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SEMI-CONTINUOUS CHANNELS WITH A PAST HISTORY

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1. Summary.

In this paper, we shall prove coding theorems for semi-continuous channels with a past history. The main object is to generalize the results obtained by Wolfowitz in [2], Section 6. 5.

2. A semi-continuous channels with a past history.

Let random variables $X_1, X_2, \cdots$ be input letters which are independently and identically distributed and take their values in the set $\{1, \cdots, a\}$, respectively. Let random variables $Y_1, Y_2, \cdots$ be output letters which take their values in the real line $(R, \mathfrak{B})$. Furthermore, let μ be a (not necessarily finite) measure such that

(1) $$P\{Y_2 \in A \mid Y_1 = y_1, X_2 = i\} = \int_A w(y_2 \mid y_1, i)\mu(dy_2) \qquad (i=1, \cdots, a)$$

hold for any set $A \in \mathfrak{B}$.

Next, let l be any positive integer. Let

(2) $$U_i = (X_{(i-1)l+1}, \cdots, X_{il}), \qquad V_i = (Y_{(i-1)l+1}, \cdots, Y_{il}), \qquad i=1, 2, \cdots, \text{ ad inf.}$$

Let Q' be the probability distribution of U_1, and for $u=(x_1, \cdots, x_l)$, $v=(y_1, \cdots, y_l)$ and $v'=(y_1', \cdots, y_{l-1}', y_0)$ define

(3) $$h(v \mid v', u) = \prod_{j=1}^{l} w(y_j \mid y_{j-1}, x_j)$$

and

(4) $$q_{Q'}(v \mid v') = \sum_u Q'(u) h(v \mid v', u).$$

Suppose that

(5) $$(U_1, V_1), (U_2, V_2), \cdots$$

is a Markov chain with the transition function $Q'(u)h(v \mid v', u)$ and

(6) $$Y_{1l}, Y_{2l}, \cdots$$

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constitute a Markov chain. Thus, V_i depend only on $Y_{(i-1)l}$ and U_i, so we have Conditional Prob. dens. $\{V_i = v | V_{i-1} = v', U_i = u\}$ = Conditional Prob. dens. $\{V_i=v | Y_{(i-1)l}=y_0, U_i=u\}$, that is, $h(v|v', u)=h(v|y_0, u)$.

ASSUMPTION I. For any l, each of Markov chains (5) and (6) has only one ergodic set and has no cyclically moving sets.

Under Assumption I, we have "stationary" distribution $\nu_{Q'}(A)$ $(A\in\mathfrak{B})$ for the Markov chain (6).

We remark here that Assumption I is satisfied for the indecomposable channel defined in [2].

Define the random variable

$$(7)\qquad \frac{J_n(Q', y_0)}{n} = \frac{1}{n}\sum_{i=1}^{n} \log \frac{h(V_i | V_{i-1}, U_i)}{q_{Q'}(V_i | V_{i-1})} = \frac{1}{n}\sum_{i=1}^{n} \log \frac{h(V_i | Y_{(i-1)l}, U_i)}{q_{Q'}(V_i | Y_{(i-1)l})}$$

where $Y_0=y_0$, and put

$$(8)\qquad C=\sup_{l}\sup_{Q'}\left[\frac{1}{l}\int_{-\infty}^{\infty}\nu_{Q'}(dy)E\left[\log\frac{h(V_1|y, U_1)}{q_{Q'}(V_1|y)}\,\middle|\, Q'\right]\right].$$

We assume that C is finite.

We shall prove the following

THEOREM 1. *Let* $\varepsilon>0$ *and* α, $0<\alpha\leqq 1$, *be arbitrary. Let* $w(\cdot\,|\,\cdot\,,\,\cdot)$ *be the channel probability density function which satisfies Assumption* I. *Then, for any fixed* $y_0\in R$, *when* n *is sufficiently large, there exists an* $(n, 2^{n(C-\varepsilon)}, \alpha)$ *code for this semi-continuous channel with* $w(\cdot\,|\,\cdot\,,\,\cdot)$.

