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ON A MATCHING METHOD FOR A LINEAR ORDINARY DIFFERFNTIAL EQUATION CONTAINING A PARAMETER, III

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§ 1. Introduction.

In the previous papers [2] and [3], the author considered the asymptotic nature of the solution of a vectorial linear differential equation of the form

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon) y \tag{1. 1}$$

in the neighborhood of a turning point which was assumed to be at the origin, proved the existence of two types of asymptotic solutions: the one is a socalled outer solution in some neighborhood of the origin which does not contain the origin itself, say in the outer domain, and the other is an inner solution in the inner domain containing the origin, and the most important result was that the outer and the inner domains are overlapped with each other for an arbitrarily small parameter ε. Because of this fact, the outer solution and the inner solution can be matched together at some suitable point which belongs to both of the domains, and once the connection matrix between them is known, we can understand the behavior of one outer solution in the direct neighborhood of the origin, and at the same time the continuation problem, or the Stokes phenomenon can be solved. A detailed description of this procedure about a second order equation is given in the lecture notes of Friedrichs [1], and the method of computing a connection matrix about a second order system of differential equation of the form

$$\varepsilon \frac{dy}{dx} = \begin{bmatrix} -ix, & \varepsilon i\gamma(x) \\ \varepsilon i\bar{\gamma}(x), & ix \end{bmatrix} y \qquad (x\text{: real, } i=\sqrt{-1})$$

was given by Wasow [5], and these are basic ideas in our subsequent study.

The purpose of this paper is to calculate the connection matrix $C(\varepsilon)$ between the outer and the inner solution of (1. 1) and also to study an asymptotic nature of it by using the results of [3]. The notations and the assumptions on the differential equation under consideration are identified with those of [3] throughout this paper, and when theorems or formulas are quoted from [3] we put the symbol "II" in front of their numbers.

Here we summarize the results of [3].

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1. Outer solution. *Let T be any sector of τ-plane with vertex at the origin and central angle less than $\pi/(m+q)h$, then there exists an actual solution of* (1.1) *of the form*

$$(1.2)\qquad \begin{aligned} &y_1(x, \varepsilon)=\Omega(x)u(\tau, \varepsilon), \qquad \tau=x^{1/mh},\\ &u(\tau, \varepsilon)=\hat{u}(\tau, \varepsilon)\exp\left\{\sum_{\nu=0}^{h}\varepsilon^{\nu-h}F^{(\nu)}(\tau)\right\}.\end{aligned}$$

Here $\hat{u}(\tau, \varepsilon)$ is a matrix function which can be asymptotically developable in powers of $\tau^{-(m+q)}\varepsilon$ such that for every positive integer r we have

$$(1.3)\qquad \hat{u}(\tau, \varepsilon)-\sum_{\nu=0}^{r}\varepsilon^{\nu}u^{(\nu)}(\tau)=E_r(\tau, \varepsilon)[\tau^{-(m+q)}\varepsilon]^{r+1}$$

in the outer domain D_1:

$$\tau\in T, \qquad 0<|\varepsilon|\leqq\varepsilon_1, \qquad |\arg\varepsilon|\leqq\delta_1, \qquad c_1|\varepsilon|^{1/(m+q)}\leqq|\tau|\leqq c_2$$

where $E_r(\tau, \varepsilon)$ is a bounded function in D_1 and $\varepsilon_1, \delta_1, c_1$ and c_2 are certain constants independent of ε but may depend on r. The matrix functions $F^{(\nu)}(\tau)$ can be written

$$(1.4)\qquad \begin{aligned} &F^{(\nu)}(\tau)=\tau^{-(m+q)(\nu-h)}\hat{F}^{(\nu)}(\tau), \qquad \nu\leqq h-1,\\ &F^{(h)}(\tau)=f^{(h)}\log+\hat{F}^{(h)}(\tau), \qquad \hat{F}^{(h)}(0)=0,\end{aligned}$$

where $\hat{F}^{(\nu)}(\tau)$ $(\nu\leqq h)$ are holomorphic in $|\tau|\leqq\tau_0$ and $f^{(h)}$ is a constant matrix, and $u^{(\nu)}(\tau)$ are of the form

$$(1.5)\qquad u^{(\nu)}(\tau)=\tau^{-(m+q)\nu}\hat{u}^{(\nu)}(\tau),$$

where $\hat{u}^{(\nu)}(\tau)$ are bounded in $|\tau|\leqq\tau_0$ and polynomials of $\log\tau$ *of degree at most ν whose coefficients are holomorphic in $|\tau|\leqq\tau_0$.*

2. Inner solution. *We denote by S any sector in the s-plane with sufficiently small central angle. Then the equation* (1.1) *has an actual solution of the form*

$$(1.6)\qquad \begin{aligned} &y_2(s, \rho)=\Omega(\varepsilon^a)v(s, \rho), \qquad s=x\varepsilon^{-a}, \qquad \rho=\varepsilon^{1/(m+q)}, \qquad a=\frac{mh}{m+q},\\ &v(s, \rho)=\Omega(s^{k(s)})\hat{v}(s, \rho)s^{k(s)\Pi}\exp[Q(s)].\end{aligned}$$

The matrix function $\hat{v}(s, \rho)$ is asymptotically developable in powers of $s^{k(s)e}\rho$ such that for every positive integer r we have

$$(1.7)\qquad \hat{v}(s, \rho)-\sum_{v=0}^{r}w^{(\nu)}(s)[s^{k(s)e}\rho]^{\nu}=E_r(s, \rho)[s^{k(s)}\rho]^{r+1}, \qquad \left(e=1+\frac{q}{m}+\frac{1}{mh}\right)$$

in the inner domain D_2 defined by

$$s\in S, \quad 0<|\rho|\leqq\rho_2, \quad |\arg|\leqq\rho_2, \quad |s^e\rho|\leqq c_3,$$

where $E_r(s,\rho)$ *is bounded in* D_2 *and* ρ_2, δ_2 *and* c_3 *are constants independent of* ρ. *The matrix functions* $w^{(\nu)}(s)$ *are bounded in the domain* D_2.

