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A REMARK CONCERNING A RENEWAL THEOREM ON (J, X)-PROCESSES

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1. The purpose of this note is to weaken the conditions assumed in our previous paper [2] in proving the following result on (J, X)-processes:

$$(1)\qquad \lim_{x\to\infty}\sum_{n=1}^{\infty} P\{x\leqq X_1+\cdots+X_n\leqq x+h\}=\frac{h}{m}\qquad \text{if}\quad 0<m<\infty,$$

where m is a constant such that

$$p\text{-}\lim_{n\to\infty}\frac{X_1+\cdots+X_n}{n}=m.$$

Let $\{(J_n, X_n);\ n=0, 1, 2, \cdots\}$ be a (J, X)-process with the state space $\boldsymbol{I}_r\times R$, where $\boldsymbol{I}_r=\{1, 2, \cdots, r\}$ and $R=(-\infty, \infty)$, i.e., a two-dimensional stochastic process satisfying the conditions $X_0\equiv 0$ (a.s.) and

$$P\{J_n=k,\ X_n\leqq x|(J_0, X_0), \cdots, (J_{n-1}, X_{n-1})\}=Q_{J_{n-1},\,k}(x)\qquad \text{(a.s.)}$$
$$\text{for all } (k, x)\in \boldsymbol{I}_r\times R,$$

where $\{Q_{jk}(\cdot);\ j, k=1, 2, \cdots, r\}$ is a family of non-decreasing functions defined on R such that $Q_{jk}(-\infty)=0$ for $j, k=1, 2, \cdots, r$, and $\sum_{k=1}^{r} Q_{jk}(+\infty)=1$ for $j=1, 2, \cdots, r$.

In [2] the following assumptions were made to prove (1): (i) There is a positive integer M such that every element of the matrix P^M is positive, where $P=(p_{jk})$ is the $r\times r$ matrix with elements $p_{jk}\overset{\text{def}}{=}Q_{jk}(+\infty)$, (ii) the conditional distribution of X_n given $J_{n-1}=j$ and $J_n=k$ is a non-lattice distribution with finite second moment, and (iii)

$$\overline{\lim_{|t|\to\infty}}\,|\psi_{jk}(t)|<1\quad \text{for all}\quad j, k\in \boldsymbol{I}_r,$$

where

$$\psi_{jk}(t)=E\{e^{itX_n}|J_{n-1}=j,\ J_n=k\}$$
$$=\frac{1}{p_{jk}}\int_{-\infty}^{\infty} e^{itx}dQ_{jk}(x).$$

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In this note we shall show that the condition (iii) is unnecessary, namely, we shall prove

THEOREM. (1) *holds true if the conditions* (i) *and* (ii) *are satisfied.*

To prove this we shall use the method of Maruyama [3], which is concerned with a renewal theorem for independent identically distributed random variables.

2. *Proof of the theorem.* Let $S_n=\sum_{\nu=1}^{n} X_\nu$ and $N_H(x, x+h)=E\{\sum_{n=1}^{\infty} H(S_n-x-h/2)\}$, where H is a real-valued even function. If $H(x)=1$ for $|x|\leqq h/2$ and $=0$ for $|x|>h/2$, then (1) may be written in the form

$$\lim_{x\to\infty} N_H(x, x+h)=\frac{1}{m}\int_{-\infty}^{\infty} H(x)dx. \tag{2}$$

Following [3], we shall prove that the relation (2) holds for a positive real-valued even function $H(x)$ such that the integrals

$$\int_{-\infty}^{\infty} H(x)e^{-itx}dx=h(t),$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{itx}dt=H(x)$$

are absolutely convergent and the equalities hold for all t and x, and further $h(t)$ vanishes outside a finite interval $[-c, c]$.

Denoting by $\varphi_n(t)$ the characteristic function of S_n and introducing a convergence factor r, we may write

$$\begin{aligned} N_H\left(x-\frac{h}{2}, x+\frac{h}{2}\right) &= \lim_{r\to 1-0} \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{-itx}\sum_{n=1}^{\infty} r^n\varphi_n(t)dt \\ &= \lim_{r\to 1-0} \frac{1}{2\pi}\left[\int_{c\geqq|t|\geqq\delta}+\int_{|t|<\delta}\right] h(t)e^{-itx}\sum_{n=1}^{\infty} r^n\varphi_n(t)dt \\ &= \lim_{r\to 1-0} [I_1(x)+I_2(x)]. \end{aligned}$$

On applying the Riemann-Lebesgue theorem, we get

$$\lim_{x\to\infty}\lim_{r\to 1-0} I_1(x)=0$$

for every $\delta>0$. To evaluate the second term $I_2(x)$ we recall the following facts (see [1], [2]);

(a) The equation $\det(\delta_{jk}-zp_{jk}\,\phi_{jk}(t))=0$ has a root $z=\zeta_0(t)$ for small t such that $\zeta_0(t)\to 1$ as $t\to 0$. Under the assumption (i), for sufficiently small $|t|<t_0$,

$$\varphi_n(t)=\frac{\sigma(t)}{\zeta_0(t)^n}+\rho_n(t) \quad \text{and} \quad |\rho_n(t)|\leqq\frac{K}{(1+\varepsilon)^n}, \tag{3}$$

where K and ε are some positive constants. The functions $\zeta_0(t)$ and $\sigma(t)$ have respectively continuous derivatives of the second order for $|t|<t_0$, and $\sigma(t)\to 1$ as $t\to 0$.

