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APPLICATION OF THE THEORY OF MARKOV PROCESSES TO COMMINUTION

I. THE CASE OF DISCRETE TIME PARAMETER

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1. Introduction.

The purpose of the present work is to study the size distribution of solid particles obtained on grinding from the standpoint of the theory of stochastic processes. The term comminution or grinding applies to any industrial operation for the production of fine powders by mechanical breaking. Most comminuting machines such as ball, tube, and rod mills, however, subject charged masses to a continued repetition of breakage mechanisms. For this reason, we are concerned only with the process of repeated fracture.

The problem of the distribution function for the dimensions of ground materials has been discussed in several mathematical papers. Kolmogorov [12], Halmos [11], Epstein [8], and Rényi [15] showed, by application of the central limit theorem to a probabilistic model, that the distribution function approaches the logarithmic-normal form asymptotically. Unfortunately, their fundamental postulate that the probability of fracture does not depend on particle size is too restrictive to serve as a realistic representation of actual grinding.

Under more general conditions, Filippov [10] deduced a limit distribution different from that of logarithmic-normal type, regarding the comminution process as a kind of purely discontinuous Markov process with a continuous time parameter. Although his treatment is relatively inaccessible and based on the introduction of not a few complicated functions, it appears to be essentially the stochastic analogue of the deterministic formulation developed by Bass [2].

We shall construct a new stochastic-process model for comminution, which includes previous theories as special cases and yields a simpler interpretation of the phenomenon. Our approach leads to fragment size distributions that are asymptotically logarithmic-normal, even when breaking probabilities of solid pieces vary considerably with their volumes. It is mathematically interesting that this result gives an example in which the central limit theorem holds for the sum of mutually dependent random variables.

In the present paper (I), we deal mainly with the case of a discrete time parameter, after providing a brief explanation of the stochastic model proposed.

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The detailed investigation of a continuous parameter process will be presented in the subsequent paper of this series (II).

2. The proposed model.

Consider a collection of particles exposed to repetitive fracture. Each particle breaks with some fracture probability and then splits into a number of fragments. This event is repeated at random intervals. In order to describe the sequence of these events, we introduce the following random variables: At the initial time mark one point in a certain particle. Let the random variable X_t denote the fineness of the fragment that contains the marked point at an arbitrary time t.

The fineness of a granular material means the logarithm of the inverse ratio of its size to a standard dimension. Here the particle size can be reasonably defined by the volume rather than by the diameter, since the total volume of particles remains unchanged during breakage. For an ensemble starting with solids originally of the same dimension, it is convenient to adopt this initial size as a reference standard. If one replaces the size of broken pieces by their fineness, logarithmic-normal distributions are transformed into normal distributions.

The stochastic process $\{X_t, t \geqq 0\}$ thus obtained is obviously a purely discontinuous process, in the sense that the sample functions jump discontinuously at the moment of fracture. The time parameter may be taken to be the set of nonnegative integers or the set of nonnegative real numbers, according as the applied forces are discrete or continuous in time. Hereafter we shall confine ourselves to the discrete parameter process, which can be thought of as composed of discrete steps.

We further assume the process $\{X_t, t=0, 1, \cdots\}$ to possess the Markov property. Intuitively, this hypothesis implies that the probability law governing the future development of the process is completely determined by a knowledge about the present value of particle fineness, regardless of the manner in which the present state has emerged from the past. In fact, experimental evidence indicates that the probability that a brittle solid will break at a particular instant depends solely on the stress at that instant and is independent of the previous loading history [6].

A discrete parameter Markov process may be described in terms of the (one-step) transition probability function $F(x, \xi, t)$, representing the conditional probability of $X_t \leqq x$ under the assumption that $X_{t-1}=\xi$. Especially $\lambda_x(t)=1-F(x, x, t)$ expresses the fracture probability of the marked particle of fineness x at time t. For an integer-valued Markov chain, it is usual to define the transition probability $f_{ij}(t)$ as the conditional probability that $X_t=j$, given that $X_{t-1}=i$. The so-called breakage matrix [5] corresponds to the matrix of transition probabilities $(f_{ij}(t))$.

We now proceed to show what relation exists between the probability distribution of the random variable X_t and the observed distribution of the fragment fineness. The distribution function of the random variable $X_t^{(Z)}$ associated with a fixed point Z will be designated by $P^{(Z)}(x, t)$; namely, $P^{(Z)}(x, t)=P\{X_t^{(Z)} \leqq x\}$. Let us introduce a new random variable $Y_t^{(Z)}(x)$ which assumes the value 1 or 0 according to whether $X_t^{(Z)} \leqq x$ or $X_t^{(Z)} > x$. Then the mathematical expectation of

$Y_t^{(Z)}(x)$ becomes $E\{Y_t^{(Z)}(x)\}=P^{(Z)}(x, t)$.