Now we take this opportunity to mention two remarks about the solution of the equation

$$\text{(II. 3. 20)} \qquad \frac{dv}{ds}=H^{(0)}(s)v,$$

where $H^{(0)}(s)$ is a matrix function whose elements are polynomials of s.

Firstly it was remarked in the previous paper [3] that the connection matrix between the solution $v_0^{(0)}$ defined in the neighborhood of the origin $s=0$ and the asymptotic solution $v_\infty^{(0)}$ defined in a certain sectorial domain containing $s=\infty$ is calculated by the asymptotic matching. But by this method we can find only a few of the elements in the connection matrix with reasonable accuracy, and the other elements remain essentialy undetermined, and consequently the successive determinations of the solutions in the neighborhood of the $s=0$ of the nonhomogeneous equations (II. 3. 21) is impossible. Therefore we must use the convergent matching, instead of the asymptotic matching, that will provide satisfactory accuracy for all elements in the connection matrix. This is accomplished by expressing $v_\infty^{(0)}$ as a convergent generalized factorial series, and in fact from a theorem of Turrittin [4], this is possible in our equation (II. 3. 20) under the assumption (II. 2. 10).

Secondly it is concenred with the definition of the sector S. In [3], the central angle of S was taken as the one which contains at least one singular direction: $\mathrm{Re}(\lambda_j-\lambda_k)s^{(m+q)/m}=0$ $(j,k=1,2,\cdots,n)$, but from a rather simple character of the coefficient matrix $H^{(0)}(s)$, we can take the central angle of S as at least $m\pi/(m+q)$ (its proof is given, for example, in Friedrichs [1]). Accordingly we give some improvements about the proofs of Lemma II. 3. 3 and Theorem II. 5. 1 in later.

In Section 2, some more detailed analyses than [3] which are necessary for our subsequent study are given and in Section 3 we obtain the desired asymptotic representation of the connection matrix.

§ 2. Lemmas.

LEMMA 2. 1. *About the exponential factors in the formulas* (1. 2) *and* (1. 6), *we have a following relation,*

$$\text{(2. 1)} \qquad \sum_{\nu=0}^{h-1}[\tau^{-(m+q)}\varepsilon]^{\nu-h}\hat{F}^{(\nu)}(0)=Q(s)$$

and the constant matrices $f^{(h)}$ *in* (1. 4) *and* Π *in* (1. 6) *are related by*

$$\text{(2. 2)} \qquad f^{(h)}=mh\Pi$$

Proof. The constant matrices $C^{(\nu)}(0)$ in the expression of coefficient matrix $C(\tau,\varepsilon)$ of the equation (II. 2. 11) are of the form

$$C^{(\nu)}(0)=mh\begin{bmatrix}\begin{bmatrix}0 & \delta_{\nu 0} & & 0\\ & 0 & \ddots & \\ & & & \delta_{\nu 0}\\ a^{(\nu)}_{1n_1,\mu n_1} & \cdots & a^{(\nu)}_{12,\mu 2} & 0\end{bmatrix} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \begin{bmatrix}0 & \delta_{\nu 0} & & 0\\ & 0 & \ddots & \\ & & & \delta_{\nu 0}\\ a^{(\nu)}_{pn_p,\mu n_p} & \cdots & a^{(\nu)}_{p2,\mu 2} & 0\end{bmatrix}\end{bmatrix}\quad (\nu=0,1,\cdots,h-1),$$

$$C^{(h)}(0)=mh\begin{bmatrix}\begin{bmatrix}0 & 0 & & 0\\ & 0 & \ddots & \\ & & & 0\\ a^{(h)}_{1n_1,\mu n_1} & \cdots & a^{(h)}_{12,\mu 2} & 0\end{bmatrix} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \begin{bmatrix}0 & 0 & & 0\\ & 0 & \ddots & \\ & & & 0\\ a^{(h)}_{pn_p,\mu n_p} & \cdots & a^{(h)}_{p2,\mu 2} & 0\end{bmatrix}\end{bmatrix}-hq\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & n-1\end{bmatrix},$$

where $\delta_{\nu 0}=1$ for $\nu=0$ and $\delta_{\nu 0}=0$ for $\nu=1,2,\cdots,h-1$, and $a^{(\nu)}_{il,\mu}$ are some constant numbers introduced in (II. 1. 7) and for nonzero such number $a^{(\nu)}_{il,\mu}$, we must have

$$\text{(2. 3)}\qquad \frac{\nu}{a}+\mu-\frac{lq}{m}=0\quad\begin{pmatrix}l=n_i,n_i-1,\cdots,2\\ i=1,2,\cdots,p\\ \mu\geqq m^{(\nu)}_{il}\\ \nu=0,1,\cdots,h\end{pmatrix},$$

On the other hand, the matrix $H^{(0)}(s)$ in (II. 3. 20) can be written

$$H^{(0)}(s)=\begin{bmatrix}H^{(0)}_{11}(s) & & 0\\ & \ddots & \\ 0 & & H^{(0)}_{pp}(s)\end{bmatrix},\qquad H^{(0)}_{jj}(s)=\begin{bmatrix}0 & 1 & & 0\\ & & \ddots & \\ & & & 1\\ h^{(0)}_{in_i}(s) & \cdots & h^{(0)}_{i2}(s) & 0\end{bmatrix}\quad (j=1,2,\cdots,p).$$

and $h^{(0)}_{il}(s)$ are polynomials of s of the form

$$h^{(0)}_{il}(s)=\sum_{(\nu,\mu)}a^{(\nu)}_{il,\mu}s^{\mu}$$

where nonzero terms $a^{(\nu)}_{il,\mu}s^{\mu}$ come from the indices of (ν,μ) such that

$$\text{(2. 4)}\qquad \nu+a\mu-\frac{alq}{m}=0$$

which are just the same pairs of (ν,μ) as that of (2. 3). By the transformation $v^{(0)}=\Omega(s)w^{(0)}$, the equation (II. 3. 20) becomes $dw^{(0)}/ds=K(s)w^{(0)}$ with $K(s)=\Omega(s)^{-1}H^{(0)}(s)\Omega(s)-\Omega(s)d\Omega(s)/ds$, and if we rearrange it by descending powers of $s^{1/mh}$, we have

