KŌDAI MATH. SEM. REP.
19 (1967), 31—34

SOME TAUBERIAN THEOREMS FOR STOCHASTIC PROCESSES

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1. In this paper some Tauberian theorems for a class of stochastic processes will be proved. We shall give the theorems in the form including also an Abelian result.

2. We state first the following

LEMMA. *Let* $\{\alpha_\lambda(t); \lambda \in \Lambda\}$ *be a class of complex-valued functions of bounded variation in every finite interval, and assume for every* $\lambda \in \Lambda$ *that*

$$f_\lambda(s) = \int_0^\infty e^{-st} d\alpha_\lambda(t)$$

converges for $s > 0$. *If* $\alpha_\lambda(t)$ *are uniformly bounded in every finite interval of* t *and if there exists a positive constant* γ *such that*

$$\lim_{t\to\infty} t^{-\gamma}\alpha_\lambda(t) = \frac{A_\lambda}{\Gamma(\gamma+1)} \tag{1}$$

uniformly in $\lambda \in \Lambda$, *where* A_λ *is bounded on* Λ, *then*

$$\lim_{s\to+0} s^\gamma f_\lambda(s) = A_\lambda \tag{2}$$

uniformly in $\lambda \in \Lambda$. *Conversely if there exist constants* K *and* $\gamma > 0$ *such that for every* $\lambda \in \Lambda$ *the functions* $\operatorname{Re}\alpha_\lambda(t) + Kt^\gamma$ *and* $\operatorname{Im}\alpha_\lambda(t) + Kt^\gamma$ *are non-decreasing in* $0 \leqq t < \infty$ *and if* (2) *holds uniformly in* $\lambda \in \Lambda$ *with* A_λ *bounded on* Λ, *then* (1) *holds uniformly in* $\lambda \in \Lambda$.

The proof of this Lemma will not be given here, since it is similar in the main to the proof of well-known Tauberian theorem (see [1]).

3. We shall now prove the following

THEOREM 1. *Let* $\{X(t); t \geqq 0\}$ *be a stochastic process such that* $\int_0^T X(t)dt$ *exists for every finite* $T > 0$, *and assume that there exist positive constants* M *and* γ *such that* $\sqrt{E\{|X(t)|^2\}} \leqq Mt^{\gamma-1}$ *for every* $t > 0$. *Then a necessary and sufficient condition that*

$$\underset{T\to\infty}{\text{l.i.m.}}\ T^{-\gamma}\int_0^T X(t)dt = \frac{Y}{\Gamma(\gamma+1)} \tag{3}$$

is that

Received May 7, 1966.

(4) $$\underset{s\to+0}{\text{l.i.m.}}\, s^{\gamma}L(s)=Y,$$

where

$$L(s)=\int_0^{\infty}e^{-st}X(t)dt \qquad (s>0).$$

Proof. Note that for proving our theorem it is sufficient to consider the case $Y=0$. Otherwise, indeed, we may consider the stochastic process $\{X_1(t); t\geqq 0\}$, where $X_1(t)=X(t)-\{\Gamma(\gamma)\}^{-1}t^{\gamma-1}Y$. We note further that by the assumption we have

(5) $$E\{|s^{\gamma}L(s)|^2\}=s^{2\gamma}\int_0^{\infty}\int_0^{\infty}e^{-st}e^{-s\tau}E\{X(t)\overline{X(\tau)}\}dtd\tau\leqq(M\Gamma(\gamma))^2,$$

and

(6) $$E\left\{\left|T^{-\gamma}\int_0^T X(t)dt\right|^2\right\}=T^{-2\gamma}\int_0^T\int_0^T E\{X(t)\overline{X(\tau)}\}dtd\tau\leqq(M\gamma^{-1})^2.$$

Hence by Schwarz's inequality

(7) $$|E\{s^{\gamma}L(s)\overline{X(\tau)}\}|\leqq M^2\Gamma(\gamma)\tau^{\gamma-1},$$