Proof. Since from Assumption I, we can apply the strong law of large number to the Markov chain (5), so for any Q'

$$\frac{J_n(Q', y_0)}{n}\to\int_{-\infty}^{\infty}\nu_{Q'}(dy)E\left[\log\frac{h(V_1|y, U_1)}{q_{Q'}(V_1|y)}\,\middle|\, Q'\right]$$

holds with probability one. Thus, we can prove the theorem by the Wolfowitz's method used in [2].

3. A special channel.

In this section, we shall consider the strong converse theorem for a special channel with the following assumption. (This channel is a generalization of the channel studied by Wolfowitz in [2], Section 6. 5.)

ASSUMPTION II. Let μ be a finite measure for which the relations (1) hold. For each i $(1\leqq i\leqq a)$, there is a constant K such that

$$(9)\qquad \frac{1}{K}\leqq w(y_2|y_1, i)\leqq K \qquad \text{for all } y_1 \text{ and } y_2.$$

If Assumption II holds, then clearly Assumption I is satisfied.

For a (finite or infinite) sequence $\bar{x}=(x_1, \cdots, x_m, \cdots)$, we put

$$p^{(m)}(y_0, y_m: \bar{x})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w(y_1|y_0, x_1)w(y_2|y_1, x_2)\cdots w(y_m|y_{m-1}, x_m)\mu(dy_1)\mu(dy_2)\cdots\mu(dy_{m-1}).$$

LEMMA 1. *Under Assumption* II,

$$\left|\frac{p^{(n)}(y, \eta: \bar{x})}{p^{(n)}(z, \eta: \bar{x})}-1\right|\leqq 2K^2\rho^{n-1} \qquad (n=1, 2, \cdots) \tag{10}$$

hold for all x *and for almost all* y, z *and* η *where*

$$\rho=1-\frac{1}{K}\mu(R)>0.$$

Proof. By a simple modification of the usual method, we easily get the lemma.

Next, for an $(l+m)$-sequence $u=(x_1, \cdots, x_{m+l})$, we define

$$h^{(m)}(y, \eta_0, \eta^{(l)}: u)=p^{(m)}(y, \eta_0: u)\prod_{j=1}^{l} w(\eta_j|\eta_{j-1}, x_j), \tag{11}$$

$$q_{Q'}^{(m)}(y, \eta_0, \eta^{(l)})=\sum_u Q'(u)h^{(m)}(y, \eta_0, \eta^{(l)}: u) \tag{12}$$

and

$$T_1^{(m+l)}(y, u)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h^{(m)}(y, \eta_0, \eta^{(l)}: u)\log\frac{h^{(m)}(y, \eta_0, y^{(l)}: u)}{q_{Q'}^{(m)}(y, \eta_0, \eta^{(l)})}\mu(d\eta_0)\cdots\mu(d\eta_l) \tag{13}$$

where $\eta^{(l)}=(\eta_1, \cdots, \eta_l)$.

LEMMA 2. *Let* $\varepsilon>0$ *be arbitrary. Let* m *be sufficiently large. Then for any* Q'

$$\Delta^{(m)}(y, z)=\frac{1}{m+l}|T_1^{(m+l)}(y, u)-T_1^{(m+l)}(z, u)|<\varepsilon. \tag{14}$$

Proof.

$$\begin{aligned}
&\Delta^{(m)}(y, z)\\
&\leqq\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h^{(m)}(y, \eta_0, \eta^{(l)}: u)\left|\log\frac{h^{(m)}(y, \eta_0, \eta^{(l)}: u)}{h^{(m)}(z, \eta_0, \eta^{(l)}: u)}\right|\mu(d\eta_0)\cdots\mu(d\eta_l)\\
&\quad+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h^{(m)}(y, \eta_0, \eta^{(l)}: u)\left|\log\frac{q_{Q'}^{(m)}(y, \eta_0, \eta^{(l)})}{q_{Q'}^{(m)}(z, \eta_0, \eta^{(l)})}\right|\mu(d\eta_0)\cdots\mu(d\eta_l)\\
&\quad+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |h^{(m)}(y, \eta_0, \eta^{(l)}: u)-h^{(m)}(z, \eta_0, \eta^{(l)}: u)|\left|\log\frac{h^{(m)}(z, \eta_0, \eta^{(l)}: u)}{q_{Q'}^{(m)}(z, \eta_0, \eta^{(l)})}\right|\mu(d\eta_0)\cdots\mu(d\eta_l)\\
&=I_1+I_2+I_3.
\end{aligned}$$