$$K(s)=\sum_{\nu=0}^{N}K^{(\nu)}s^{(q/m)-((m+q)\nu/mh)}\qquad (N\text{ some positive integer}),$$

where $K^{(\nu)}$ are constant matrices such that

$$mhK^{(\nu)}=C^{(\nu)}(0), \qquad \nu=0,1,\cdots,h.$$

Suppose that after successive diagonalizations of $K(s)$ of h-times we get

$$\frac{dw^{(0)}}{ds}=L(s)w^{(0)},$$

$$L(s)=\sum_{\nu=0}^{h}L^{(\nu)}s^{(q/m)-((m+q)\nu/mh)}+0(s^{-1-(1/mh)}),$$

then the constant matrices $D^{(\nu)}(0)$ in the formula (II. 2. 15) and $L^{(\nu)}$ satisfy

$$mhL^{(\nu)}=D^{(\nu)}(0) \qquad (\nu-0,1,\cdots,h),$$

hence we have

$$\begin{aligned}\sum_{\nu=1}^{h-1}[\tau^{-(m+q)}\varepsilon]^{\nu-h}\hat{F}^{(\nu)}(0)&=\sum_{\nu=0}^{h-1}\int[\tau^{-(m+q)}\varepsilon]^{\nu-h}\tau^{-1}D^{(\nu)}(0)d\tau\\&=\sum_{\nu=0}^{h-1}D^{(\nu)}(0)\frac{[\tau^{-(m+q)}\varepsilon]^{\nu-h}}{(m+q)(h-\nu)}\\&=\sum_{\nu=0}^{h-1}\frac{mh}{(m+q)(h-\nu)}L^{(\nu)}s^{(m+q)(h-\nu)/mh}=Q(s),\end{aligned}$$

and

$$f^{(h)}=D^{(h)}(0)=mhL^{(h)}=mh\Pi.$$

This proves the lemma.

Next we consider the nonhomogeneous equation

$$\text{(II. 3. 39)} \qquad \frac{dt}{ds}=H^{(0)}(s)t+F(s).$$

The homogeneous equation (II. 3. 20) has, as staded in the introduction, a fundamental solution which is asymptotically developable in powers of $s^{-1/m}$ of the form:

$$\text{(II. 3. 37)} \qquad v^{(0)}(s)\simeq\Omega(s)\left\{\sum_{\nu=0}^{\infty}v_{\nu}^{(0)}s^{-\nu/m}\right\}s^{\Pi}\exp[Q(s)] \qquad (s\to\infty)$$

in every sector S of central angle at least $m\pi/(m+q)$. A solution of (II. 3. 39) is given by the integral

$$\text{(II. 3. 40)} \qquad t(s)=\int_{\Gamma(s)}v^{0}(s)v^{(0)}(\sigma)^{-1}F(\sigma)d\sigma.$$

As in [2], if we define $\hat{t}(s)$, $\hat{v}^{(0)}(s)$ and $\hat{F}(s)$ by

$$\text{(II. 3. 41)} \qquad \begin{aligned}t(s)&=\Omega(s)\hat{t}(s)s^{\Pi}\exp[Q(s)],\\v^{(0)}(s)&=\Omega(s)\hat{v}^{(0)}(s)s^{\Pi}\exp[Q(s)],\\F(s)&=\Omega(s)\hat{F}(s)s^{\Pi}\exp[Q(s)],\end{aligned}$$

then (II. 3. 37) becomes

(II. 3. 42)

$$\hat{t}(s)=\hat{v}^{(0)}(s)\int_{\Gamma(s)}\exp[Q(s)-Q(\sigma)]\left(\frac{s}{\sigma}\right)^{\Pi}\hat{v}^{(0)}(\sigma)^{-1}\hat{F}(\sigma)\left(\frac{\sigma}{s}\right)^{\Pi}\exp\{Q(\sigma)-Q(s)\}d\sigma.$$

Suppose that $\hat{F}(s)s^{-b}$ is bounded for some constant b and has the asymptotic expansion in powers of $s^{-1/mh}$ in the sence that for every integer $r \geqq 0$

$$\hat{F}(s)s^{-b} - \sum_{\nu=0}^{r} \hat{F}^{(\nu)} s^{-\nu/mh} = R(s)s^{-r/mh} \qquad (s \to \infty),$$

where $R(s) \to 0$ as $s \to \infty$ in a domain $S(s_1)$, and $\hat{F}^{(\nu)}$ are polynomials of $\log s$. Here $S(s_1)$ denotes a domain such that

$$s \in S, \qquad |s| > s_1$$

for sufficiently large positive number s_1. In this case we can prove a following lemma.

LEMMA 2.2. *Let S be any sector in s-plane of central angle $m\pi/(m+q)$ whose boundary lines don't coincide with any singular direction, then by choosing the integral path $\gamma_{jk}(s)$ appropriately for each element of the integral* (II. 3. 40), *we have*

$$\hat{t}(s) = s^{h+1} t^*(s), \qquad s \in S(s_0) \quad (s_0 \geqq s_1),$$

where $t^*(s)$ is bounded in $S(s_0)$, and asymptotically developable in powers of $s^{-1/mh}$ as $s \to \infty$ such that for every positive integer r,

$$t^*(s) - \sum_{\nu=0}^{r} t^{(\nu)}(s) s^{-\nu/mh} = o(s^{-r/mh}),$$

where $t^{(\nu)}(s)$ are polynomials of $\log s$.

