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CYLINDERS IN EUCLIDEAN SPACE E^{2+N}

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Introduction. Massey [1] proved that a complete surface of Gaussian curvature zero in Euclidean space E^3 of dimension 3 is a cylinder. This theorem was extended to a surface of the principal and the secondary curvatures $\lambda=\mu=0$ in Euclidean space E^4 of dimension 4. In this paper we shall prove the following theorem:

THEOREM A. *A connected, oriented and complete surface M^2 of class C^4 in Euclidean space E^{2+N} $(N\geqq 1)$ of dimension $2+N$ with the curvatures $\lambda_1=\lambda_2=\cdots=\lambda_N=0$ is a cylinder.*

As usual, a cylinder means a surface which is generated by a moving straight line with a fixed direction through a curve in E^{2+N}. The author expresses his deep gratitude to Professor T. Ōtsuki who gave him a lot of useful suggestions.

1. In the following we consider a connected, oriented and complete surface M^2 of class C^4 in Euclidean space E^{2+N}. We shall make use of Frenet-frames in the sense of Ōtsuki. In our case we cannot define the uniquely determined Frenet-frame, but we can take suitably such a frame $(p, e_1, e_2, e_3, \cdots, e_{2+N})$ from the first. Then we have the following:

$$dp=e_1\omega_1+e_2\omega_2, \qquad de_A=\sum_B \omega_{AB}e_B, \qquad \omega_{AB}+\omega_{BA}=0, \qquad A, B=1, 2, \cdots, 2+N. \tag{1.1}$$

$$\begin{cases} d\omega_i=\omega_{ij}\wedge\omega_j+\sum_r \omega_{ir}\wedge\omega_r, \qquad d\omega_{12}=\sum_r \omega_{1r}\wedge\omega_{r2}, \\ d\omega_{ir}=\omega_{ij}\wedge\omega_{jr}+\sum_s \omega_{is}\wedge\omega_{sr}, \\ d\omega_{rs}=\sum_j \omega_{rj}\wedge\omega_{js}+\sum_t \omega_{rt}\wedge\omega_{ts}, \end{cases} \tag{1.2}$$
$$i, j=1, 2, \qquad r, s, t=3, \cdots, 2+N.$$
$$\omega_{ir}=\sum_r A_{rij}\omega_j, \qquad A_{rij}=A_{rji},$$

where ω_1, ω_2 and ω_{12} are the basic forms and the connection form of M^2 with respect to the induced metric. And we have

$$\omega_{1r}\wedge\omega_{2r}=\lambda_{r-2}\omega_1\wedge\omega_2, \tag{1.3}$$

$$\omega_{1r}\wedge\omega_{2s}+\omega_{1s}\wedge\omega_{2r}=0, \tag{1.4}$$

$$G(p)=\sum_r \lambda_{r-2}. \tag{1.5}$$

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By the hypothesis of $\lambda_1=\lambda_2=\cdots=\lambda_N=0$, we could define two sets:

$$M_0=\{p\in M^2\colon \mathrm{rank}(A_{rij}(p))=0, \text{ for all } r\}, \tag{1. 6}$$

$$M_1=\{p\in M^2\colon \text{there exists } r \text{ such that } \mathrm{rank}(A_{rij}(p))=1\}. \tag{1. 7}$$

It is clear that M_0 is closed and M_1 is open in M^2.

Now suppose that $M_1\neq\phi$ and take a point $p\in M_1$, then there exists r such that rank $(A_{rij}(p))=1$. The asymptotic direction with respect to e_r at p is defined by the direction of the tangent vector $v=\sum_i v^i e_i$, which satisfies $\sum_{i,j} A_{rij}v^i v^j=0$. Let e_1 be the unit tangent vector field defined in a neighborhood U_p of p in M_1 which has the asymptotic direction with respect to e_r and let e_2 be the unit tangent vector field orthogonal to e_1 where the orientation of (e_1, e_2) is coherent with the one of M^2. Then it follows that

$$A_{rij}=\begin{pmatrix} 0 & 0 \\ 0 & f_r \end{pmatrix} \quad \text{or,} \quad \omega_{1r}=0, \quad \omega_{2r}=f_r\omega_2, \tag{1. 8}$$

where f_r is a continuous everywhere non-zero function defined in U_p. By virtue of (1. 4), (1. 8) and $\lambda_1=\lambda_2=\cdots=\lambda_N=0$ we get the following:

$$A_{sij}=\begin{pmatrix} 0 & 0 \\ 0 & f_s \end{pmatrix} \quad \text{or,} \quad \omega_{1s}=0, \quad \omega_{2s}=f_s\omega_2, \quad (s\neq r) \tag{1. 9}$$

where f_s is a continuous function defined in U_p. For any unit normal vector $e=\sum_t a^t e_t$ at p, the second fundamental form with respect to e is given by

$$d^2p\cdot e=\sum_t a^t f_t\omega_2\omega_2. \tag{1. 10}$$

The above relation shows that the asymptotic direction at p is independent of the choice of the unit normal vector at p.

