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TOTALLY UMBILICAL HYPERSURFACES OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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Introduction. It is well known that a totally umbilical hypersurface with non-vanishing mean curvature of a Euclidean space is isometric with a sphere. To prove this theorem we use, among others, the fact that in a Euclidean space the mean curvature of a totally umbilical hypersurface is a constant.

In more general Riemannian manifolds, however, there does not exist the position vector and so the validity of the similar theorem to the above mentioned one can not always be expected, even if we assume that the mean curvature of the hypersurface is a constant.

Now it is natural to ask for what Riemannian manifold we can prove a totally umbilical hypersurface to be isometric with a sphere. In the present paper, the author tries to prove the above theorem in a locally product Riemannian manifold using the results which were obtained by Obata recently.

In § 1 we give definitions of a locally product Riemannian manifold and of some types of locally product Riemannian manifolds.

In § 2 we give preliminaries of the theory of hypersurface. In this paragraph we introduce some tensor fields which satisfy analogous conditions to that of Sasaki's almost contact structure.[1] However, in this paper, the theory of these tensor fields is not studied in detail because we study it in another paper.

In § 3 we prove the above stated theorem under the additional condition that the hypersurface has constant mean curvature.

In § 4 we show non-existence of umbilical hypersurface with constant mean curvature in certain locally product Riemannian manifolds.

§ 1. Locally product Riemannian manifold.

An $(n+1)$-dimensional Riemannian manifold M^{n+1} is called a locally product Riemannian manifold if there exists a system of coordinate neighbourhoods $\{U_\alpha\}_{\alpha\in A}$ such that in each U_α the first fundamental form of M^{n+1} is given by

$$(1.1)\qquad ds^2 \overset{\text{def}}{=} G_{\kappa\lambda}dX^\kappa dX^\lambda = \sum_{a,b=1}^{p} G_{ab}(X^c)dX^a dX^b + \sum_{t,s=1}^{q} G_{ts}(X^r)dX^t dX^s,$$
$$p\geqq 2,\ q\geqq 2,\ p+q=n+1,$$

and in $U_\alpha \cap U_\beta$ the coordinate transformation

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1) Sasaki [4].

$$(X^a, X^t) \to (X^{a'}, X^{t'})$$

is given by

(1. 2) $$X^{a'} = X^{a'}(X^a), \qquad X^{t'} = X^{t'}(X^t).$$

If we define $F_\lambda^{\ \kappa}$ by

(1. 3) $$(F_\lambda^{\ \kappa}) = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{p} & 0 \\ 0 & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{q} \end{pmatrix}, \qquad (p \neq 0, q \neq 0),$$

in each U_α, $F_\lambda^{\ \kappa}$ is a tensor field over M^{n+1} and satisfies

(1. 4) $$F_\lambda^{\ \kappa} F_\mu^{\ \lambda} = \delta_\mu^\kappa,$$

(1. 5) $$F_{\lambda\kappa} \overset{\text{def}}{=} G_{\mu\kappa} F_\lambda^{\ \mu} = F_{\kappa\lambda},$$

(1. 6) $$\bar{\nabla}_\mu F_\lambda^{\ \kappa} = 0,$$

where $\bar{\nabla}$ denotes the operator of the covariant differentiation with respect to the Riemannian metric $G_{\lambda\kappa}$. The tensor field $F_\lambda^{\ \kappa}$ is called an almost product structure of M^{n+1}.[2)]

Suppose that M^p and M^q are both Einstein spaces and that the first fundamental form of a locally product Riemannian manifold M^{n+1} be identical with that of $M^p \times M^q$, then we have

$$R_{ab} = \lambda G_{ab}, \qquad R_{st} = \mu G_{st},$$

for certain constants λ and μ. The above equations can be written in the form

(1. 7) $$R_{\lambda\kappa} = A G_{\lambda\kappa} + B F_{\lambda\kappa},$$

where $A = (1/2)(\lambda + \mu)$, $B = (1/2)(\lambda - \mu)$.