Proof. We can assume that every element of the matrix in the integrand of (II. 3. 42) is of the form

$$\text{(II. 3. 45)} \qquad p_{jk}(\sigma)\sigma^b \left(\frac{s}{\sigma}\right)^{\pi_j - \pi_k} \exp\left[q_{jk}(s) - q_{jk}(\sigma)\right],$$

where $p_{jk}(s)$ is bounded and asymptotically developable in powers of $s^{-1/mh}$ in $S(s_1)$, π_j is the diagonal element of the matrix Π, and $q_{jk}(s) = q_j(s) - q_k(s)$. $q_j(s)$ denotes the diagonal element of $Q(s)$ of the form

$$\text{(2. 5)} \qquad q_j(s) = \frac{m\lambda_j}{m+q} s^{(m+q)/m} + l_{j1} s^{(m+q)(h-1)/mh} + \cdots + l_{jh-1} s^{(m+q)/mh}.$$

Here we introduce the variables ζ and ξ by

$$\text{(2. 6)} \qquad \zeta = \frac{m}{m+q} \sigma^{(m+q)/m}, \qquad \xi = \frac{m}{m+q} s^{(m+q)/m},$$

then the function (II. 3. 45) must be changed into

$$\text{(2. 7)} \qquad \tilde{p}_{jk}(\zeta) \left(\frac{\xi}{\zeta}\right)^{\tilde{\pi}_j - \tilde{\pi}_k} \zeta^{(mb-q)/(m+q)} \exp\left[\tilde{q}_{jk}(\xi) - \tilde{q}_{jk}(\zeta)\right],$$

where $\tilde{p}_{jk}(\zeta)$ is bounded and has the asymptotic expansion in powers of $\zeta^{-1/(m+q)h}$ and

$$\text{(2. 8)} \qquad \tilde{q}_{jk}(\xi) = (\lambda_j - \lambda_k)\xi + \sum_{\nu=1}^{h-1} (\tilde{l}_{j\nu} - \tilde{l}_{k\nu}) \xi^{1-\nu/h}.$$

If we denote by Σ a sector in ζ-plane corresponding to S under the transformation (2. 6), the central angle of Σ is π and every line

$$\mathrm{Re}\,(\lambda_j-\lambda_k)\zeta=0, \qquad j, k=1, 2, \cdots, n; \quad j\neq k$$

does not coincide with the boundary lines of Σ from the assumption on S. From this, we can draw a line α_{jk} through the origin into the interior of Σ on which $\mathrm{Re}\,(\lambda_j-\lambda_k)\zeta>0$ for every (j, k) $j\neq k$. Let $\lambda_{jk}(\xi)$ be a line which is parallel to α_{jk} and extends from ξ to infinity, then the integral path $\gamma_{jk}(s)$ $(j\neq k)$ is to be taken as the invrse image of $\lambda_{jk}(\xi)$ under the transformation (2. 6). Now we prove the first part of the lemma. Let $\Sigma(\xi_0)$ be the domain of ξ-plane such that

$$\xi\in\Sigma, \qquad |\xi|\geqq\xi_0,$$

where ξ_0 is a positive number, then we have

$$\left|\frac{\zeta}{\xi}\right|\geqq\delta>0, \qquad \xi\in\Sigma(\xi_0), \qquad \zeta\in\lambda_{jk}(\xi) \qquad (j\neq k, 1, 2, \cdots, n)$$

and then for some positive constant C independent of ξ and ζ

$$\left|\frac{(\tilde{l}_{j\nu}-\tilde{l}_{k\nu})(\xi^{1-\nu/h}-\zeta^{1-\nu/h})}{(\lambda_j-\lambda_k)(\xi-\zeta)}\right|\leqq C|\xi|^{-\nu/h},$$

$$\left|\frac{(\tilde{\pi}_j-\tilde{\pi}_k)\log\xi/\zeta}{(\lambda_j-\lambda_k)(\xi-\zeta)}\right|\leqq C|\xi|^{-1}.$$

Hence we have

$$\mathrm{Re}\,\{\tilde{q}_{jk}(\xi)-\tilde{q}_{jk}(\zeta)+(\tilde{\pi}_j-\tilde{\pi}_k)\log\xi/\zeta\}=\mathrm{Re}\,\{(\lambda_j-\lambda_k)(\xi-\zeta)+O(\xi^{-1/h})\}$$

and from this it follows that if we choose ξ_0 sufficiently large, the integral of (2. 7) is of the order $O(\xi^{(mb-q)/(m+q)})$ in $\Sigma(\xi_0)$ which implies that the integral of (II. 3. 45) is of the order $O(s^{b-q/m})$ in $S(s_0)$. For $j=k$, we take some indefinite integral specified in later and we can see that the estimate of it is $O(s^{b+1})$. Next we examine the asymptotic property of $t^*(s)$.

Let $j\neq k$. Let us write for simplifications

$$q_{jk}(s)-q_{jk}(\sigma)+(\pi_j-\pi_k)(\log s-\log\sigma)=q(s)-q(\sigma),$$

and for every integer $r(\geqq r_0)$, we assume

$$\sigma^b p_{jk}(\sigma)=\sum_{\nu=r_0}^{r} p_\nu(\log\sigma)\sigma^{-\nu/mh}+o(\sigma^{-r/mh}) \qquad (\sigma\to\infty), \tag{2. 9}$$

where r_0 may be a negative integer, and $p_\nu(z)$ are polynomials of z. Since

$$\int_{\gamma_{jk}(s)}\sigma^{k/mh}(\log\sigma)^l\,\{\exp\,[q(s)-q(\sigma)]\}\,d\sigma$$
$$=\frac{m}{(m+q)(\lambda_j-\lambda_k)}s^{(k-hq)/mh}(\log s)^l\{1+g_1(s)\}$$
$$-\int_{\gamma_{jk}(s)}s^{k/mh-1}\left\{\frac{k}{mh}(\log s)^l+l(\log s)^{l-1}+g_2(s)\right\}\{\exp\,[q(s)-q(\sigma)]\}\,d\sigma,$$

where k and l are some integers, $g_1(s)$ is a convergent power series of $s^{-1/mh}$, and $g_2(s)$ is also a convergent power series of $s^{-1/mh}$ whose coefficients are polynomials of $\log s$, we have by repeated integration by parts that

$$\int_{\gamma_{jk}(s)} \sigma^b p_{jk}(\sigma)\{\exp[q(s)-q(\sigma)]\}d\sigma$$
$$=\sum_{\nu=r_0-hq}^{r} p_\nu(\log s)s^{-\nu/mh}+o(s^{-r/mh})+\int_{\gamma_{jk}(s)} R_r(s)\{\exp[q(s)-q(\sigma)]\}d\sigma,$$

where $p_\nu(z)$ and $p(z)$ are polynomials of z, and

$$R_r(s)=s^{-(r+1-hq)/mh}p(\log s)+O(s^{-(r+1-hq)/mh}).$$

From the first part of the proof of this lemma, we have

$$\int_{\gamma_{jk}(s)} R_r(s)\{\exp[q(s)-q(\sigma)]\}d\sigma=s^{-(r+1)/mh}\tilde{p}(\log s)+O(s^{-(r+1)/mh})$$
$$=o(s^{-r/mh}) \qquad (s\to\infty).$$