Needless to say, real particle fineness distributions are in general subject to to random fluctuations. The volume fraction of fragments up to fineness x after time t may be represented by the random variable

(2. 1) $$M(x, t)=\frac{1}{V}\int_V Y_t^{(Z)}(x)dv(Z),$$

the integration extending over the volume of all particles V. Hence we get

(2. 2) $$E\{M(x, t)\}=\frac{1}{V}\int_V P^{(Z)}(x, t)dv(Z)=P(x, t).$$

In words, the expectation of $M(x, t)$ is identical with the volume average of $P^{(Z)}(x, t)$, which we shall rewrite simply as $P(x, t)$.

In many practical applications it is of primary importance to examine the ideal case where there are N uniform particles at $t=0$. Let $M_k(x, t)$ be the random variable $M(x, t)$ defined with respect to the initial particle labeled k, and suppose that $M_1(x, t)$, $M_2(x, t)$, $\cdots$, $M_N(x, t)$ have a common distribution with finite expectation $P(x, t)$. Their arithmetic mean

(2. 3) $$M_N^*(x, t)=\frac{1}{N}\sum_{k=1}^{N} M_k(x, t)$$

stands for the proportion by volume of fragments not finer than x to the whole assembly of particles.

If the random variables $M_k(x, t)$, $k=1, 2, \cdots, N$, are mutually independent, the strong law of large numbers holds, so that

(2. 4) $$P\{\lim_{N\to\infty} M_N^*(x, t)=P(x, t)\}=1.$$

The law of large numbers in the sense of convergence with probability one is not necessarily valid for identically distributed but dependent random variables. If the common expectation $P(x, t)$ exists, however, the dependent sequence $\{M_N(x, t)\}$ obeys the weak law of large numbers, such that for every $\varepsilon>0$

(2. 5) $$\lim_{N\to\infty} P\{|M_N^*(x, t)-P(x, t)|>\varepsilon\}=0.$$

3. Markov processes with independent increments.

The stochastic process introduced above is a discrete parameter Markov process $\{X_t\}$ with distribution function $P(x, t)$. In addition to the random variables X_t, we define a sequence of random variables $\{Z_t\}$ in such a way that

(3. 1) $$Z_0=X_0;\ Z_t=X_t-X_{t-1},\ t=1, 2, \cdots.$$

The random variable Z_t refers to the fineness change during the tth step of the process. Therefore,

(3. 2) $$X_t=Z_0+Z_1+\cdots+Z_t$$

is the sum of $t+1$ jointly distributed random variables.

We first consider the simplest case that has been treated by Kolmogorov, Halmos, Epstein, and Rényi. According to Epstein [8], their basic assumptions may be stated as follows: the probability of fracture of any piece is independent of the size of the piece and of the presence of other particles; and the fraction by volume of material having dimension less than $ky(0\leqq k\leqq 1)$ arising from the breakage of a unit volume of size y is independent of y itself.

With our notation the above two postulates reduce to

$$F(x, \xi, t)=G(x-\xi, t), \tag{3. 3}$$

which asserts that the transition probability function $F(x, \xi, t)$ depends on x and ξ only through the difference $x-\xi$. A Markov process satisfying (3. 3) is said to be spatially homogeneous or to have independent increments. In this case, the increments Z_t are mutually independent random variables with distribution functions

$$P\{Z_t\leqq x\}=G(x, t), \qquad G(x, 0)=P(x, 0). \tag{3. 4}$$

The process $\{X_t\}$ is thus a sequence of the consecutive sums of independent random variables.

Now one can apply the central limit theorem to prove that the random variable X_t is asymptotically normally distributed. Suppose that the independent random variables Z_s, $s=0, 1, \cdots, t$, with distribution functions $G(x, s)$ have finite means μ_s and variances σ_s^2, and put

$$M_t=E\{X_t\}=\sum_{s=0}^{t}\mu_s, \qquad S_t^2=\mathrm{Var}\{X_t\}=\sum_{s=0}^{t}\sigma_s^2. \tag{3. 5}$$

The central limit theorem gives conditions under which the distribution function for the reduced variable $(X_t-M_t)/S_t$ converges as $t\to\infty$ to the standardized normal distribution function

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-z^2/2}dz. \tag{3. 6}$$

In order that

$$\lim_{t\to\infty}P\left\{\frac{X_t-M_t}{S_t}\leqq x\right\}=\Phi(x) \tag{3. 7}$$

and that

$$\lim_{t\to\infty}\max_{0\leqq s\leqq t}\frac{\sigma_s}{S_t}=0, \tag{3. 8}$$

it is necessary and sufficient that Lindeberg's condition

$$\lim_{t\to\infty}\frac{1}{S_t^2}\sum_{s=0}^{t}\int_{|x-\mu_s|\geqq\varepsilon S_t}x^2dG(x, s)=0 \tag{3. 9}$$

be satisfied for every $\varepsilon>0$. In particular, the sequence $\{Z_t\}$ obeys the normal convergence law (3. 7), provided that

$$\lim_{t\to\infty}\frac{1}{S_t^{2+\delta}}\sum_{s=0}^{t}E\{|Z_s-\mu_s|^{2+\delta}\}=0 \tag{3.10}$$

for some fixed $\delta>0$. (Ljapunov's sufficient condition)