(b) $1/\zeta_0(t)$ may be written as

$$\frac{1}{\zeta_0(t)}=\zeta_0(t)^{-1}=1+imt-\left(m^2+\frac{m'}{2}\right)t^2+o(t^2),$$

where $m=i\zeta_0'(0)$ and $m'=\zeta_0''(0)$, both being real constants.

From (3) and applying the Riemann-Lebesgue theorem, we obtain

$$\lim_{x\to\infty}\lim_{r\to 1-0}\int_{-\delta}^{\delta} h(t)e^{-itx}\sum_{n=1}^{\infty} r^n\rho_n(t)dt=0$$

and thus it suffices to consider the integral

$$J(x)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\delta}^{\delta}\text{Re}\left\{\frac{r\sigma(t)\zeta_0(t)^{-1}}{1-r\zeta_0(t)^{-1}}h(t)e^{-itx}\right\}dt.$$

Now the following relations follows from (a) and (b);

$$
\begin{aligned}
&R=R(t)\overset{\text{def}}{=}\text{Re}(1-\zeta_0(t)^{-1})=O(t^2),\\
&I=I(t)\overset{\text{def}}{=}\text{Im}(imt-\zeta_0(t)^{-1})=o(t^2),\\
&R_1=R_1(t)\overset{\text{def}}{=}\text{Re}(\sigma(t))=1+O(t),\\
&I_1=I_1(t)\overset{\text{def}}{=}\text{Im}(\sigma(t))=O(t),\\
&Q(r,t)=|1-r\zeta_0(t)^{-1}|^2\\
&\quad=(1-r)^2+2(1-r)rR+r^2R^2+r^2m^2t^2-2r^2mtI+r^2I^2\\
&\quad=(1-r)^2+r^2m^2t^2+(1-r)O(t^2)+o(t^2),\\
&Q(1,t)=m^2t^2+O(t^3).
\end{aligned}
\tag{4}
$$

Note also that $R(t)\geqq 0$ (see [2]).

We rewrite $J(x)$ as

$$
\begin{aligned}
J(x)=&\frac{r}{2\pi}\int_{-\delta}^{\delta}\frac{1-r}{Q(r,t)}\text{Re}\{\sigma(t)\zeta_0(t)^{-1}h(t)e^{-itx}\}dt\\
&+\frac{r^2}{2\pi}\int_{-\delta}^{\delta}\frac{1}{Q(r,t)}\text{Re}\{\sigma(t)(1-\overline{\zeta_0(t)^{-1}})\zeta_0(t)^{-1}h(t)e^{-itx}\}dt.
\end{aligned}
$$

Using (4) and the argument in [3] we can show that the first integral tends to $h(0)/(2m)$ as $r\to 1-0$ and $\delta\to +0$.

The second integral can be rewritten as

$$\frac{r^2}{2\pi}\int_{-\delta}^{\delta}\frac{1}{Q(r,t)}[R_1\{R(1-R)-(mt-I)^2\}-I_1(mt-I)]h(t)\cos xt\,dt$$

$$+\frac{r^2}{2\pi}\int_{-\delta}^{\delta}\frac{1}{Q(r,t)}[-R_1I+I_1\{R(1-R)-(mt-I)^2\}]h(t)\sin xt\,dt$$

$$+\frac{r^2}{2\pi}\int_{-\delta}^{\delta}\frac{1}{Q(r,t)}R_1\,mt\,h(t)\sin xt\,dt$$

$$=J_1(x)+J_2(x)+J_3(x).$$

The functions in brackets in the integrands of $J_1(x)$ and $J_2(x)$ are respectively of order $O(t^2)$ and $o(t^2)$. Hence the Riemann-Lebesgue theorem is applicable to $J_1(x)$ and $J_2(x)$, thus obtaining

$$\lim_{x\to\infty}\lim_{r\to1-0}(J_1(x)+J_2(x))=0.$$

Now consider $J_3(x)$.

$$\lim_{r\to1-0}J_3(x)=\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{R_1\,mt}{Q(1,t)}h(t)\sin xt\,dt$$

$$=\frac{1}{2\pi m}\int_{-\delta}^{\delta}h(t)\frac{\sin xt}{t}dt+\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{1}{t}\left\{\frac{R_1\,mt^2}{Q(1,t)}-\frac{1}{m}\right\}h(t)\sin xt\,dt$$

$$=\frac{1}{2\pi m}\int_{-c}^{c}h(t)\frac{\sin xt}{t}dt+o(1)=\frac{1}{4\pi m}\int_{-x}^{x}d\xi\int_{-c}^{c}h(t)e^{it\xi}dt+o(1)$$

$$=\frac{1}{2m}\int_{-x}^{x}H(\xi)d\xi+o(1)=\frac{1}{2m}\int_{-\infty}^{\infty}H(x)dx+o(1)=\frac{h(0)}{2m}+o(1)$$

as $x\to\infty$, since

$$\frac{1}{t}\left\{\frac{R_1\,mt^2}{Q(1,t)}-\frac{1}{m}\right\}$$

is bounded for $|t|<\delta$. Thus (2) holds for $H(x)$ satisfying our regularity conditions.

The theorem will be proved, if we show that (2) holds for the indicator function $H(x)$ of the interval $[-h/2, h/2]$. The proof of this part, however, has already been given by Maruyama [3].

References

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