Let $\delta<0$ be arbitrary. We choose m_0 so large that

$$\left|\frac{p^{(m_0)}(y,\eta:u)}{p^{(m_0)}(z,\eta:u)}-1\right|\leqq\delta \tag{15}$$

(The existence of m_0 is assured by Lemma 1.) From (15), we have that for all $m\geqq m_0$

$$\left|\frac{h^{(m)}(y,\eta_0,\eta^{(l)}:u)}{h^{(m)}(z,\eta_0,\eta^{(l)}:u)}-1\right|\leqq\delta \tag{16}$$

and

$$\left|\frac{q_{Q'}^{(m)}(y,\eta_0,\eta^{(l)})}{q_{Q'}^{(m)}(z,\eta_0,\eta^{(l)})}-1\right|\leqq\delta. \tag{17}$$

We are now in a position to evaluate I_1, I_2 and I_3. Let $m\geqq m_0$ be fixed arbitrarily. We have that from (16) and (17)

$$I_1\leqq|\log(1-\delta)| \qquad\text{and}\qquad I_2\leqq|\log(1-\delta)|. \tag{18}$$

On the other hand, by Assumption II, we have that $h^{(m)}(y,\eta_0,\eta^{(l)}:u)\leqq K^{l+1}$ and $q_{Q'}^{(m)}(y,\eta_0,\eta^{(l)})\geqq 1/K^{l+1}$ for all $y, \eta_0, \eta^{(l)}$ and u. Thus, from (16), we have

$$\begin{aligned}I_3&\leqq\delta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h^{(m)}(z,\eta_0,\eta^{(l)}:u)\left|\log\frac{h^{(m)}(z,\eta_0,\eta^{(l)}:u)}{q_{Q'}^{(m)}(z,\eta_0,\eta^{(l)})}\right|\mu(d\eta_0)\cdots\mu(d\eta_l)\\&\leqq 2\delta(l+1)\log K.\end{aligned} \tag{19}$$

Combining (18) and (19), we obtain

$$\Delta^{(m)}(y,z)\leqq\frac{1}{m+l}\{2|\log(1-\delta)|+2\delta(l+1)\log K\}$$

for all $m\geqq m_0$. Thus, we have the lemma.

LEMMA 3.

$$\begin{aligned}T_2^{(l+m)}(y,z)&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=m+1}^{l+m}w(y_j|y_{j-1},x_j)\mu(dy_{m+1})\cdots\mu(dy_{m+l})\\&\times\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{m}w(y_j|y_{j-1},x_j)\log\frac{\dfrac{\prod_{j=1}^{m}w(y_j|y_{j-1},x_j)}{p^{(m)}(y_0,y_m:u)}}{\dfrac{\sum_{u_i'}Q'(u_i')\prod_{j=1}^{m+l}w(y_j|y_{j-1},x_{ij}')}{q_{Q'}^{(m)}(y_0,y_m;(y_{m+1},\cdots,y_{m+l}))}}\right\}\mu(dy_1)\cdots\mu(dy_m)\\&\leqq 4(m+1)\log K\end{aligned} \tag{20}$$

for all y and $u=(x_1,\cdots,x_{m+l})$

Proof. To prove this inequality, it is sufficient to show that

$$\left|\log \frac{\prod_{j=1}^{m} w(y_j|y_{j-1}, x_j)}{p^{(m)}(y_0, y_m: u)}\right| \leqq (m+1) \log K \tag{21}$$

and

$$\left|\log \frac{\sum_{u_i} Q'(u_i) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_{ij})}{q_{Q'}^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}))}\right| \leqq (m+1) \log K \tag{22}$$

for any $(y_0, \cdots, y_{m+l})$ and u.

