x=\sum_{j=1}^{n} a_j x_{A_j}$ and $\hat{x}=\sum_{j=1}^{n} a_j \hat{x}_{A_j}$ for any $A_j\in\mathfrak{M}$. Then, for any unit vector $\psi\in\mathfrak{H}$, there exists a probability measure Q^{ψ} on $R^{\mathfrak{M}}$ such that

$$P_A^{\psi}(M)=Q_x^{\psi}(M) \qquad M\in\mathfrak{B}, \qquad X=\Pi x_T\in R^{\mathfrak{M}}.$$

By taking $\hat{x}_A$ to be the random variables on a probability space $(R^{\mathfrak{M}}, \mathfrak{B}^{\mathfrak{M}}, Q^{\psi})$

$$\hat{x}_A(X)=\hat{x}_A(\Pi x_T)=x_A \text{ (=a real number)},$$

we obtain $Q^{\psi}(\hat{x}(X)\in M)=Q_x^{\psi}(M)$ for $M\in\mathfrak{B}$. Therefore from the above equality

$$(p(A)\psi, \psi)=\int_R p(\lambda)dP_A^{\psi}(\lambda)=\int_R p(\lambda)dQ_x^{\psi}(\lambda)=\int_{R^{\mathfrak{M}}} p(\hat{x}(X))dQ^{\psi}(X).$$

Thus we can find $(R^{\mathfrak{M}}, \mathfrak{B}^{\mathfrak{M}}, Q^{\psi})$ as a probability space $(\Omega, \mathfrak{B}, \mu^{\psi})$ and $\{\hat{x}_T\colon T\in\mathfrak{M}\}$ as a set of random variables.

Conversely, suppose that

$$(p(A)\psi, \psi)=\int_{\Omega} p(\hat{x}(\omega))d\mu^{\psi}(\omega)$$

for every polynomial p, where $A=\sum_{j=1}^{n} a_j A_j$ and $\hat{x}=\sum_{j=1}^{n} a_j \hat{x}_{A_j}$. Taking $X(\omega)=\Pi\hat{x}_T(\omega)$ as a random vector from a probability space $(\Omega, \mathfrak{B}, \mu^{\psi})$ to a measurable space $(R^{\mathfrak{M}}, \mathfrak{B}^{\mathfrak{M}})$, we may define a probability measure on Q^{ψ} on $(R^{\mathfrak{M}}, \mathfrak{B}^{\mathfrak{M}})$ by

2) By "bijectively" is meant "onto and one to one". When M is linear, so is this bijection.

$$Q^{\phi}(\tilde{M})=\mu^{\phi}(X^{-1}(\tilde{M})) \qquad \text{for} \qquad \tilde{M}\in\mathfrak{B}^{\mathfrak{M}}$$

and, in particular, for $M\in\mathfrak{B}$ and $x=\sum_{j=1}^{n}a_jx_{A_j}$

$$Q_x^{\phi}(M)=Q^{\phi}(\{\Pi x_T\in R^{\mathfrak{M}}\colon\ x\in M\})=\mu^{\phi}(\hat{x}(\omega)\in M).$$

Then, since $A=\sum_{j=1}^{n}a_jA_j$ is self-adjoint operator,

$$\int_R p(\lambda)dP_A^{\phi}(\lambda)=\int_R p(\lambda)dQ_x^{\phi}(\lambda) \qquad x=\sum_{j=1}^{n}a_jx_{A_j}$$

for every polynomial p, and hence we obtain

$$P^{\phi}{}_{\sum_n a_jA_j}(M)=Q^{\phi}{}_{\sum_n a_jx_{A_j}}(M).$$

Proof of (2). Suppose $\mathfrak{M}$ is probabilistically semi-definite. Let us denote random variables on $(R^{\mathfrak{M}},\mathfrak{B}^{\mathfrak{M}})$ by $\hat{x}_A(X)=\hat{x}_A(\Pi x_T)=x_A$. Then the quite similar argument may be applicable to this case and the required three equations in the theorem are obtained.

The converse is also similar to the preceding: Define a probability measure Q^{ϕ} on $(R^{\mathfrak{M}},\mathfrak{B}^{\mathfrak{M}})$ and degenerate its measure to $Q_{(x_A)^2\pm x_B}$, $Q_{x_A\pm x_B}$ and Q_{x_B} on $(R,\mathfrak{M})$. And then, probabilistical semi-definiteness is obtained.