Conversely suppose that the Ricci tensor of a locally product Riemannian manifold has the form (1. 7). If we choose a coordinate system in which the first fundamental form takes the form (1. 1), (1. 7) gives

$$R_{ab} = (A+B) G_{ab}, \qquad R_{ts} = (A-B) G_{ts}.$$

Thus, if $p > 2$, $q > 2$, then G_{ab}, G_{ts} are both Riemannian metrics of Einstein spaces and $A+B$, $A-B$ are both constants. We call the locally product Riemannian manifold with the Ricci tensor of the form (1. 7) a separately Einstein space.[3)]

2) Legrand [1], [2].

3) Tachibana [5]. Some writers called this manifold an almost Einstein space [6].

We next suppose that M^p and M^q are both spaces of constant curvature, then in $M^p \times M^q$ we get

$$R_{dcba}=\lambda(G_{cb}G_{da}-G_{ca}G_{db}), \qquad R_{tsrq}=\mu(G_{sr}G_{tq}-G_{tr}G_{sq})$$

for certain constants λ and μ. This equation can be written in the form

$$\begin{aligned}(1.8) \qquad R_{\nu\mu\lambda\kappa}=&A(G_{\nu\kappa}G_{\mu\lambda}-G_{\nu\lambda}G_{\mu\kappa}+F_{\nu\kappa}F_{\mu\lambda}-F_{\mu\kappa}F_{\nu\lambda})\\ &+B(F_{\nu\kappa}G_{\mu\lambda}-F_{\mu\kappa}G_{\nu\lambda}+G_{\nu\kappa}F_{\mu\lambda}-G_{\mu\kappa}F_{\nu\lambda}),\end{aligned}$$

where $A=(1/4)(\lambda+\mu)$, $B=(1/4)(\lambda-\mu)$.

Conversely suppose that the metric tensor of a locally product Riemannian manifold has the form (1.1), equation (1.8) gives

$$R_{dcba}=2(A+B)(G_{da}G_{cb}-G_{ca}G_{db}), \qquad R_{tsrq}=2(A-B)(G_{sr}G_{tq}-G_{tr}G_{sq}).$$

Thus, if $p>2, q>2$, then M^p and M^q are both spaces of constant curvature. We call such a Riemannian manifold a space of separately constant curvature.[4)]

§ 2. Hypersurfaces of locally product Riemannian manifolds.

Let $\bar{M}^{n+1}$ be a locally product orientable Riemannian manifold with local coordinates $\{X^\kappa\}$ and M^n be its orientable hypersurface represented parametrically by the equation

$$(2.1) \qquad X^\kappa=X^\kappa(x^h),$$

where $\{x^h\}$ be local coordinates of M^n. We put $B_i^\kappa=\partial X^\kappa/\partial x^i$, then B_i^κ $(i=1,2,\cdots,n)$ are linearly independent tangent vectors at each point of M^n.

The induced Riemannian metric g_{ji} of the hypersurface is given by

$$(2.2) \qquad g_{ji}=G_{\lambda\kappa}B_j^\lambda B_i^\kappa$$

Since the Riemannian manifold $\bar{M}^{n+1}$ and the hypersurface M^n are both orientable, we choose a unit normal vector C^κ to the hypersurface in such a way that C^κ, B_i^κ form the positive sense of $\bar{M}^{n+1}$ and that B_i^κ form the positive sense of M^n. Then we get

$$(2.3) \qquad G_{\lambda\kappa}B_h^\lambda C^\kappa=0, \qquad G_{\lambda\kappa}C^\lambda C^\kappa=1.$$

The transform $F_\lambda^\kappa B_i^\lambda$ of B_i^λ by F_λ^κ and $F_\lambda^\kappa C^\lambda$ of C^λ by F_λ^κ can be expressed as linear combinations of B_i^κ and C^κ, we put

$$(2.4) \qquad F_\lambda^\kappa B_i^\lambda=f_i^h B_h^\kappa+f_i C^\kappa,$$

$$(2.5) \qquad F_\lambda^\kappa C^\lambda=f^h B_h^\kappa+f C^\kappa,$$

4) Tachibana [5]. Some writers called this manifold a space of almost constant curvature [6].