The first inequality is the direct consequence of Assumption II. We shall prove the second inequality. Since, by Assumption II,

$$\log \frac{\sum_u Q'(u) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_j)}{q_{Q'}^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}))}$$

$$= \log \frac{\sum_u Q'(u) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_j)}{\sum_u Q'(u) h^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}): u)}$$

$$\leqq \frac{1}{\sum_u Q'(u) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_j)} \sum_u Q'(u) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_j) \left|\log \frac{\prod_{j=1}^{m} w(y_j|y_{j-1}, x_j)}{p^{(m)}(y_0, y_m: u)}\right|$$

$$\leqq 2(m+1) \log K$$

and

$$-\log \frac{\sum_u Q'(u) \prod_{j=1}^{m+l} w(y_j|y_{j-1}, x_j)}{q_{Q'}^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}))}$$

$$\leqq \frac{1}{q_{Q'}^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}))} \sum_u Q'(u) h^{(m)}(y_0, y_m, (y_{m+1}, \cdots, y_{m+l}): u)$$

$$\times \left|\log \frac{p^{(m)}(y_0, y_m: u)}{\prod_{j=1}^{m} w(y_j|y_{j-1}, x_j)}\right| \leqq 2(m+1) \log K,$$

so we obtain the inequality (22). From (21) and (22), we have the lemma.

LEMMA 4. *Let $\varepsilon > 0$ be arbitrary. Then, for sufficiently large l and m*

$$\frac{1}{l+m}\left|E\left[\log \frac{h(V_1|y, u)}{q_{Q'}(V_1|y)}\middle| Q'\right] - E\left[\log \frac{h(V_1|z, u)}{q_{Q'}(V_1|z)}\middle| Q'\right]\right| < \varepsilon \tag{23}$$

for all $y, z, u=(x_1, \cdots, x_{l+m})$ *and* Q'.

Proof. Since

$$E\left[\log \frac{h(V_1|y, u)}{q_{Q'}(V_1|y)} \middle| Q'\right]=T_1^{(l+m)}(y, u, Q')+T_2^{(l+m)}(y, u, Q'),$$

so, (23) is easily obtained by Lemmas 2 and 3, for sufficiently large l and m.

To prove the theorem 2, we use the following theorem proved in [3].

THEOREM A. *Let* α, $0\leqq\alpha<1$, *be arbitrary. Let* E *be a set of inputs and let* $\{(u_1, B_1), \cdots, (u_N, B_N)\}$ $(u_i\in E,\ i=1, \cdots, N)$ *be any code. If we can choose a positive number* θ *such that*

$$\int_{A_{u_i}(\theta)} h(v|u_i)\mu(dv)\leqq\frac{1-\alpha}{2} \qquad (i=1, \cdots, N),$$

then N *must satisfy the relation*

$$N\leqq\frac{2}{1-\alpha}\cdot 2^{\theta}.$$

Here,

$$A_u(\theta)=\{v|\log\frac{h(v|u)}{q(v)}>\theta\}, \qquad \text{(cf. [3])}.$$

We now prove the strong converse theorem.

THEOREM 2. *Let* $\varepsilon>0$ *and* α, $0\leqq\alpha<1$, *be arbitrary. Let* $w(\cdot|\cdot,\cdot)$ *be the channel probability density function for which Assumption* II *is satisfied. For all* n *sufficiently large, any* (n, N, α) *code for the semi-continuous channel with* $w(\cdot|\cdot,\cdot)$ *must satisfy*

$$N<2^{n(C+\varepsilon)}$$

Proof. At first, we choose two positive integers l and m such that Lemma 4 holds for $\varepsilon/8$. Put $u_i=(x_{(i-1)(l+m)+1}, \cdots, x_{i(l+m)})$ and $u^{(k)}=(u_1, \cdots, u_k)$ and, similarly, $V_i=(Y_{(i-1)(l+m)+1}, \cdots, Y_{i(l+m)})$ and $V^{(k)}=(V_1, \cdots, V_k)$.