§ 3. **Applications.** In what follows, we assume the operator to be bounded if the contrary is not explicitly stated. The following lemma is immediate from the theorem of Kleinecke [2] and we adopt his method.

LEMMA. *Let A and B be self-adjoint operators. If $AB-BA$ commutes with A, then A and B are commutative.*

Proof. We denote $D_AB=AB-BA$, the inner derivative D_A of B with respect to A. By Leibniz rule,

$$D_A^nB^n=D_A^n(B\cdot B^{n-1})=\sum_{j=0}^{n}\binom{n}{j}(D_A^jB)(D_A^{n-j}B^{n-1})$$

where $D_A^0C=C$, $D_A^1C=D_AC$ and $D_A^nC=D_A(D_A^{n-1}C)$. Since D_AB and A is commutative, $D_A^k(D_AB)^i=0$ for $k\geqq 1$ and $i\geqq 1$, which implies

$$D_A^nB^n=n!(D_AB)^n, \qquad n=1,2,\cdots.$$

On the other hand $\|D_AB\|\leqq 2\|A\|\cdot\|B\|$. Regarding D_A as a bounded operator on the uniform normed algebra generated by B,

$$\|(AB-BA)^n\|^{1/n}=\|(D_AB)^n\|^{1/n}\leqq\frac{2}{(n!)^{1/n}}\|A\|\cdot\|B\|.$$

From this inequality we know that the spectral radius of a self-adjoint operator $i(AB-BA)$ reduces to zero. Thus $\|AB-BA\|=\|i(AB-BA)\|=0$, that is, $AB=BA$.

Unfortunately, it is not possible to extend the above lemma to the unbounded case. Because, we have already known by Heisenberg's uncertainty principle that $PQ-QP\subset -i\hbar I$ for a position operator Q and a momentum operator P, so that $i(PQ-QP)$ and P are commutative but P and Q do not commute with each other, where the commutativity of two self-adjoint operators means that of their corresponding spectral projections.

REMARK. It follows immediately from the above lemma, that if a set $\mathfrak{M}$ of bounded self-adjoint operators on a Hilbert space has the Jordan product $A\circ B=(AB+BA)/2$ with the property $(A\circ A)\circ B=A\circ(A\circ B)$ for each pair A, $B\in\mathfrak{M}$, then $\mathfrak{M}$ is commutative [3]. Indeed, $(A\circ A)\circ B=(A^2B+BA^2)/2$ and $A\circ(A\circ B)=(A^2B+BA^2)/4+ABA/2$ which imply $A^2B+BA^2=2ABA$ or $A(AB-BA)=(AB-BA)A$, and hence by Lemma $AB=BA$. By this fact, an associative Jordan algebra of bounded self-adjoint operators is necessarily commutative.

THEOREM 2. *Let $\mathfrak{M}$ be a set of bounded self-adjoint operators. Then $\mathfrak{M}$ is commutative if and only if $\mathfrak{M}$ is probabilistically semi-definite.*

Proof. If $\mathfrak{M}$ is probabilistically semi-definite, then there exists a probability space $(\Omega, \mathfrak{B}, \mu^{\psi})$ and a set of random variables $\{x_T\colon T\in\mathfrak{M}\}$. Further we get for any pair of operators A and B in $\mathfrak{M}$,

$$
\begin{aligned}
(2ABA\psi, \psi)&=((A+B)^3\psi, \psi)-((A-B)^3\psi, \psi)-2(B^3\psi, \psi)\\
&\qquad -((A^2+B)^2\psi, \psi)+((A^2-B)^2\psi, \psi)\\
&=\int_\Omega (x_A(\omega)+x_B(\omega))^3 d\mu^{\psi}(\omega)-\int_\Omega (x_A(\omega)-x_B(\omega))^3 d\mu^{\psi}(\omega)\\
&\quad -2\int_\Omega x_B(\omega)^3 d\mu^{\psi}(\omega)-\int_\Omega (x_A(\omega)^2+x_B(\omega))^2 d\mu^{\psi}(\omega)+\int_\Omega (x_A(\omega)^2-x_B(\omega))^2 d\mu^{\psi}(\omega)\\
&=\int_\Omega x_A(\omega)^2 x_B(\omega) d\mu^{\psi}(\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
((A^2B+BA^2)\psi, \psi)&=\frac{1}{2}((A^2+B)^2\psi, \psi)-\frac{1}{2}((A^2-B)^2\psi, \psi)\\
&=\frac{1}{2}\int_\Omega (x_A(\omega)^2+x_B(\omega))^2 d\mu^{\psi}(\omega)-\frac{1}{2}\int_\Omega (x_A(\omega)^2-x_B(\omega))^2 d\mu^{\psi}(\omega)\\
&=\int_\Omega x_A(\omega)^2 x_B(\omega) d\mu^{\psi}(\omega).
\end{aligned}
$$

