AN IMPROVEMENT OF A LIMIT THEOREM ON (J, X)-PROCESSES

BY HIROHISA HATORI AND TOSHIO MORI

1. Let $\boldsymbol{I}_r=\{1, 2, \cdots, r\}$ and $R=(-\infty, \infty)$, and let $\{(J_n, X_n);\ n=0, 1, 2, \cdots\}$ be a (J, X)-process with the state space $\boldsymbol{I}_r\times R$, or a two-dimensional stochastic process that statisfies $X_0\equiv 0$, and

$$P\{J_n=k,\ X_n\leqq x|(J_0, X_0), \cdots, (J_{n-1}, X_{n-1})\}=Q_{J_{n-1},\,k}(x) \qquad \text{(a. s.)}$$

for all $(k, x)\in\boldsymbol{I}_r\times R$, where $\{Q_{jk}(\cdot);\ j, k=1, 2, \cdots, r\}$ is a family of non-decreasing functions defined on R such that $Q_{jk}(-\infty)=0$ for $j, k=1, 2, \cdots, r$, and $\sum_{k=1}^{r}Q_{jk}(+\infty)=1$ for $j=1, 2, \cdots, r$. Let I be the $r\times r$ identity matrix and let $P=(p_{jk})$ be the $r\times r$ matrix with elements $p_{jk}\overset{\text{def}}{=}Q_{jk}(+\infty)$. Throughout this paper we assume that there exists a positive integer m for which every element of the matrix P^m is positive. Then the equation

$$\det(I-zP)=0 \tag{1}$$

has the root $\alpha_0=1$ as a simple root and the remaining roots $\alpha_1, \cdots, \alpha_{k-1}$ are greater than 1 in absolute value. In the previous paper [1], we have proved a limit theorem concerning the (J, X)-process by assuming that (i) the polynomial $\det(I-zP)$ is of r-th degree and (ii) all roots of (1) are simple. In this paper we shall prove the conclusion of this theorem without these two assumptions.

REMARK. In the previous paper, we derived a sequence $\{Y_n\}$ of random variables from the (J, X)-process and proved a theorem for the sequence $\{Y_n\}$. However, it is sufficient to consider only the sequence $\{X_n\}$ of random variables in place of the sequence $\{Y_n\}$. Because we may show that the stochastic process $\{(J_n, Y_n);\ n\geqq 0\}$ is also a (J, X)-process. In what follows, we shall prove this fact. Since we have from the definition of $\{Y_n\}$ that

$$P\{Y_1\leqq y_1, \cdots, Y_n\leqq y_n|J_0=k_0, \cdots, J_n=k_n\}$$

$$=\int\cdots\int P\{R_1(J_0, J_1, X_1)\leqq y_1, \cdots, R_n(J_{n-1}, J_n, X_n)\leqq y_n|J_0=k_0, \cdots, J_n=k_n, X_1=x_1, \cdots, X_n=x_n\}$$

Received March 17, 1966.

$$\times P\{x_1 \leqq X_1 < x_1+dx_1, \cdots, x_n \leqq X_n < x_n+dx_n | J_0=k_0, \cdots, J_n=k_n\}$$

$$=\int\cdots\int P\{R_1(k_0, k_1, x_1)\leqq y_1, \cdots, R_n(k_{n-1}, k_n, x_n)\leqq y_n\}$$

$$\times P\{x_1\leqq X_1<x_1+dx_1 | J_0=k_0, J_1=k_1\}\cdots P\{x_n\leqq X_n<x_n+dx_n | J_{n-1}=k_{n-1}, J_n=k_n\}$$

$$=\frac{1}{p_{k_0k_1}}\int P\{R_1(k_0, k_1, x_1)\leqq y_1\}dQ_{k_0k_1}(x_1)\cdots\frac{1}{p_{k_{n-1}k_n}}\int P\{R_n(k_{n-1}, k_n, x_n)\leqq y_n\}dQ_{k_{n-1}k_n}(x_n)$$

$$=F_{k_0k_1}(y_1)\cdots F_{k_{n-1}k_n}(y_n),$$

where

$$F_{jk}(y)\overset{\text{def}}{=}P\{Y_n\leqq y | J_{n-1}=j, J_n=k\}=\frac{1}{p_{jk}}\int P\{R_n(j, k, x)\leqq y\}dQ_{jk}(x)$$

is the conditional distribution of Y_n given that $J_{n-1}=j$ and $J_n=k$, it follows that Y_n and $(Y_1, \cdots, Y_{n-1})$ are conditionally independent under the condition that $(J_0, \cdots, J_n)$ is given, and therefore

$$P\{Y_n\leqq y | J_0, \cdots, J_n, Y_1, \cdots, Y_{n-1}\}=P\{Y_n\leqq y | J_0, \cdots, J_n\}=F_{J_{n-1}J_n}(y).$$