Moreover, for a sequence of identically distributed independent random variables, the central limit theorem holds without additional requirements. When the process $\{X_t\}$ with $X_0=0$ is homogeneous in both space and time, that is, when $F(x, \xi, t)$ is a function of $x-\xi$ alone, the random variables Z_t have a common distribution with mean μ and variance σ^2, so that

$$\lim_{t\to\infty}P\left\{\frac{X_t-\mu t}{\sigma t^{1/2}}\leqq x\right\}=\Phi(x). \tag{3.11}$$

It should be noted that this result corresponds to the special case where the probability of fracture is constant irrespective of the number of steps that have occurred prior to the given step.

4. Time-homogeneous Markov chains in which direct transitions are possible only to neighhoring states.

In the following we shall investigate the limiting behavior of discrete parameter Markov processes that are not spatially homogeneous. For these processes Lindeberg's condition (3. 9) is neither necessary nor sufficient, because the random variable X_t can no longer be conceived of as the sum of mutually independent random variables. Nevertheless, it will be verified that under appropriate conditions the Markov process $\{X_t\}$ with dependent increments actually satisfies the central limit theorem (3. 7).

For simplicity we shall restrict our attention to the Markov chain with stationary transition probabilities, whose state space is the set of nonnegative integers 0, 1, 2, $\cdots$. The one-step transition probability f_{ij} from state i to state j may be written in the form

$$f_{ij}=\begin{cases}0 & \text{for} \quad j<i,\\ 1-\lambda_i & \text{for} \quad j=i,\\ \lambda_i\mu_{ij} & \text{for} \quad j>i,\end{cases} \tag{4.1}$$

where $0\leqq\lambda_i\leqq1$, $0\leqq\mu_{ij}\leqq1$, $\sum_{j=i+1}^{\infty}\mu_{ij}=1$. As physical quantities λ_i and μ_{ij} represent the fracture probability of a particle of fineness i and the fineness distribution of its fragments, respectively.

Kolmogorov's hypothesis discussed in the preceding section states that λ_i is a constant independent of i and μ_{ij} depends only on $j-i$; in symbols,

$$\lambda_i=\lambda, \qquad \mu_{ij}=\nu_{j-i}. \tag{4.2}$$

On the other hand, Filippov [10] assumed that λ_i decreases exponentially with increasing i; more precisely,

$$\lambda_i=\lambda e^{-ai}, \qquad \mu_{ij}=\nu_{j-i}. \tag{4.3}$$

Here λ and a are positive constants. It was already mentioned that his model does not generate asymptotically normal distributions.

We are concerned with the limiting behavior of X_t under the condition that λ_i varies as a power function of i

$$\lambda_i=\lambda(h+i)^{-b}, \tag{4. 4}$$

where λ, h, and b are positive constants. In this section it is further supposed that

$$\mu_{ij}=\nu_{j-i}=\begin{cases} 1 & \text{if } j=i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4. 5}$$

In other words, the system is permitted to change only through transitions from states to their immediate neighbors.

Such a process $\{X_t\}$ is constructed with reference to the following experimental setup: At $t=0$ let there be merely particles of the same dimension y_0 and put $X_0=0$. Assume that each particle produces κ identical fragments upon splitting. Define the fineness of a broken piece of size y as $\log_\kappa(y_0/y)$. Then the sample function of X_t is increased by one through a single breakage. As the process continues, we get a succession of transitions $0\to1\to2\to\cdots$. Consequently, all the particles that have undergone exactly i splittings are in the ith state.

It is convenient to introduce a new sequence $\{T_i\}$, whose random variable T_i indicates the number of steps required for the process to visit state i. Denoting by S_j, $j=0, 1, \cdots, i-1$, the sojourn times in state j, we have

$$T_i=S_0+S_1+\cdots+S_{i-1}. \tag{4. 6}$$

In contrast to the original variable X_t, T_i is the sum of mutually independent random variables. The probability distributions of X_t and T_i are related by the obvious identity

$$P\{X_t\geqq i\}=P\{T_i\leqq t\}. \tag{4. 7}$$

LEMMA 1. *Suppose that*

$$\lim_{i\to\infty} P\left\{\frac{T_i-Ai^\alpha}{Bi^\beta}\leqq x\right\}=F(x), \tag{4. 8}$$

where A, α, B, β are nonnegative constants such that $B\neq0$, $\alpha>\beta$, and $F(x)$ is a proper and continuous distribution function. Then it follows from (4. 7) *that*

$$\lim_{t\to\infty} P\left\{\frac{X_t-Ct^\gamma}{Dt^\delta}>-x\right\}=F(x) \tag{4. 9}$$

with

$$\begin{aligned} &C=A^{-1/\alpha}, \qquad \gamma=1/\alpha; \\ &D=\alpha^{-1}A^{-(1+\beta)/\alpha}B, \qquad \delta=(1-\alpha+\beta)/\alpha. \end{aligned} \tag{4. 10}$$