The proof of theorem A will complete if we prove the following theorem.

THEOREM B. *Each connected component of M_1 is a proper cylinder.*

A point of a cylinder is called *proper* if there does not exist any neighborhood of the point which is contained in a plane. A cylinder is called proper if all the points of it are proper. In fact we can easily show that if $\mathring{M}_0\neq\phi$, then each connected component of $\mathring{M}_0$ is a piece of plane, where $\mathring{M}_0$ is the largest open set contained in M_0. Since M^2 is complete and flat by (1. 5), the universal covering space is E^2. And because the covering map π is a local isometry, we get the following:

$$\pi^{-1}(\mathring{M}_0)=\overset{\circ}{\widehat{\pi^{-1}(M_0)}}, \tag{1. 11}$$

$$\pi^{-1}(M_1')=(\pi^{-1}(M_1))' \tag{1. 12}$$

where M_1' is the set of all boundary points of M_1 in M^2. By virtue of theorem B, $\mathring{M}_0$ consists of plane stripes in E^{2+N} and we can prove that through each point of M_1', there passes a unique straight line contained entirely in M_1'. These facts show that through each point of M^2, there passes a unique straight line contained entirely in M^2, and we can prove that these straight lines are all mutually parallel, to others which implies theorem A.

2. Let us prove theorem B. From (1. 8) and (1. 9) we get the following;

$$\omega_{12}=g\omega_2, \tag{2. 1}$$

where g is a continuous function in U_p. Here we consider U_p as an open ball. By (2. 1), we get at once

$$\omega_1=du, \tag{2. 2}$$

where u is a continuous function in U_p. It is clear that the asymptotic line through p is a straight line segment. In fact it is given by $\omega_2=0$, then we get along it, $dp=e_1du$, $de_1=de_2=0$ by (1. 8), (1. 9) and (2. 1). By (1. 8), (1. 9), (2. 1) and the structure equations of M^2, we get along it

$$\frac{dg}{du}+g^2=0, \tag{2. 3}$$

$$\frac{1}{2}\frac{d}{du}\sum_s f_s^2+\sum_s f_s^2\cdot g=0, \tag{2. 4}$$

we may consider that $u=0$ corresponds to p. Solving these differential equations, we get the following:

$$g(u)=\frac{g(0)}{g(0)u+1}, \tag{2. 5}$$

$$h(u)=\frac{h(0)}{[g(0)u+1]^2}, \tag{2. 6}$$

where $h(u)=\sum_s f_s^2$. It is obvious that $h(u)$ is independent of the choice of Frenet-frames in U_p, and so we may consider it on M_1. On the other hand, let us consider the following function:

$$\bar{h}=\sum_s(\mathrm{trace}(A_{s\imath j}))^2, \tag{2. 7}$$

making use of any field of frames of $M^2\subset E^4$, $\bar{h}$ is a continuous function defined on M^2, and by the definitions of M_1 and M_0, we get the following:

$$\bar{h}|M_1=h, \tag{2. 8}$$

$$\bar{h}|M_0=0. \tag{2. 9}$$

Making use of $\bar{h}$ and the expression of h, we can prove that the asymptotic line through p is a full straight line. In fact, otherwise it is written as $x=x(u)$, $0\leqq u<u_0$. By completeness, it follows that $\lim_{u\uparrow u_0}x(u)\in M^2$. By virtue of (2. 6) and (2. 8) we get $\lim_{u\uparrow u_0}\bar{h}(u)\neq 0$, which implies that $\lim_{u\uparrow u_0}x(u)\in M_1$.

Since $g(u)$ is continuous, we must have $g(0)=0$, i.e., $g=0$. And since $de_1=0$,

$de_2=\sum_s f_s e_s \omega_2 \neq 0$, each connected component of M_1 is a proper cylinder. Then the proof of theorem B is completed.

References

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