from which we have obviously

(2. 6) $$f_i^h = B^h{}_\kappa F_\lambda{}^\kappa B_i{}^\lambda,$$

(2. 7) $$f_i = C_\kappa F_\lambda{}^\kappa B_i{}^\lambda = g_{hi} f^h,$$

(2. 8) $$f = C_\kappa F_\lambda{}^\kappa C^\lambda,$$

where we denote by $(B^i{}_\kappa, C_\kappa)$ the dual basis of $(B_i{}^\kappa, C^\kappa)$.

The tensor field $F_{\lambda\kappa}$ being symmetric with respect to its indices, we can easily see that

(2. 9) $$f_{ji} \overset{\text{def}}{=} g_{ih} f_j{}^h = f_{ij}.$$

Transforming again the both members of (2. 4) and (2. 5) by F and taking account of (2. 4) and (2. 5), we find

$$B_i{}^\mu = f_i{}^h (f_h{}^j B_j{}^\mu + f_h C^\mu) + f_i (f^j B_j{}^\mu + f C^\mu)$$

and

$$C^\mu = f^h (f_h{}^i B_i{}^\mu + f_h C^\mu) + f (f^i B_i{}^\mu + f C^\mu),$$

from which we have

(2. 10) $$f_i{}^h f_h{}^j = \delta_i^j - f^j f_i,$$

(2. 11) $$f_i{}^h f_h = -f f_i, \qquad f^i f_i{}^h = -f f^h,$$

(2. 12) $$f_i f^i = 1 - f^2.$$

Since we denote by $(B^i{}_\kappa, C_\kappa)$ the dual basis of $(B_i{}^\kappa, C^\kappa)$, we have

(2. 13) $$f_i{}^i = F_\kappa{}^\kappa - f = (p-q) - f.$$

Now denoting ∇_j the symbol of covariant differentiation along the hypersurface, we have the equation of Gauss

(2. 14) $$\nabla_j B_i{}^\kappa \overset{\text{def}}{=} \partial_j B_i{}^\kappa + B_j{}^\mu B_i{}^\lambda \overline{\left\{ {\kappa \atop \mu\lambda} \right\}} - B_h{}^\kappa \left\{ {h \atop ji} \right\} = H_{ji} C^\kappa,$$

where $\left\{ {h \atop ji} \right\}$ are the Christoffel symbol formed from the induced Riemannian metric and H_{ji} are components of the second fundamental tensor of the hypersurface.

We have also the equation of Weingarten.

(2. 15) $$\nabla_j C^\kappa \overset{\text{def}}{=} \partial_j C^\kappa + B_j{}^\lambda C^\mu \overline{\left\{ {\kappa \atop \lambda\mu} \right\}} = -H_j{}^i B_i{}^\kappa,$$

where $H_j{}^i = g^{ih} H_{jh}$.

We differentiate (2. 4) and (2. 5) covariantly along the hypersurface and obtain

(2.16) $$(\nabla_j f_i^h - f^h H_{ji} - f_i H_j^h) B_h^\kappa + (\nabla_j f_i - f H_{ji} + f_i^h H_{jh}) C^\kappa = 0,$$

(2.17) $$(\nabla_j f^h - f H_j^h + H_j^i f_i^h) B_h^\kappa + (\nabla_j f + 2 f^h H_{jh}) C^\kappa = 0,$$

from which, transvecting with $B_{k\kappa}$,

(2.18) $$\nabla_j f_{ih} = f_i H_{jh} + f_h H_{ji},$$

(2.19) $$\nabla_j f_h = f H_{jh} - H_j^i f_{ih},$$

and transvecting with C_κ,

(2.20) $$\nabla_j f = -2 f^h H_{jh}.$$

§ 3. Totally umbilical hypersurface.