Proof. Let $t\to\infty$ and $i\to\infty$ in such a way that

$$\frac{t-Ai^{\alpha}}{Bi^{\beta}}\to x. \tag{4.11}$$

Since $i^{\beta}/t=o(i^{\alpha}/t)\to 0$, we find

$$\begin{aligned} i&=A^{-1/\alpha}[t-Bi^{\beta}\{x+o(1)\}]^{1/\alpha} \\ &=\left(\frac{t}{A}\right)^{1/\alpha}-\frac{Bt^{(1-\alpha+\beta)/\alpha}}{\alpha A^{(1+\beta)/\alpha}}\{x+o(1)\} \\ &=Ct^{\gamma}-Dt^{\delta}\{x+o(1)\}. \end{aligned} \tag{4.12}$$

Therefore the probability functions on both sides of (4.7) lead to

$$\begin{aligned} P\{X_t\geqq i\}&\to P\left\{\frac{X_t-Ct^{\gamma}}{Dt^{\delta}}>-x\right\}, \\ P\{T_i\leqq t\}&\to P\left\{\frac{T_i-Ai^{\alpha}}{Bi^{\beta}}\leqq x\right\}\to F(x), \end{aligned} \tag{4.13}$$

which complete the proof. This argument is due to Feller [9], who studied the case of $\alpha=2\beta=1$.

Next we shall show that T_i possesses a nearly normal distribution given by

$$\lim_{i\to\infty} P\left\{\frac{T_i-\lambda^{-1}(b+1)^{-1}i^{b+1}}{\lambda^{-1}(2b+1)^{-1/2}i^{b+1/2}}\leqq x\right\}=\Phi(x). \tag{4.14}$$

From the definition of S_j it is evident that

$$P\{S_j=m\}=\lambda_j(1-\lambda_j)^{m-1}. \tag{4.15}$$

Accordingly we obtain

$$\begin{aligned} E\{T_i\}&=\sum_{j=0}^{i-1}\frac{1}{\lambda_j}=\frac{i^{b+1}}{\lambda(b+1)}+o(i^{b+1}), \\ \operatorname{Var}\{T_i\}&=\sum_{j=0}^{i-1}\frac{1-\lambda_j}{\lambda_j^2}=\frac{i^{2b+1}}{\lambda^2(2b+1)}+o(i^{2b+1}), \\ \sum_{j=0}^{i-1}E\{[S_j-E\{S_j\}]^4\}&=\sum_{j=0}^{i-1}\frac{(1-\lambda_j)(9-9\lambda_j+\lambda_j^2)}{\lambda_j^4}=\frac{9i^{4b+1}}{\lambda^4(4b+1)}+o(i^{4b+1}) \end{aligned} \tag{4.16}$$

as $i\to\infty$. This means that T_i satisfies Ljapunov's sufficient condition

$$\lim_{i\to\infty}\sum_{j=0}^{i-1}E\{[S_j-E\{S_j\}]^4\}/[\operatorname{Var}\{T_i\}]^2=0. \tag{4.17}$$

It can be easily seen that X_t is also asymptotically normally distributed. Combination of Lemma 1 and (4.14) yields

$$\lim_{t\to\infty} P\left\{\frac{X_t-[\lambda(b+1)t]^{1/(b+1)}}{(2b+1)^{-1/2}[\lambda(b+1)t]^{1/2(b+1)}}>-x\right\}=\Phi(x). \tag{4.18}$$

Taking account of $\Phi(x)=1-\Phi(-x)$, we have the following theorem.

THEOREM 1. *Under the assumptions* (4. 4) *and* (4. 5),

$$\lim_{t\to\infty} P\left\{\frac{X_t-[\lambda(b+1)t]^{1/(b+1)}}{(2b+1)^{-1/2}[\lambda(b+1)t]^{1/2(b+1)}}\leqq x\right\}=\Phi(x) \tag{4. 19}$$

holds for any real number x.

Notice that in (4. 19) we have the central limit theorem applied to the sum of mutually dependent random variables Z_s.