(8) $$\left|E\left\{T^{-\gamma}\int_0^T X(t)dt\cdot\overline{X(\tau)}\right\}\right|\leqq M^2\gamma^{-1}\tau^{\gamma-1}$$

and

(9) $$\left|E\left\{s^{\gamma}L(s)\cdot T^{-\gamma}\int_0^T\overline{X(t)}dt\right\}\right|\leqq\left[E\{|s^{\gamma}L(s)|^2\}\cdot E\left\{\left|T^{-\gamma}\int_0^T X(t)dt\right|^2\right\}\right]^{1/2}.$$

First we suppose that (3) holds with $Y=0$ and prove (4) with $Y=0$. It follows from (5) and (9) that

(10) $$\lim_{T\to\infty}E\left\{s^{\gamma}L(s)T^{-\gamma}\int_0^T\overline{X(t)}dt\right\}=\lim_{T\to\infty}T^{-\gamma}\int_0^T E\{s^{\gamma}L(s)\overline{X(t)}\}dt=0$$

uniformly in $s>0$. It can be seen from (7) that the class $\{\alpha_s(t); s>0\}$ of functions $\alpha_s(t)=\int_0^t E\{s^{\gamma}L(s)\overline{X(\tau)}\}d\tau$ satisfies the conditions of the first part of Lemma with $A_\lambda\equiv 0$. Hence we have that

(11) $$\lim_{\sigma\to+0}\sigma^{\gamma}\int_0^{\infty}e^{-\sigma t}E\{s^{\gamma}L(s)\overline{X(t)}\}dt=\lim_{\sigma\to+0}E\{s^{\gamma}L(s)\cdot\sigma^{\gamma}\overline{L(\sigma)}\}=0$$

uniformly in $s>0$, and therefore we have

(12) $$\lim_{s\to+0}E\{|s^{\gamma}L(s)|^2\}=0$$

which implies (4) with $Y=0$. Next we suppose that (4) holds with $Y=0$ and prove (3) with $Y=0$. From (6) and (9) we have that

(13) $$\lim_{s\to+0}E\left\{s^{\gamma}L(s)\cdot T^{-\gamma}\int_0^T\overline{X(\tau)}d\tau\right\}=\lim_{s\to+0}s^{\gamma}\int_0^{\infty}e^{-st}E\left\{X(t)\cdot T^{-\gamma}\int_0^T\overline{X(\tau)}d\tau\right\}dt=0$$

uniformly in $T>0$. From (8) we see that the conditions of the second part of Lemma are satisfied with $A_\lambda\equiv 0$ by the class $\{\alpha_T(t);\, T>0\}$ of functions

$$\alpha_T(t)=\int_0^t E\left\{X(u)\cdot T^{-\gamma}\int_0^T \overline{X(\tau)}d\tau\right\}du.$$

Hence

$$\text{(14)}\qquad \lim_{T'\to\infty} T'^{-\gamma}\int_0^{T'} E\left\{X(t)\cdot T^{-\gamma}\int_0^T \overline{X(\tau)}d\tau\right\}dt=\lim_{T'\to\infty} E\left\{T'^{-\gamma}\int_0^{T'} X(t)dt\cdot T^{-\gamma}\int_0^T \overline{X(\tau)}d\tau\right\}=0$$

uniformly in $T>0$, and therefore we have

$$\text{(15)}\qquad \lim_{T\to\infty} E\left\{\left|T^{-\gamma}\int_0^T X(t)dt\right|^2\right\}=0,$$

which implies (3) with $Y=0$. Thus our theorem is proved.