Let $j=k$. Assume that $\sigma^b p_{jj}(\sigma)$ possesses the asymptotic representation of the form (2. 9), and we can easily show by integration by parts that every term of the series (2. 9) has an indefinite integral of the form

$$\int p_r(\log\sigma)\sigma^{r/mh}d\sigma=P_{r+mh}(\log\sigma)\sigma^{(r+mh)/mh},$$

$$\int p_{-mh}(\log\sigma)\sigma^{-1}d\sigma=P_0(\log\sigma),$$

where $P_{r+mh}(z)$ is a polynomial of z which is determined uniquely and its degree is the same as that of $p_r(z)$, $P_0(z)$ is also a polynomial of z which may be added an arbitrary constant. We fix the definition of $P_0(z)$ by requiring that its constant term be zero. The degree of $P_0(z)$ is one more than $p_{-mh}(z)$.

Now let

$$\sigma^b p_{jj}(\sigma)=p_1(\sigma)+p_2(\sigma),$$

where

$$p_1(\sigma)=\begin{cases}0, & \text{if } r_0>mh,\\ \sum_{\nu=r_0}^{mh} p_\nu(\log\sigma)\sigma^{-\nu/mh}, & \text{if } r_0\leqq mh,\end{cases}$$

then the function

$$P_1(s)=\begin{cases}0, & \text{if } r_0>mh,\\ \sum_{r_0-mh}^{0} P_\nu(\log s)s^{-\nu/mh} & \text{if } r_0\leqq mh.\end{cases}$$

is an indefinite integral of $p_1(\sigma)$. Define $P_2(s)$ by

$$P_2(s)=\int_s^\infty p_2(\sigma)d\sigma$$

Then, for every $r\geqq mh+1$

$$P_2(s)-\int_s^{\infty}\sum_{\nu=r_1}^{r+mh} p_\nu(\log\sigma)\sigma^{-\nu/mh}d\sigma=\int_s^{\infty}R_r(\sigma)d\sigma,$$

where $r_1=\max(r_0, mh+1)$, and

$$R_r(\sigma)=o(\sigma^{-(r+mh)/mh}) \qquad (\sigma\to\infty),$$

and hence we have

$$P_2(s)-\sum_{\nu=r_1-mh}^{r} P_\nu(\log s)s^{-\nu/mh}=o(s^{-r/mh}) \qquad (s\to\infty).$$

Therefore the particular indefinite integral $P_1(s)+P_2(s)$ of $\sigma^b p_{jj}(\sigma)$ defined above has the asymptotic expansion of the form

$$\int\sigma^b p_{jj}(\sigma)d\sigma\simeq\sum_{\nu=r_0-mh}^{\infty} P_\nu(\log s)s^{-\nu/mh} \qquad (s\to\infty),$$

this proves our lemma.

From the above lemma, we can give an improvement for Theorem II. 5. 1.

THEOREM II. 5. 1′. *Let S be a sector in the s-plane of central angle $m\pi/(m+q)$ whose boundary lines do not coincide with any singular direction, and let*

(II. 3. 53) $$v\sim\Omega(s^{k(s)})\left\{\sum_{\nu=0}^{\infty} w^{(\nu)}(s)\,[s^{k(s)e}\rho]^\nu\right\}s^{k(s)\Pi}\exp[Q(s)]$$

be a formal solution of (II. 3. 10) *where $w^{(\nu)}(s)$ are asymptotically developable in powers of $s^{-1/mh}$ as $s\to\infty$. Then the differential equation*

(II. 3. 10) $$\frac{dv}{ds}=H(s,\rho)v$$

has an actual solution $v(s,\rho)$ of the form

$$v(s,\rho)=\Omega(s^{k(s)})\hat{v}(s,\rho)s^{k(s)\Pi}\exp[Q(s)].$$

and for every $r\geqq 0$, there exists a domain D_2 defined by

$$s\in S,\qquad 0<|\rho|\leqq\rho_2,\qquad |\arg\rho|\leqq\delta_2,\qquad |s^e\rho|\leqq c_3$$

in which it holds that

$$\hat{v}(s,\rho)-\sum_{\nu=0}^{r} w^{(\nu)}(s)[s^{k(s)e}\rho]^\nu=E_r(s,\rho)[s^{k(s)}\rho]^{r+1}$$

where $E_r(s,\rho)$ is bounded in D_2.

Proof. This is proved by the same method as in the proof of Theorem 5. 1 in [2] and the same argument as in the proof of Theorem II. 5. 1.

§ 3. Matching matrices.

We have found that the differential equation (1. 1) has a fundamental system $y_1(x,\varepsilon)$ that can be effectively calculated from its asymptotic representation (1. 2) for all pairs of values of x and ε in D_1 and also has a fundamental system $y_2(s, \rho)$ that can be similarly calculated from (1. 6) for s and ρ in D_2. Since the domain D_1 does not contain the origin $x=0$, if we want to know the behavior of $y_1(x, \varepsilon)$ at the origin, it can not be read off from the formulas (1. 2) and (1. 3). But the domain D_2 contains the origin $s=0$ which corresponds to $x=0$, then if we can find the connection matrix $C(\varepsilon)$ between $y_1(x, \varepsilon)$ and $y_2(s, \rho)$, this problem can be solved. Since the two domains overlap, we can give the asymptotic representation of $C(\varepsilon)$ which satisfies

$$y_1(x, \varepsilon)=y_2(s, \rho)C(\varepsilon). \tag{3. 1}$$

Now

$$C(\varepsilon)=y_2(s, \rho)^{-1}y_1(x, \varepsilon), \tag{3. 2}$$

and since this is independent of x, we can take for the calculation of $C(\varepsilon)$ any convenient point which belongs to both of the domains D_1 and D_2.