5. General time-homogeneous Markov chains.

In this section we consider more general time-homogeneous Markov chains in which direct transitions from a state i are possible to all succeeding states $j>i$. Instead of the condition (4. 5), we shall assume

$$\mu_{ij}=\begin{cases}\nu_{j-i} & \text{whenever } j>i,\\ 0 & \text{otherwise,}\end{cases} \tag{5. 1}$$

where the mean ν and the variance ρ^2 of the distribution $\{\nu_k\}$ exist and

$$\sum_{k=1}^{\infty} k^{[4b]}\nu_k<\infty, \tag{5. 2}$$

the symbol $[r]$ referring to the smallest integer not less than r. The restriction (4. 4) on λ_i, together with the initial condition $X_0=0$, will be imposed as in § 4.

Let the random variables $N(t)$, U_m and $X_n{}^*$ represent the number of breakages up to time t, the fineness change during the mth fracture, and the fineness after the nth fracture, respectively. Then we have the relations

$$X_t=Z_1+Z_2+\cdots+Z_t=U_1+U_2+\cdots+U_{N(t)}, \tag{5. 3}$$

$$X_n{}^*=U_1+U_2+\cdots+U_n. \tag{5. 4}$$

Since U_m are mutually independent and have the same distribution as $\{\nu_k\}$, X_t is the sum of independent identically distributed random variables, but their number $N(t)$ is a random variable depending on U_m.

For the present case the random variable T_i must be interpreted as the time taken to reach or pass the ith state, so that

$$P\{X_t\geqq i\}=P\{T_i\leqq t\} \tag{5. 5}$$

may hold. If we define a random variable $M(i)$ by the number of breakages required for attaining to or jumping over state i, furthermore, a similar identity can be derived as follows:

$$P\{M(i)\geqq n\}=P\{X_n{}^*\leqq i\}. \tag{5. 6}$$

Denoting by R_m the time difference between the $(m-1)$st and mth fracture, we obtain

$$T_i = R_1 + R_2 + \cdots + R_{M(i)}, \tag{5.7}$$

$$T_n^* = R_1 + R_2 + \cdots + R_n, \tag{5.8}$$

where T_n^* stands for the time elapsed until the nth breakage takes place. In the particular case when the condition (4.5) is satisfied, (5.7) reduces to (4.6), because the sojourn time S_{m-1} equals R_m for every positive integer m.

It is worth while noting that the nonnegative integer-valued random variable $M(i)$ is independent of all the components $R_1, R_2, \cdots$. The random variables R_m, $m = 1, 2, \cdots$, although not identically distributed, are mutually independent. Hence T_i is the sum of independent random variables R_m, whose number is itself a random variable independent of the R_m. We shall verify that the probability distribution associated with such a sequence $\{T_i\}$ is nearly normal, employing a useful theorem due to Dobrušin [7].

THEOREM 2. *Let $\xi(n)$ be a random variable such that*

$$\lim_{n\to\infty} P\left\{\frac{\xi(n)-An^{\alpha}}{Bn^{\beta}} \leqq x\right\} = F(x), \tag{5.9}$$

where $F(x)$ is a proper distribution function, $A \neq 0$, $B \neq 0$, and $\alpha > \beta$, and let η_i be a random variable independent of $\xi(n)$ such that

$$\lim_{i\to\infty} P\left\{\frac{\eta_i - Ci^{\gamma}}{Di^{\delta}} \leqq x\right\} = G(x), \tag{5.10}$$

where $G(x)$ is a proper distribution function, $C \neq 0$, $D \neq 0$, and $\gamma > \delta > 0$. Suppose that

$$(\alpha-1)\gamma + \delta = \beta\gamma, \tag{5.11}$$

and set

$$H(x) = P\{BC^{\beta}\xi + \alpha AC^{\alpha-1}D\eta \leqq x\}, \tag{5.12}$$

in which ξ and η are mutually independent random variables with distribution functions $F(x)$ and $G(x)$, respectively. Then

$$\lim_{i\to\infty} P\left\{\frac{\xi(\eta_i) - AC^{\alpha}i^{\alpha\gamma}}{i^{\beta\gamma}} \leqq x\right\} = H(x). \tag{5.13}$$

It is now necessary to investigate the limiting behavior of T_n^* and $M(i)$. First we shall show that T_n^* satisfies the condition (5.9) of Theorem 2. If the sequence of independent random variables $\{R_m\}$ obey the central limit theorem, their consecutive sum T_n^* would be asymptotically normally distributed. Therefore, we prove that R_m really satisfy Ljapunov's sufficient condition

$$\lim_{n\to\infty} \sum_{m=1}^{n} E\{[R_m - E\{R_m\}]^4 / [\mathrm{Var}\{T_n^*\}]^2 = 0. \tag{5.14}$$