It follows immediately the following

COROLLARY 1. *Let $\{X(t);\, t>0\}$ be a stochastic process such that $\int_0^T X(t)dt$ exists for every finite $T>0$, and let $E\{|X(t)|^2\}$ be bounded for $t\geqq 0$. Then a necessary and sufficient condition that*

$$\text{(16)}\qquad \underset{T\to\infty}{\text{l.i.m.}}\frac{1}{T}\int_0^T X(t)dt=0$$

is that

$$\text{(17)}\qquad \lim_{s\to+0} s^2\int_0^\infty\int_0^\infty e^{-st}e^{-s\tau}\rho(t,\tau)dtd\tau=0,$$

where

$$\rho(t,\tau)=E\{X(t)\overline{X(\tau)}\}.$$

We state the discrete analogue of Theorem 1 in the following

THEOREM 2. *Let $\{X_n;\, n\geqq 1\}$ be a sequence of random variables and assume that there exists a constant $\gamma>0$ such that $n^{2-2\gamma}E\{|X_n|^2\}$ is bounded for $n\geqq 1$. Then a necessary and sufficient condition that*

$$\text{(18)}\qquad \underset{n\to\infty}{\text{l.i.m.}}\, n^{-\gamma}\sum_{n=1}^{n} X_k=\frac{Y}{\Gamma(\gamma+1)}$$

is that

$$\text{(19)}\qquad \underset{s\to1-0}{\text{l.i.m.}}\,(1-s)^\gamma\sum_{k=1}^{\infty} s^k X_k=Y.$$

Proof. Define a stochastic process $\{X(t);\, t\geqq 0\}$ by $X(t)=X_n$ for $n\leqq t<n+1$, where $X_0\equiv 0$, and apply Theorem 1.

COROLLARY 2. *Let $\{X_n;\, n\geqq 1\}$ be a sequence of random variables. Suppose that $E\{|X_n|^2\}$ is bounded for $n\geqq 1$. Then a necessary and sufficient condition that*

$$\text{(20)}\qquad \underset{n\to\infty}{\text{l.i.m.}}\frac{1}{n}\sum_{k=1}^{n} X_k=0$$

is that

$$\lim_{s\to 1-0}(1-s)^2\sum_{k,l=1}^{\infty}\rho_{k,l}s^{k+l}=0, \tag{21}$$

where

$$\rho_{k,l}=E\{X_k\bar{X}_l\}.$$

REMARK 1. When $\gamma>1$, the assumption in Theorem 1 that $\sqrt{E\{|X(t)|^2\}}\leqq Mt^{\gamma-1}$ for every t may be weakened. In fact, we have the result of Theorem 1 under the assumption that $\sqrt{E\{|X(t)|^2\}}\leqq M(1+t^{\gamma-1})$ for every t.

REMARK 2. In the case $\gamma=0$, we have also theorems analogous to Theorem 1 and Theorem 2.

REMARK 3. The weak law of large numbers for the class of weakly stationary processes follows from our theorems. In fact, let $\{X(t); t\geqq 0\}$ be a weakly stationary process, and let $\rho(t)=E\{X(t+\tau)\overline{X(\tau)}\}$ be its covariance function with spectral representation

$$\rho(t)=\int_{-\infty}^{\infty}e^{i\lambda t}dF(\lambda),$$

where $F(\lambda)$ is the spectral distribution function of $\{X(t); t\geqq 0\}$. Then we have that

$$\begin{aligned}
&s^2\int_0^{\infty}\int_0^{\infty}e^{-st}e^{-s\tau}\rho(t-\tau)dtd\tau\\
&=s^2\int_0^{\infty}\int_0^{\infty}e^{-st}e^{-s\tau}\left\{\int_{-\infty}^{\infty}e^{i\lambda(t-\tau)}dF(\lambda)\right\}dtd\tau\\
&=s^2\int_{-\infty}^{\infty}\left|\int_0^{\infty}e^{-(s-i\lambda)t}dt\right|^2dF(\lambda)\\
&=\int_{-\infty}^{\infty}\frac{s^2}{s^2+\lambda^2}dF(\lambda)
\end{aligned}$$

converges to zero as $s\to +0$ if and only if $F(\lambda)$ is continuous at $\lambda=0$. Hence by Corollary 1, (16) holds if and only if $F(\lambda)$ is continuous at $\lambda=0$. The discrete analogue is obtained in a similar way.

REFERENCE

[1] WIDDER, D. V., The Laplace Transform, Princeton (1946).

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