The most symmetrically located such point is

$$x_\eta=\eta^{mh}\rho^{mh-1/2e}, \qquad s_\eta=\eta^{mh}\rho^{-1/2e}$$

and then

$$\tau_\eta=\eta\rho^{(\delta-1)/\delta}, \qquad \tau_\eta^{-(m+q)}\varepsilon=\eta^{-(m+q)}\rho^{(m+q)/\delta}, \tag{3. 3}$$

$$s_\eta^{1/mh}=\eta\rho^{-1/\delta}, \qquad s_\eta^{e}\rho=\eta^{emh}\rho^{emh/\delta}, \tag{3. 4}$$

where $\delta=2emh$. Here η is any constant such that $mh \arg \eta \in T \cap S$, and $|\eta|$ might be taken equal to 1. However, for the study of the structure of the matrix $C(\varepsilon)$ we consider η as an additional parameter in the calculation, and in fact this procedure simplifies a practical calculation of it very much.

In order to study the matrix

$$C(\varepsilon)=y_2(s_\eta, \rho)^{-1}y_1(x_\eta, \varepsilon) \tag{3. 5}$$

in detail, we begin by remembering the structure of the function

$$\begin{aligned} y_1(x_\eta, \varepsilon)&=\Omega(x_\eta)u(\tau_\eta, \varepsilon)\\ &=\Omega(x_\eta)\hat{u}(\tau_\eta, \varepsilon) \exp \sum_{\nu=0}^{h} \varepsilon^{\nu-h}F^{(\nu)}(\tau_\eta)\\ &=\Omega(x_\eta)\hat{u}(\tau_\eta, \varepsilon)\cdot\exp\left\{\sum_{\nu=0}^{h}[\tau_\eta^{-(m+q)}\varepsilon]^{\nu-h}\hat{F}^{(\nu)}(\tau_\eta)+f^{(h)}\log \tau_\eta\right\}, \end{aligned} \tag{3. 6}$$

where $\hat{u}(\tau, \varepsilon)$ has the asymptotic expansion of the form

$$\hat{u}(\tau, \varepsilon)\simeq \sum_{\nu=0}^{\infty} \hat{u}^{(\nu)}(\tau)[\tau^{-(m+q)}\varepsilon]^{\nu}. \tag{3. 7}$$

The functions $\hat{F}^{(\nu)}(\tau)$ are holomorphic in τ and $\hat{u}^{(\nu)}(\tau)$ are developable in powers of τ such that for every positive integer r

$$\hat{u}^{(\nu)}(\tau)= \sum_{\mu=0}^{r} \hat{u}^{(\nu)}{}_{\mu}\tau^{\mu}+o(\tau^{r}), \tag{3. 8}$$

where $\hat{u}^{(\nu)}{}_{\mu}$ are polynomials of $\log \tau$.

Now define the matrix function $U(\eta, \varepsilon)$ by

(3.9)
$$\begin{aligned} U(\eta, \varepsilon) &= u(\tau_\eta, \varepsilon)\left[\exp\left\{\sum_{\nu=0}^{h-1}[\tau_\eta{}^{-(m+q)}\varepsilon]^{\nu-h}\hat{F}^{(\nu)}(0)+f^{(h)}\log \tau_\eta\right\}\right]^{-1} \\ &= \hat{u}(\tau_\eta, \varepsilon)\exp\left\{\sum_{\nu=0}^{h}[\tau_\eta{}^{-(m+q)}\varepsilon]^{\nu-h}(\hat{F}^{(\nu)}(\tau_\eta)-F^{(\nu)}(0))\right\}. \end{aligned}$$

Combining (3.7) and (3.8), and replace τ by τ_η, we have formally the iterated series

$$\hat{u}(\tau_\eta, \varepsilon) \sim \sum_{\nu=0}\sum_{\mu=0}\hat{u}^{(\nu)}{}_{\mu}(\log \eta\rho^{(\delta-1)/\delta})\eta^{-(m+q)\nu+\mu}\rho^{\{(m+q)\nu+(\delta-1)\mu\}/\delta}.$$

If we rearrange formally this series by collecting all terms for which $r=(m+q)\nu+(\delta-1)\eta$ has the same value, we have

(3.10)
$$\hat{u}(\tau_\eta, \varepsilon) \sim \sum_{r=0}^{\infty} u^{(r)}(\eta, \rho)\rho^{r/\delta},$$

where

(3.11)
$$u^{(r)}(\eta, \rho) = \sum_{(m+q)\nu+(\delta-1)\mu=r} u^{(\nu)}{}_{\mu}(\log \eta\rho^{(\delta-1)/\delta})\eta^{-(m+q)\nu+\mu},$$

in particular we have

(3.12)
$$u^{(0)}(\eta, \rho) = \hat{u}^{(0)}(0).$$

LEMMA 3.1

$$\hat{u}(\tau_\eta, \varepsilon) \simeq \sum_{r=0}^{\infty} u^{(r)}(\eta, \rho)^{r/\rho}.$$

Proof. From the asymptotic representations (3.7) and (3.8),

$$\begin{aligned} &\hat{u}(\tau_\eta, \varepsilon) - \sum_{\nu=0}^{r} u^{(\nu)}(\eta, \rho)\rho^{\nu/\delta} \\ &\simeq \sum_{\nu>r/(m+q)} \hat{u}^{(\nu)}(\tau_\eta)[\tau_\eta{}^{-(m+q)}\varepsilon]^{\nu} + \sum_{\nu\leqq r/(m+q)}\ \sum_{\mu>\{r-(m+q)\nu\}/(\delta-1)} \hat{u}^{(\nu)}(\log \tau_\eta)\tau^{\mu}[\tau^{-(m+q)}\varepsilon]^{\nu} \\ &= o(\rho^{r/\delta}) \qquad (\rho\to 0). \end{aligned}$$

in the same way we have

(3.13)
$$\sum_{\nu=0}^{h}[\tau^{-(m+q)}\varepsilon]^{\nu-h}(\hat{F}^{(\nu)}(\tau_\eta)-\hat{F}^{(\nu)}(0)) \simeq \sum_{r\geqq 1} F^{(r)}(\eta, \rho)\rho^{r/\delta}$$

where

(3.14)
$$F^{(r)}(\eta, \rho) = \sum_{(m+q)(\nu-h)+(\delta-1)\mu=r} \hat{F}^{(\nu)}{}_{\mu}\eta^{-(m+q)(\nu-h)+\mu},$$

with $\hat{F}^{(\nu)}{}_{\mu}$ constant matrices.