The moments of R_m may be calculated from those of S_j. As stated in the foregoing section, the sojourn time in the jth state S_j is characterized by the probability (4.15). Under the hypothesis (4.4), the first four moments of S_j become for $j \to \infty$

$$
(5.15)\quad
\begin{aligned}
E\{S_j\} &= \frac{1}{\lambda_j} = \frac{j^b}{\lambda} + o(j^b),\\
E\{S_j^2\} &= \frac{2-\lambda_j}{\lambda_j^2} = \frac{2j^{2b}}{\lambda^2} + o(j^{2b}),\\
E\{S_j^3\} &= \frac{6-6\lambda_j+\lambda_j^2}{\lambda_j^3} = \frac{6j^{3b}}{\lambda^3} + o(j^{3b}),\\
E\{S_j^4\} &= \frac{24-36\lambda_j+14\lambda_j^2-\lambda_j^3}{\lambda_j^4} = \frac{24j^{4b}}{\lambda^4} + o(j^{4b}).
\end{aligned}
$$

The rth moment of R_m is clearly given by

$$
(5.16)\quad E\{R_m^r\} = \sum_{j=m-1}^{\infty} P\{X_{m-1}^*=j\} E\{S_j^r\}.
$$

Considering that the assumption (5. 2) ensures the existence of $E\{(X_n^*)^b\}$, $E\{(X_n^*)^{2b}\}$, $E\{(X_n^*)^{3b}\}$, and $E\{(X_n^*)^{4b}\}$, substitution of (5. 15) into (5. 16) yields

$$
(5.17)\quad
\begin{aligned}
E\{R_m\} &= \frac{E\{(X_{m-1}^*)^b\}}{\lambda} + o[E\{(X_{m-1}^*)^b\}],\\
E\{R_m^2\} &= \frac{2E\{(X_{m-1}^*)^{2b}\}}{\lambda^2} + o[E\{(X_{m-1}^*)^{2b}\}],\\
E\{R_m^3\} &= \frac{6E\{(X_{m-1}^*)^{3b}\}}{\lambda^3} + o[E\{(X_{m-1}^*)^{3b}\}],\\
E\{R_m^4\} &= \frac{24E\{(X_{m-1}^*)^{4b}\}}{\lambda^4} + o[E\{(X_{m-1}^*)^{4b}\}]
\end{aligned}
$$

as $m\to\infty$.

To estimate $E\{(X_n^*)^{rb}\}$ for $r=1, 2, 3, 4$, we recall that X_n^* is the sum of a constant number of mutually independent random variables with the common distribution $\{\nu_k\}$. In view of the strong law of large numbers, $X_n^*/(\nu n)$ or $(X_n^*)^{rb}/(\nu n)^{rb}$ should converge to unity with probability one as n tends to infinity. Making use of the following lemma, we can readily observe that for large n

$$
(5.18)\quad E\{(X_n^*)^{rb}\} = (\nu n)^{rb} + o(n^{rb}).
$$

LEMMA 2. *Let $\{\zeta_n\}$ be an infinite sequence of random variables and $\{w_n\}$ a sequence of real numbers tending to infinity. It is assumed that*

$$
(5.19)\quad \frac{\zeta_n}{w_n} \to 1 \qquad \textit{in probability}
$$

as $n\to\infty$. Then

$$
(5.20)\quad \lim_{n\to\infty} \frac{E\{\zeta_n\}}{w_n} = 1,
$$

provided that ζ_n has a finite expectation $E\{\zeta_n\}$.

Proof. One can expand the characteristic function $\phi_n(\theta)$ of ζ_n as

$$\phi_n(\theta)=1+i\theta E\{\zeta_n\}+i\theta\psi_n(\theta), \quad \lim_{\theta\to 0}\psi_n(\theta)=0 \tag{5. 21}$$

for any θ. Since the characteristic function $\phi_n'(\theta)$ of ζ_n/w_n is equal to $\phi_n(\theta/w_n)$,

$$\phi_n'(\theta)=1+i\theta\frac{E\{\zeta_n\}}{w_n}+i\theta\frac{\psi_n(\theta/w_n)}{w_n}. \tag{5. 22}$$

On the other hand, (5. 19) implies

$$\lim_{n\to\infty}\phi_n'(\theta)=e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\cdots, \tag{5. 23}$$

whence (5. 20) follows immediately. This lemma does not mean that (5. 20) is true for sequences $\{\zeta_n\}$ without expectations [9].