Let $u^{(k)}$ be any sequence. Define $N(u|u^{(k)})$ as the number of element u in $u^{(k)}$ and define Q' by

$$kQ'(u)=N(u|u^{(k)}).$$

Then, we have

$$E\left[\sum_{i=1}^{k}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}\middle|y_0,u^{(k)},Q'\right]$$

$$=E\left[E\left[\sum_{i=1}^{k-1}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}+\log\frac{h(V_k|V_{k-1},u_k)}{q_{Q'}(V_k|V_{k-1})}\middle|y_0,V^{(k-1)},u^{(k)},Q'\right]\middle|y_0,u^{(k)},Q'\right]$$

$$\leqq E\left[\sum_{i=1}^{k-1}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}\middle|y_0,u^{(k-1)},Q'\right]+E\left[\log\frac{h(V_k|y_0,u_k)}{q_{Q'}(V_k|y_0)}\middle|Q'\right]+\frac{\varepsilon(l+m)}{8}$$

and consequently,

$$E\left[\sum_{i=1}^{k}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}\middle|y_0,u^{(k)},Q'\right]$$

$$\leqq\sum_{i=1}^{k}E\left[\log\frac{h(V_i|y_0,u_i)}{q_{Q'}(V_i|y_0)}\middle|Q'\right]+\frac{k(l+m)\varepsilon}{8}$$

$$(24)\qquad =k\sum_{u}Q'(u)E\left[\log\frac{h(V_1|y_0,u)}{q_{Q'}(V_1|y_0)}\middle|Q'\right]+\frac{k(l+m)\varepsilon}{8}$$

$$\leqq k(l+m)\left(C+\frac{\varepsilon}{8}\right)+k(l+m)\frac{\varepsilon}{8}$$

$$=k(l+m)\left(C+\frac{\varepsilon}{4}\right).$$

On the other hand, since, from Assumption II,

$$\left|\log\frac{h(v_2|v_1,u_2)}{q_{Q'}(v_2|v_1)}\right|\leqq 2(l+m)\log K,$$

so, we have

$$(25)\qquad D\left[\sum_{i=1}^{k}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}\middle|y_0,u^{(k)},Q'\right]\leqq 16k(l+m)^2(\log K)^2.$$

Combining (24) and (25), we obtain

$$P\left\{\sum_{i=1}^{k}\log\frac{h(V_i|V_{i-1},u_i)}{q_{Q'}(V_i|V_{i-1})}\geqq k(l+m)\left(C+\frac{\varepsilon}{2}\right)\right\}\leqq\frac{1-\alpha}{2}$$

for sufficiently large k.

Let $\{(u_{01}^{(k)},A_{01}),\cdots,(u_{0M}^{(k)},A_{0M})\}$ be a (k,M,α)-code such that, for all u

$$N(u|u_{0j}^{(k)})=N(u|u^{(k)}),\qquad j=1,\cdots,M.$$

It follows from Theorem A that

$$(26)\qquad M\leqq\frac{2}{1-\alpha}\exp_2\left\{k(l+m)\left(C+\frac{3\varepsilon}{4}\right)\right\}.$$

Now the total number of Q'-vectors whose components are integral multiples of $1/k$ is less than $(k+1)$. For each such Q'-vector (26) holds. Consequently, we can conclude that for sufficiently large k

$$
\begin{aligned}
N &< (k+1)^{a^{l+m}} \frac{2}{1-\alpha} \exp_2\left\{k(l+m)\left(C+\frac{3\varepsilon}{4}\right)\right\} \\
&< \exp_2\left\{k(l+m)\left(C+\frac{4}{5}\varepsilon\right)\right\}.
\end{aligned} \tag{27}
$$

Thus the theorem is proved for sufficiently large n of the form $k(l+m)$.

Finally, suppose $n=k(l+m)+t$, with k an integer and $1 \leqq t < l+m$. Then, writing $n'=(k+1)(l+m)$, from (27) we have that

$$N < 2^{n'(C+4\varepsilon/5)} < 2^{n(1+(l+m)/n)(C+4\varepsilon/5)} < 2^{n(C+\varepsilon)}$$

for n sufficiently large. This completes the proof.

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