Here we must notice that for every power r of ρ, the power of η in the expressions (3.11) and (3.14) satisfy

$$-(m+q)\nu+\mu=\delta\mu-r, \qquad -(m+q)(\nu-h)+\mu=\delta\mu-r,$$

which mean that the functions $u^{(r)}(\eta, \rho)$ and $F^{(r)}(\eta, \rho)$ contain only a finite number of terms $(\log \eta\rho^{(\delta-1)/\delta})^{\nu}\eta^{\mu}$ with $\mu\equiv -r \pmod{\delta}$. If we expand the exponential function

$$\exp \sum_{\nu=0}^{h} [\tau_\eta^{-(m+q)}\varepsilon]^{\nu-h}(\hat{F}^{(\nu)}(\tau_\eta)-\hat{F}^{(\nu)}(0)) \simeq \sum_{r=0}^{\infty} F^{(\nu)}(\eta)\rho^{\nu/\delta}$$

where $F^{(\nu)}(\eta)$ are polynomials of η each of whose terms has a power $\mu\equiv -\nu \pmod{\delta}$, in particular $F^{(0)}$ is a unit matrix.

Hence we can write

$$\begin{aligned} &U(\eta, \rho)\simeq \sum_{\nu=0} U^{(\nu)}(\eta, \rho)\rho^{\nu/\delta} \qquad (\rho\to 0), \\ (3.15)\qquad &U^{(0)}(\eta, \rho)=\hat{u}^{(0)}(0), \\ &U^{(\nu)}(\eta, \rho)=\sum_{\mu} U^{(\nu)}{}_{\mu}(\log \eta\rho^{(\delta-1)/\delta})\eta^{\mu}, \end{aligned}$$

where the summation with respect to μ consists of a finite number of terms for which $\mu\equiv -\nu \pmod{\delta}$, and $U^{(\nu)}{}_{\mu}(z)$ are polynomials of z.

Next we consider the function $y_2(s_\eta, \rho)$. Since the value of s_η becomes infinite as $\rho\to 0$ for any fixed η, we use the asymptotic expansion (1. 6) and (1. 7) for $y_2(s, \rho)$ with $k(s)=1$, and then from (1. 6)

$$(3.16)\qquad y_2(s_\eta, \rho)=\Omega(\varepsilon^{\alpha}s_\eta)\hat{v}(s_\eta, \rho)s_\eta{}^{\Pi} \exp [Q(s_\eta)].$$

Let $V(\eta, \rho)$ be

$$(3.17)\qquad V(\eta, \rho)\equiv\hat{v}(s_\eta, \rho)\simeq \sum_{\nu=0}^{\infty} w^{(\nu)}(s_\eta)[s_r{}^{e}\rho]^{\nu},$$

and from Lemma 2. 2 $w^{(\nu)}(s)$ has an asymptotic expansion of the form

$$(3.18)\qquad w^{(\nu)}(s)\simeq \sum_{\nu=0} w^{(\nu)}{}_{\mu}(\log s)s^{-\nu/mh},$$

where $w^{(\nu)}{}_{\mu}(z)$ are polynomials of z. Insert (3. 18) into (3. 17) and substitute (3. 4) for $s^{-1/mh}$ and $s^{e}\rho$, then we have

$$\begin{aligned} V(\eta, \rho) &\simeq \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} w^{(\nu)}{}_{\mu}(\log \eta\rho^{-1/\delta})\eta^{emh\nu-\mu}\rho^{(emh\nu+\mu)/\delta} \\ &\simeq \sum_{r=0}^{\infty}\left\{\sum_{emh\nu+\mu=r} w^{(\nu)}{}_{\mu}(\log \eta\rho^{-1/\delta})\eta^{emh\nu-\mu}\right\}\rho^{r/\delta} \end{aligned}$$

Here it is easily verified that if $r\equiv 0 \pmod{\delta}$ then $emh\nu-\mu\equiv 0 \pmod{\delta}$ and if $r\equiv l \pmod{\delta}$ then $emh\nu-\mu\equiv -l \pmod{\delta}$. Therefore we have

$$V(\eta, \rho)\simeq \sum_{\nu=0}^{\infty} V^{(\nu)}(\eta, \rho)\rho^{\nu/\delta} \qquad (\rho\to 0),$$

where $V^{(\nu)}(\eta, \rho)$ are polynomials of η and η^{-1}, the power of each term of which equals to $-\nu \pmod{\delta}$, and in particular $V^{(0)}(\eta, \rho)=w^{(0)}(0)$ which is nonsingular and we can suppose that $w^{(0)}(0)=\hat{u}^{(0)}(0)$. From this it follows that

$$V^{-1}(\eta,\rho)\simeq\sum_{\nu=0}^{\infty}\check{V}^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}\qquad(\rho\to 0),$$

(3.19)
$$\check{V}^{(0)}=[\hat{u}^{(0)}(0)]^{-1}$$
$$\check{V}^{(\nu)}(\eta,\rho)=\sum_{\mu}\check{V}^{(\nu)}{}_{\mu}(\log\eta\rho^{-1/\delta})\eta^{\mu},$$

where $\check{V}^{(\nu)}{}_{\mu}(z)$ are polynomials of z and the summation with respect to μ is taken over a finite number of integers μ such that $\mu=-\nu\ (\mathrm{mod}\ \delta)$.