When, at each point of the hypersurface M^n, the second fundamental tensor is proportional to the first fundamental tensor of M^n, that is, when it satisfies that

(3.1) $$H_{ji} = H g_{ji},$$

the hypersurface is called a totally umbilical hypersurface. The proportional factor H is the mean curvature of the hypersurface. A totally umbilical hypersurface with the vanishing mean curvature is called a totally geodesic hypersurface.

First of all, we prove the

LEMMA 3.1. *Let M^n be a totally umbilical hypersurface of a locally product Riemannian manifold. If the mean curvature H does not vanish at each point of M^n, the function f is not constant.*

Proof. Since M^n is totally umbilical, because of (2.20), we have

(3.2) $$\nabla_j f = -2 H f_j.$$

Suppose that f is a constant. Then, from our assumptions, it follows that $f_j = 0$ and so

(3.3) $$f = \pm 1,$$

because of (2.12). On the other hand, making use of (2.19), we have

(3.4) $$f_{ih} = f g_{ih}.$$

These imply that

(3.5) $$f_{ih} = \pm g_{ih}.$$

Substituting (3.5) into (2.4) and (2.5) and regarding that $f_i = 0$, we have

$$F_\lambda^\kappa B_i^\lambda = \pm B_i^\kappa, \qquad F_\lambda^\kappa C^\lambda = \pm C^\kappa \quad \text{(resp.)},$$

from which

$$F_\lambda{}^\kappa = \pm\delta_\lambda^\kappa.$$

This contradicts the fact that $F_\lambda{}^\kappa$ is a non-trivial almost product structure over $\bar{M}^{n+1}$. So f can not be constant over M^n. This completes the proof.

Let M^n be a totally umbilical hypersurface with constant mean curvature of $\bar{M}^{n+1}$. Differentiating (3. 2) covariantly and making use of (2. 19), we have

$$\nabla_i\nabla_h f = 2H^2(f_{hi} - f g_{hi}). \tag{3. 6}$$

Again, differentiating covariantly, we get by (2. 18),

$$\nabla_j\nabla_i\nabla_h f = -H^2(2\nabla_j f g_{ih} + \nabla_i f g_{jh} + \nabla_h f g_{ji}). \tag{3. 7}$$

From the last equation and the Ricci's identity with respect to $\nabla_h f$, we have

$$\underset{\nabla f}{\mathcal{L}}\left\{ {h \atop ji} \right\} \overset{\text{def}}{=} \nabla_j\nabla_i\nabla^h f + R_{kji}{}^h\nabla^k f = -2H^2(\delta_i^h\nabla_j f + \delta_j^h\nabla_i f). \tag{3. 8}$$

This means that the gradient vector $\nabla^h f$ is an infinitesimal projective transformation[5] over M^n. Thus we have, from Lemma 3. 1, the

THEOREM 3. 2. *Let M^n be a totally umbilical hypersurface with constant mean curvature of a locally product Riemannian manifold. If M^n is not totally geodesic, the gradient vector field of f is an infinitesimal projective transformation.*

On the other hand, owing to Obata,[6] we know that if, in a complete simply connected Riemannian manifold, there exists a scalar function f which satisfies (3. 7), the Riemannian manifold is isometric with a sphere of radius $1/H$. Since Lemma 3. 1 shows us that f is not constant we have the

THEOREM 3. 3. *Let M^n be a complete, simply connected totally umbilical hypersurface of a locally product Riemannian manifold. If M^n has a non-zero constant mean curvature H, M^n is isometric with a sphere of radius $1/H$.*

Now, regarding that a Euclidean space E^{n+1} is a product Riemannian manifold $E^p\times E^q$ $(p+q=n+1)$, we consider the vector field $\nabla_i f$ on a sphere S^n in E^{n+1}.

Suppose that S^n be defined by the equation

$$S^n\colon\ \sum_{\kappa=1}^{n+1}(X^\kappa)^2 = 1. \tag{3. 9}$$

Then, the unit normal vector C^κ to S^n has the components

$$C^\kappa = (X^1, \cdots, X^{n+1}), \tag{3. 10}$$

5) Yano [6].