Hence, denoting by $\chi_{[J_n=k]}$ the indicator function of the event $[J_n=k]$, we have

$$\begin{aligned}&P\{J_n=k, Y_n\leqq y | J_0, \cdots, J_n, Y_1, \cdots, Y_{n-1}\}\\&=\chi_{[J_n=k]}\cdot P\{Y_n\leqq y | J_0, \cdots, J_n, Y_1, \cdots, Y_{n-1}\}\\&=\chi_{[J_n=k]}\cdot F_{J_{n-1}k}(y) \qquad \text{(a. s).}\end{aligned}$$

Therefore

$$\begin{aligned}&P\{J_n=k, Y_n\leqq y | J_0, \cdots, J_{n-1}, Y_1, \cdots, Y_{n-1}\}\\&=E\{P\{J_n=k, Y_n\leqq y | J_0, \cdots, J_n, Y_1, \cdots, Y_{n-1}\} | J_0, \cdots, J_{n-1}, Y_1, \cdots, Y_{n-1}\}\\&=E\{\chi_{[J_n=k]}\cdot F_{J_{n-1}k}(y) | J_0, \cdots, J_{n-1}, Y_1, \cdots, Y_{n-1}\}\\&=F_{J_{n-1}k}(y)\cdot P\{J_n=k | J_0, \cdots, J_{n-1}\}\\&=p_{J_{n-1}k}\cdot F_{J_{n-1}k}(y)=\bar{Q}_{J_{n-1}, k}(y),\end{aligned}$$

where $\bar{Q}_{jk}(y)\overset{\text{def}}{=}p_{jk}F_{jk}(y)$. It can easily be shown that $\{\bar{Q}_{jk}(\cdot); j, k\in \boldsymbol{I}_r\}$ satisfies the conditions mentioned above.

2. Let us define $\eta_{jk}(\theta) \overset{\text{def}}{=} E\{e^{i\theta X_n} | J_{n-1}=j, J_n=k\} \cdot P\{J_n=k | J_{n-1}=j\}$, and denote by $H(\theta)$ the $r\times r$ matrix $(\eta_{jk}(\theta))$. Let $\varphi_{kn}(\theta) \overset{\text{def}}{=} E\{e^{i\theta(X_1+\cdots+X_n)} | J_0=k\}$ be the characteristic function of $X_1+\cdots+X_n$ given that $J_0=k$, and define

$$\Phi_k(\theta, z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \varphi_{kn}(\theta) z^n. \tag{2}$$

It has been shown in [1] that

$$\boldsymbol{\varphi}_n(\theta) = H(\theta) \cdot \boldsymbol{\varphi}_{n-1}(\theta) \qquad (n=1, 2, \cdots), \tag{3}$$

and so

$$\boldsymbol{\Phi}(\theta, z) = \sum_{n=0}^{\infty} \boldsymbol{\varphi}_n(\theta) z^n = (I - zH(\theta))^{-1} \boldsymbol{e} \tag{4}$$

for small z, where

$$\boldsymbol{\varphi}_n(\theta) \overset{\text{def}}{=} \begin{pmatrix} \varphi_{1n}(\theta) \\ \vdots \\ \varphi_{rn}(\theta) \end{pmatrix}, \qquad \boldsymbol{e} \overset{\text{def}}{=} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Phi}(\theta, z) \overset{\text{def}}{=} \begin{pmatrix} \Phi_1(\theta, z) \\ \vdots \\ \Phi_r(\theta, z) \end{pmatrix}$$

are r-dimensional vectors. Denote by $\boldsymbol{\pi} \overset{\text{def}}{=} [\pi_1, \cdots, \pi_r]$ the vector of initial probabilities $\pi_k = P\{J_0=k\}$ $(k=1, \cdots, r)$, and denote by $\varphi_n(\theta)$ the characteristic function of $X_1+\cdots+X_n$. Then we have that

$$\begin{aligned} \varphi_n(\theta) &\overset{\text{def}}{=} E\{e^{i\theta(X_1+\cdots+X_n)}\} \\ &= \sum_{k=1}^{r} \pi_k \varphi_{kn}(\theta) = \boldsymbol{\pi} \cdot \boldsymbol{\varphi}_n(\theta), \end{aligned} \tag{5}$$