Since $\Omega(x_\eta)=\Omega(\varepsilon^a)\Omega(s_\eta)$, we have from (3.5) (3.6) (3.9) (3.16) and (3.17)

$$s_\eta{}^{\Pi}\exp[Q(s_\eta)]C(\varepsilon)\left[\exp\left\{\sum_{\nu=0}^{h-1}[\tau_\eta{}^{-(m+q)}\varepsilon]^{\nu-h}\hat{F}^{(\nu)}(0)+f^{(h)}\log\tau_\eta\right\}\right]^{-1}=V(\eta,\rho)^{-1}U(\eta,\rho). \tag{3.20}$$

On the basis of (3.15) and (3.19), the right hand term has the asymptotic expansion of the form

$$V(\eta,\rho)^{-1}U(\eta,\rho)\simeq\sum_{\nu=0}\Lambda^{(\nu)}(\eta,\rho)\rho^{\nu/\delta}\qquad(\rho\to 0),$$

(3.21)
$$\Lambda^{(0)}(\eta,\rho)=E\ \text{(unit matrix)},$$
$$\Lambda^{(\nu)}(\eta,\rho)=\sum_{\mu}\Lambda^{(\nu)}{}_{\mu}(\eta,\rho)\eta^{\mu},$$

where $\Lambda^{(\nu)}{}_{\mu}(\eta,\rho)$ are polynomials of $\log(\eta\rho^{(\delta-1)/\delta})$ and $\log(\eta\rho^{-1/\delta})$. The summation with respect to μ is over a finite number of integers for which $\mu=-\nu\ (\mathrm{mod}\ \delta)$.

If we denote by $c_{jk}(\varepsilon)$ the $j-k$ element of the matrix $C(\varepsilon)$, each element of the left member of (3.20) can be written from Lemma 2.1 that

$$c_{jk}(\varepsilon)\{\exp[q_j(s_\eta)-q_k(s_\eta)]\}\eta^{mh(\pi_j-\pi_k)}\rho^{-mh\{\pi_j+(\delta-1)\pi_k\}/\delta}, \tag{3.22}$$

where π_j and $q_j(s)$ are the diagonal elements of the matrix Π and $Q(s)$ respectively.

Now we prove the main theorem.

THEOREM 3.1. *Let the outer solution $y_1(x,\varepsilon)$ and the inner solution $y_2(s,\rho)$ be defined in the domain D_1 and D_2 respectively, and let the sector T in the τ-plane and S in the s-plane correspond to the sector $\hat{T}$ and $\hat{S}$ in the x-plane for which we assume that the axes of symmetry of $\hat{T}$ and $\hat{S}$ coincide for positive ε and that the sectors $\hat{T}$ and $\hat{S}$ contain in their insides every singular directions:* $\mathrm{Re}(\lambda_j-\lambda_k)s^{(m+q)/m}=0$ *$(j\neq k,\ jk=1,2,\cdots,n)$ for all ε in $D_1\cap D_2$ by taking δ_1 and δ_2 sufficiently small. Then the connection matrix $C(\varepsilon)$ defined in* (3.1) *has the asymptotic expansion of the form*

$$C(\varepsilon)\simeq\varepsilon^{a\Pi}\sum_{\nu=0}^{\infty}C_\nu\rho^\nu$$

in the domain

$$0<|\varepsilon|\leqq\varepsilon_0,\qquad|\arg\varepsilon|\leqq\delta_0$$

for sufficiently small ε_0 and δ_0, where C_ν are constant diagonal matrices. In particular $C_0=E$.

Proof. Under the above situations of the sectors $\hat{T}$ and $\hat{S}$, we prove first that $C(\varepsilon)$ is asymptotically diagonal. For $j \neq k$, from (3. 20), (3. 21) and (3. 22) we have

$$c_{jk}(\varepsilon) \simeq \{\exp [q_k(s_\eta) - q_j(s_\eta)]\} \eta^{mh(\pi_k - \pi_j)} \rho^{mh\{\pi_j + (\delta-1)\pi_k\}/\delta} \sum_{\nu=0}^{\infty} \Lambda_{jk}^{(\nu)}(\eta, \rho) \rho^{\nu/\delta}.$$

From the assumptions on $\hat{T}$ and S, we can choose η for every $j \neq k$ such that

$$\mathrm{Re}\,[q_k(s_\eta) - q_j(s_\eta)] \to -\infty \qquad \text{as} \quad \rho \to 0,$$

and then we have $c_{jk}(\varepsilon) \simeq 0$ $(\rho \to 0)$ for such η. Since $c_{jk}(\varepsilon)$ does not depend on η, we have

$$c_{jk}(\varepsilon) \simeq 0 \qquad (\rho \to 0), \quad j \neq k,$$

this implies that $C(\varepsilon)$ is asymptotically diagonal.

For $j = k$, (3. 20), (3. 21) and (3. 22) give

$$\rho^{-mh} c_{jj}(\varepsilon) \simeq \sum_{\nu=0}^{\infty} \Lambda_{jj}^{(\nu)}(\eta, \rho) \rho^{\nu/\rho} \qquad (\rho \to 0).$$

The left term of the above equation is independent of η, and hence all of the terms $\Lambda_{jj}^{(\nu)}(\eta, \rho)$ must be independent of η. For, otherwise, we can suppose that there is a first term which depend on η, say $\Lambda_{jj}^{(r)}(\eta, \rho)$.

As $\rho \to 0$,

$$\rho^{-r/\delta} \left[\rho^{-mh} c_{jj}(\varepsilon) - \sum_{\nu=0}^{r-1} \Lambda_{jj}^{(\nu)}(\eta, \rho) \rho^{\nu/\delta} \right] = \Lambda_{jj}^{(r)}(\eta, \rho) + o(1).$$

Since the left term in the above equation is independent of η, this is impossible. Because of the structure of the function $\Lambda_{jjk}^{(\nu)}(\eta, \rho)$, it is independent of η if and only if $\Lambda_{jj}^{(\nu)}(\eta, \rho)$ is a constant, and this is possible if and only if $\nu = 0 \pmod{\delta}$. Above all we can conclude that

$$\Lambda^{(\nu)}(\eta, \rho) \equiv 0 \qquad \nu \neq 0 \pmod{\delta},$$

$$\Lambda^{(\nu)}(\eta, \rho) = \Lambda^{(\nu)} \qquad \nu = 0 \pmod{\delta},$$

where $\Lambda^{(\nu)}$ are diagonal constant matrices, and in particular $C^{(0)} = \Lambda^{(0)}(\eta, \rho) = E$. This proves our theorem.

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