Whence we obtain an equality $(2ABA\psi, \psi)=((A^2B+BA^2)\psi, \psi)$ for all unit vectors ψ, thus $2ABA=A^2B+BA^2$, for the both sides are self-adjoint. This equality shows the commutativity of $AB-BA$ and A, and hence that of A and B.

Conversely, suppose that $\mathfrak{M}$ is commutative. Let $\mathfrak{A}$ be a C^*-algebra generated by $\mathfrak{M}$ and the unit operator I, then $\mathfrak{A}$ is commutative. The Gelfand-representation theorem tells us that there is a compact space Ω and that $\mathfrak{A}$ is isometrically isomorphic to the space $C(\Omega)$ of all continuous functions on Ω. Then we denote by x_T the element of $C(\Omega)$ which corresponds to the element T of $\mathfrak{M}$. For a unit vector ϕ, putting

$$f^{\phi}(x_A)=(A\phi, \phi) \qquad \text{for} \qquad A\in\mathfrak{A},$$

then, by Riesz-Markov-Kakutani theorem, a probability measure μ^{ϕ} exists on Ω and

$$(A\phi, \phi)=\int_{\Omega} x_A(\omega)d\mu^{\phi}(\omega) \qquad \text{for} \qquad A\in\mathfrak{A}.$$

Hence, in particular,

$$(p(A^2\pm B)\phi, \phi)=\int_{\Omega} p(x_A(\omega)^2\pm x_B(\omega))d\mu^{\phi}(\omega),$$

$$(p(A\pm B)\phi, \phi)=\int_{\Omega} p(x_A(\omega)\pm x_B(\omega))d\mu^{\phi}(\omega), \qquad (p(B)\phi, \phi)=\int_{\Omega} p(x_B(\omega))d\mu^{\phi}(\omega).$$

It follows from these equalities and Theorem 1 that $\mathfrak{M}$ is probabilistically semi-definite. Q.E.D.

REMARK Urbanik, however, showed, if $\mathfrak{M}$ is a set of observables, its commutativity implies the probabilistical definiteness under the separability condition of $\mathfrak{H}$, which is proved through the method of the characteristic functions. But in case of bounded operators, this separability condition can be omitted by the same method as above. Here we show another proof of the Urbanik's without using the characteristic function. If $\mathfrak{M}$ is a commuting set of self-adjoint operators not necessarily bounded on a separable Hilbert space $\mathfrak{H}$, then there is a generating self-adjoint operator A with the spectral representation $\int\lambda dE_{\lambda}$, that is, every element B of $\mathfrak{M}$ is expressed in the form $B=x_B(A)=\int x_B(\lambda)dE_{\lambda}$ by a real valued Baire measurable function x_B defined on R. Thus for any unit vector $\phi\in\mathfrak{H}$, we can take a probability measure P_A^{ϕ} over $(R, \mathfrak{B})$ and a set of random variables $\{x_T: T\in\mathfrak{M}\}$ such that

$$(A\phi, \phi)=\int_R \lambda d(E_{\lambda}\phi, \phi)=\int_R \lambda dP_A^{\phi}(\lambda)$$

and

$$\left(p\left(\sum_{j=1}^{n} a_j A_j\right)\phi, \phi\right)=\left(p\left(\sum_{j=1}^{n} a_j x_{A_j}(A)\right)\phi, \phi\right)$$
$$=\int_R p\left(\sum_{j=1}^{n} a_j x_{A_j}(\lambda)\right)dP_A^{\phi}(\lambda).$$

Hence, by Theorem 1, $\mathfrak{M}$ is probabilistically definite.

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