and therefore from (4), we have that

$$\begin{aligned} \Phi(\theta, z) &\overset{\text{def}}{=} \sum_{n=0}^{\infty} \varphi_n(\theta) z^n = \boldsymbol{\pi} \cdot \sum_{n=0}^{\infty} \boldsymbol{\varphi}_n(\theta) z^n \\ &= \boldsymbol{\pi} \cdot \boldsymbol{\Phi}(\theta, z) = \boldsymbol{\pi} \cdot (I - zH(\theta))^{-1} \boldsymbol{e} \end{aligned} \tag{6}$$

holds for small z. We shall now prove the following

THEOREM. *Under the assumption that the functions $\eta_{jk}(\theta)$ $(j, k=1, 2, \cdots, r)$ have continuous derivatives of 2nd order in a neighborhood of $\theta=0$, respectively, $(X_1+\cdots+X_n-n\mu)/\sqrt{n}$ converges in distribution to a normal distribution, where μ*

is a constant.

Proof. Let the polynomial $\det(I-zP)$ be of k-th degree, and let $\alpha_0=1, \alpha_1, \cdots, \alpha_{k-1}$ be the roots of (1) which need not all be distinct. Since $\lim_{\theta\to 0}H(\theta)=H(0)=P$, the degree $\kappa=\kappa(\theta)$ of the equation

$$\det(I-zH(\theta))=0 \tag{7}$$

is not less than k for small θ, and the equation (7) has k roots $\zeta_0(\theta), \zeta_1(\theta), \cdots, \zeta_{k-1}(\theta)$ which satisfy that

$$\zeta_0(\theta)\to 1, \zeta_1(\theta)\to\alpha_1, \cdots, \zeta_{k-1}(\theta)\to\alpha_{k-1} \quad \text{as} \quad \theta\to 0.$$

If the remaining roots $\zeta_k(\theta), \cdots, \zeta_{\kappa-1}(\theta)$ of (7) exist, then they tend to infinity as $\theta\to 0$. Therefore we may choose two positive numbers ε and θ_0 such that $\zeta_0(\theta)$ has the continuous derivative of the 2nd order in $|\theta|<\theta_0$, and

$$|\zeta_0(\theta)-1|<\varepsilon, \qquad |\zeta_l(\theta)|>1+\varepsilon \qquad (l\neq 0) \tag{8}$$

holds for $|\theta|<\theta_0$. Since $\zeta_0(\theta)$ is a simple root of (7) for $|\theta|<\theta_0$, we have the following expression from (6) by partial fraction expansion:

$$\Phi(\theta, z)=\tau_0(\theta)\Big/\left(1-\frac{z}{\zeta_0(\theta)}\right)+g(\theta, z)\Big/\left(1-\frac{z}{\zeta_1(\theta)}\right)\left(1-\frac{z}{\zeta_2(\theta)}\right)\cdots\left(1-\frac{z}{\zeta_{\kappa-1}(\theta)}\right) \tag{9}$$

where $\tau_0(\theta)$ is continuous at $\theta=0$, and

$$g(\theta, z)=g_0(\theta)z^{r-2}+g_1(\theta)z^{r-3}+\cdots+g_{r-2}(\theta) \tag{10}$$

is a polynomial of at most $(r-2)$-nd degree for any fixed θ ($|\theta|<\theta_0$). Now, we shall show that the coefficients $g_l(\theta)$ ($l=0, 1, \cdots, r-2$) are continuous functions of θ for $|\theta|<\theta_0$. Putting $f(\theta, z)\overset{\text{def}}{=}\boldsymbol{\pi}\cdot\operatorname{adj}(I-zH(\theta))\cdot\boldsymbol{e}$ and

$$\phi(\theta, z)\overset{\text{def}}{=}\left(1-\frac{z}{\zeta_1(\theta)}\right)\cdots\left(1-\frac{z}{\zeta_{\kappa-1}(\theta)}\right)=\frac{\det(I-zH(\theta))}{1-z/\zeta_0(\theta)},$$

we see that they are polynomials of at most $(r-1)$-st degree in z whose coefficients are continuous functions of θ. Then we have

$$g(\theta, z)=\frac{f(\theta, z)-\tau_0(\theta)\phi(\theta, z)}{1-z/\zeta_0(\theta)}.$$