From (5. 17) and (5. 18), it turns out that as $m\to\infty$

$$\begin{aligned} E\{R_m\}&=\frac{\nu^b m^b}{\lambda}+o(m^b),\\ E\{R_m{}^2\}&=\frac{2\nu^{2b}m^{2b}}{\lambda^2}+o(m^{2b}),\\ E\{R_m{}^3\}&=\frac{6\nu^{3b}m^{3b}}{\lambda^3}+o(m^{3b}),\\ E\{R_m{}^4\}&=\frac{24\nu^{4b}m^{4b}}{\lambda^4}+o(m^{4b}). \end{aligned} \tag{5. 24}$$

The variance and the fourth central moment of R_m are

$$\begin{aligned} \mathrm{Var}\{R_m\}&=\frac{\nu^{2b}m^{2b}}{\lambda^2}+o(m^{2b}),\\ E\{[R_m-E\{R_m\}]^4\}&=\frac{9\nu^{4b}m^{4b}}{\lambda^4}+o(m^{4b}). \end{aligned} \tag{5. 25}$$

Consequently, we get as $n\to\infty$

$$\begin{aligned} E\{T_n{}^*\}&=\sum_{m=1}^{n}E\{R_m\}=\frac{\nu^b n^{b+1}}{\lambda(b+1)}+o(n^{b+1}),\\ \mathrm{Var}\{T_n{}^*\}&=\sum_{m=1}^{n}\mathrm{Var}\{R_m\}=\frac{\nu^{2b}n^{2b+1}}{\lambda^2(2b+1)}+o(n^{2b+1}),\\ \sum_{m=1}^{n}E\{[R_m-E\{R_m\}]^4\}&=\frac{9\nu^{4b}n^{4b+1}}{\lambda^4(4b+1)}+o(n^{4b+1}), \end{aligned} \tag{5. 26}$$

which confirm the validity of (5. 14). We have thus proved

THEOREM 3. *If* (4. 4), (5. 1), *and* (5. 2) *hold,* T_n^* *is asymptotically normally distributed; that is,*

(5. 27)
$$\lim_{n\to\infty} P\left\{\frac{T_n^*-\lambda^{-1}\nu^b(b+1)^{-1}n^{b+1}}{\lambda^{-1}\nu^b(2b+1)^{-1/2}n^{b+1/2}}\leqq x\right\}=\Phi(x)$$

for every fixed x.

Next we shall consider the limiting distribution of $M(i)$. The ordinary central limit theorem for equi-distributed components U_m asserts that

(5. 28)
$$\lim_{n\to\infty} P\left\{\frac{X_n^*-\nu n}{\nu^{1/2}n^{1/2}}\leqq x\right\}=\Phi(x).$$

Then it follows from (5. 6) that the conditions of Lemma 1 are satisfied by $M(i)$. Hence we find

(5. 29)
$$\lim_{i\to\infty} P\left\{\frac{M(i)-\nu^{-1}i}{\nu^{-3/2}\rho i^{1/2}}\leqq x\right\}=\Phi(x).$$

THEOREM 4. *Under the hypothesis* (5. 1), $M(i)$ *has an approximately normal distribution as indicated in* (5. 29).

The above two theorems concerning the asymptotic normality of T_n^* and $M(i)$ permit us to apply Dobrušin's theorem to T_i. In this case we have

(5. 30)
$$\begin{aligned} &A=\lambda^{-1}\nu^b(b+1)^{-1}, \qquad \alpha=b+1;\\ &B=\lambda^{-1}\nu^b(2b+1)^{-1/2}, \qquad \beta=b+1/2;\\ &C=\nu^{-1}, \qquad \gamma=1;\\ &D=\nu^{-3/2}\rho, \qquad \delta=1/2;\\ &F(x)=G(x)=\Phi(x),\end{aligned}$$

so that (5. 11) becomes identically

(5. 31)
$$(\alpha-1)\gamma+\delta=\beta\gamma=b+1/2.$$

According to Theorem 2, substitution of (5. 30) into (5. 13) gives

(5. 32)
$$\lim_{i\to\infty} P\left\{\frac{T_i-\lambda^{-1}\nu^{-1}(b+1)^{-1}i^{b+1}}{i^{b+1/2}}\leqq x\right\}$$
$$=P\left\{\frac{\xi}{\lambda\nu^{1/2}(2b+1)^{1/2}}+\frac{\rho\eta}{\lambda\nu^{3/2}}\leqq x\right\}.$$

Since ξ and η are mutually independent random variables with the standardized normal distribution function $\Phi(x)$, the following theorem may be easily obtained.

THEOREM 5. *The assumptions* (4. 4), (5. 1), *and* (5. 2) *lead to an asymptotically normal distribution such that*

$$\lim_{i\to\infty} P\left\{\frac{T_i-\lambda^{-1}\nu^{-1}(b+1)^{-1}i^{b+1}}{\sqrt{\nu^2/(2b+1)+\rho^2}\,\lambda^{-1}\nu^{-3/2}i^{b+1/2}}\leqq x\right\}=\Phi(x) \tag{5. 33}$$

Combining (5. 5) and (5. 33) with Lemma 1, we have finally

THEOREM 6. *Under the conditions of Theorem* 5, X_t *is asymptotically normally distributed in the sense that*

$$\lim_{t\to\infty} P\left\{\frac{X_t-[\lambda\nu(b+1)t]^{1/(b+1)}}{\sqrt{\nu/(2b+1)+\rho^2/\nu}\,[\lambda\nu(b+1)t]^{1/2(b+1)}}\leqq x\right\}=\Phi(x) \tag{5. 34}$$

for every fixed x.