6) Obata [3].

and so we get

$$f = C_\kappa F_\lambda^\kappa C^\lambda = \sum_{\lambda=1}^{p}(X^\lambda)^2 - \sum_{\lambda=p+1}^{n+1}(X^\lambda)^2 = 2\sum_{\lambda=1}^{p}(X^\lambda)^2 - 1, \tag{3.11}$$

from which, we have

$$\nabla_j f = (4X^1, \cdots, 4X^p, 0, \cdots, 0). \tag{3.12}$$

COROLLARY 3.4. *Let S^n be a unit sphere of E^{n+1}. Then, for any p, $1<p<n+1$, the vector field defined by the components $(X^1, \cdots, X^p, 0, \cdots, 0)$ is an infinitesimal projective transformation over S^n.*

§ 4. Totally umbilical hypersurfaces of certain locally product Riemannian manifolds.

In this paragraph we obtain some results on totally umbilical hypersurfaces of special kind of locally product Riemannian manifolds.

At first we consider a totally umbilical hypersurface with constant mean curvature of a separately Einstein space. Then, from the Codazzi equation of the hypersurface

$$\nabla_j H_{ih} - \nabla_i H_{jh} = B_j^\nu B_i^\mu B_h^\lambda C^\kappa \bar{R}_{\nu\mu\lambda\kappa}, \tag{4.1}$$

we have

$$\nabla^r H_{ir} - \nabla_i H_r^r = -(G^{\lambda\mu} - C^\lambda C^\mu) C^\kappa B_i^\nu \bar{R}_{\nu\mu\lambda\kappa} = -C^\kappa B_i^\nu \bar{R}_{\nu\kappa}.$$

From (1.7) this can be rewritten as

$$\nabla^r H_{ir} - \nabla_i H_r^r = -(AG_{\nu\kappa} + BF_{\nu\kappa}) C^\nu B_i^\kappa = -Bf_i.$$

Since the hypersurface is umbilical and has constant mean curvature we get $B=0$ because of Lemma 3.1. Thus we have the

THEOREM 4.1. *If, in a separately Einstein space, there exists a totally umbilical hypersurface with non-zero constant mean curvature, the separately Einstein space is an Einstein space.*

COROLLARY 4.2. *Let $\bar{M}^{n+1}$ be a separately Einstein space. If $\bar{M}^{n+1}$ is not an Einstein space, there is no totally umbilical hypersurface with non-zero mean curvature.*

Next we consider a totally umbilical hypersurface with non-zero constant mean curvature of a space of separately constant curvature.

Substituting (1.8) into (4.1), we have

$$\nabla_j H_{ih} - \nabla_i H_{jh} = A(f_j f_{ih} - f_i f_{jh}) + B(g_{ih} f_j - g_{jh} f_i).$$

Since the hypersurface has constant mean curvature and is umbilical we have

$$A(f_j f_{ih} - f_i f_{jh}) + B(g_{ih} f_j - g_{jh} f_i) = 0, \tag{4.2}$$

from which, together with (2. 11) and (2. 13), we get

$$\{A(p-q) + B(n-1)\} f_j = 0. \tag{4.3}$$

On the other hand, transvecting (4. 2) with f^{ih} and making use of (2. 10), (2. 11), (2. 12) and (2. 13), we have easily

$$\{A(n-1) + B(p-q)\} f_j = 0. \tag{4.4}$$

According to Lemma 3. 1, $f_j \neq 0$. So (4. 3) and (4. 4) imply that, for $p, q > 2$, $A = B = 0$. Thus we have the

THEOREM 4. 3. *In a space of separately constant curvature, if there exists a totally umbilical hypersurface with non-zero constant mean curvature, the space of separately constant curvature is a locally Euclidean space.*

COROLLARY 4. 4. *Let* $\bar{M}^{n+1}$ *be a space of separately constant curvature. If* $\bar{M}^{n+1}$ *is not a locally Euclidean space, there is no totally umbilical hypersurface with non-zero constant mean curvature.*

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