The conclusion of this theorem points out that the process $\{X_t\}$ obeys the central limit theorem, despite the fact that X_t is the sum of the dependent random variables Z_s.

6. Concluding remarks.

In Theorem 5 we have presented an illustrative example of the central limit theorem for a random number of mutually independent but not necessarily identically distributed random variables. The asymptotic distribution of the sum of a random number of independent random variables with a common distribution was first determined by Robins [18]. It should be mentioned that our result (5. 33) for $b=0$ is substantially equivalent to the theorem of Robins.

The central limit theorem of this type has recently been extended by Anscombe [1], Rényi [16, 17], Mogyoródi [13, 14], Billingsley [3], and Blum, Hanson, and Rosenblatt [4] to cover more general cases. Their papers were devoted to the study of sequences of random variables whose expectations are zero, while they made no assumption about the dependence of the number of the random variables on each component.

We refer to the following theorem due to Mogyoródi [13].

THEOREM 7. *Suppose that* $\{\zeta_i\}$ $(i=1, 2, \cdots)$ *is a sequence of independent random variables and there exists a sequence of positive numbers* $\{B_i\}$ *tending to infinity such that* ζ_j/B_i $(j=1, 2, \cdots, i)$ *are infinitesimal and*

$$\lim_{i\to\infty} P\left\{\sum_{j=1}^{i}\zeta_j/B_i\leqq x\right\}=\Phi(x) \tag{6. 1}$$

for any real number x. *Let* $\{M(i)\}$ *be a sequence of random variables taking positive integer values and* $\{w(i)\}$ *a sequence of positive integers tending to infinity such that*

$$(6.2) \qquad \frac{M(i)}{w(i)} \to 1 \qquad \textit{in probability}$$

as $i\to\infty$. *In order that*

$$(6.3) \qquad \lim_{i\to\infty} P\left\{\sum_{j=1}^{i} \zeta_{M(i)}/B_{w(i)} \leqq x\right\} = \Phi(x),$$

it is necessary and sufficient that

$$(6.4) \qquad \lim_{\varepsilon\to 0}\limsup_{i\to\infty} \frac{B_{w(i)}}{B_{[w(i)(1+\varepsilon)]}} = 1.$$

If we put in (5.7)

$$T_i = E\{T_i\} + \tau_1\{M(i)\} + \tau_2\{M(i)\},$$

$$(6.5) \qquad \tau_1\{M(i)\} = [R_1 - E\{R_1\}] + [R_2 - E\{R_2\}] + \cdots + [R_{M(i)} - E\{R_{M(i)}\}],$$

$$\tau_2\{M(i)\} = [E\{R_1\} + E\{R_2\} + \cdots + E\{R_{M(i)}\}] - E\{T_i\},$$

then $T_i - E\{T_i\}$, $\tau_1\{M(i)\}$, and $\tau_2\{M(i)\}$ are random variables with zero mean. Mogyoródi's central limit theorem gives a necessary and sufficient condition for the asymptotic normality of $\tau_1\{M(i)\}$. Moreover, we have proved that in the present case $\tau_2\{M(i)\}$ and T_i are also asymptotically normally distributed.

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Note added in proof. Needless to say, Theorem 5 holds only for $b>0$, but applies to the case of $b=0$ with minor modification. Letting $b=0$ in (5. 33), we obtain

$$\lim_{t\to\infty} P\left\{\frac{T_i-\lambda^{-1}\nu^{-1}i}{\sqrt{\nu^2+\rho^2\lambda^{-1}\nu^{-3/2}i^{1/2}}}\leqq x\right\}=\Phi(x).$$

On the other hand, Robins' theorem shows that the correct result is

$$\lim_{t\to\infty} P\left\{\frac{T_i-\lambda^{-1}\nu^{-1}i}{\sqrt{(1-\lambda)\nu^2+\rho^2\lambda^{-1}\nu^{-3/2}i^{1/2}}}\leqq x\right\}=\Phi(x).$$

In this case the factor $1-\lambda$ appears in variance terms, because

$$\mathrm{Var}\{S_j\}=\frac{1-\lambda_j}{\lambda_j^2}=\begin{cases}(1-\lambda)/\lambda^2\fallingdotseq\lambda^{-2} & \text{if } \lambda_j=\lambda,\\ j^{2b}/\lambda^2+o(j^{2b}) & \text{if } \lambda_j=\lambda(h+j)^{-b}.\end{cases}$$

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