Since $\tau_0(\theta)$ is determined in such a way that the polynomial which is the numerator of the right hand side of the above equation has a factor $1-z/\zeta_0(\theta)$, we may conclude that the coefficients $g_0(\theta), g_1(\theta), \cdots, g_{r-2}(\theta)$ are continuous functions of θ and so there exists a positive constant K such that

$$|g_l(\theta)|<K \tag{11}$$

for $|\theta|<\theta_0$ and every $l=0, 1, \cdots, r-2$. From (8) and (9), we have

$$\Phi(\theta, z)=\tau_0(\theta)\sum_{n=0}^{\infty}\frac{z^n}{\xi_0(\theta)^n}+g(\theta, z)\cdot\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_1(\theta)^n}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_2(\theta)^n}\right)\cdots\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_{\kappa-1}(\theta)^n}\right) \tag{12}$$

for $|\theta|<\theta_0$ and small z. The coefficient of z^n in the expansion of

$$\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_1(\theta)^n}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_2(\theta)^n}\right)\cdots\left(\sum_{n=0}^{\infty}\frac{z^n}{\zeta_{\kappa-1}(\theta)^n}\right)$$

as a power series in z is given by

$$\sum_{1\leqq j_1\leqq\cdots\leqq j_n\leqq\kappa-1}\frac{1}{\zeta_{j_1}(\theta)\zeta_{j_2}(\theta)\cdots\zeta_{j_n}(\theta)},$$

which, because of (8), does not exceed

$${}_{\kappa-1}H_n\cdot\frac{1}{(1+\varepsilon)^n}\leqq\binom{n+r-2}{n}\frac{1}{(1+\varepsilon)^n}\leqq\frac{(n+1)^{r-2}}{(1+\varepsilon)^n}$$

in absolute value. It follows from (10), (11) and (12) that

$$\begin{aligned}\left|\varphi_n(\theta)-\frac{\tau_0(\theta)}{\zeta_0(\theta)^n}\right|&\leqq K\cdot\left\{\frac{(n+1)^{r-2}}{(1+\varepsilon)^n}+\frac{n^{r-2}}{(1+\varepsilon)^{n-1}}+\cdots+\frac{(n-r+3)^{r-2}}{(1+\varepsilon)^{n-r+2}}\right\}\\&\leqq K\cdot\frac{(r-1)\,(n+1)^{r-2}}{(1+\varepsilon)^{n-r+2}}\end{aligned} \tag{13}$$

for $|\theta|<\theta_0$ and $n=0, 1, 2, \cdots$. For every θ, we have $|\theta/n|<\theta_0$ for sufficiently large n, and so from (13)

$$\left|\varphi_n\left(\frac{\theta}{n}\right)-\frac{\tau_0(\theta/n)}{\zeta_0(\theta/n)^n}\right|=O\left(\frac{(n+1)^{r-2}}{(1+\varepsilon)^n}\right). \tag{14}$$

On the other hand, since

$$\zeta_0(\theta)=1+\zeta_0'(0)\theta+\frac{\zeta_0''(0)}{2}\theta^2+o(\theta^2) \tag{15}$$

holds, we have

$$\left\{\zeta_0\left(\frac{\theta}{n}\right)\right\}^n=\left\{1+\zeta_0'(0)\frac{\theta}{n}+o\left(\frac{1}{n}\right)\right\}^n\to e^{\zeta_0'(0)\theta}\qquad\text{as}\quad n\to\infty.$$

Therefore it follows from (14) that for every θ

$$\varphi_n\left(\frac{\theta}{n}\right) \to \tau_0(0)e^{-\zeta_0'(0)\theta} \qquad \text{as} \quad n\to\infty.$$

Since the continuous function $\tau_0(0)e^{-\zeta_0'(0)\theta}$ is the limit of the sequence $\{\varphi_n(\theta/n)\}_{n\geqq 1}$ of characteristic functions, it follows that $\tau_0(0)e^{-\zeta_0'(0)\theta}$ is also a characteristic function, and therefore $\tau_0(0)=1$ and $\zeta_0'(0)$ is a pure imaginary number. Let us define a real number μ by $\mu \overset{\text{def}}{=} i\zeta_0'(0)$, and consider the characteristic function $\psi_n(\theta)$ of the random variable $(X_1+\cdots+X_n-n\mu)/\sqrt{n}$. By (5), we have

$$\psi_n(\theta)=e^{-i\sqrt{n}\mu\theta}\varphi_n(\theta/\sqrt{n}),$$

and therefore from (13)

$$\text{(16)} \qquad \left|\psi_n(\theta)-e^{-i\sqrt{n}\mu\theta}\frac{\tau_0(\theta/\sqrt{n})}{\zeta_0(\theta/\sqrt{n})^n}\right| = O\left(\frac{n^{r-2}}{(1+\varepsilon)^n}\right)$$

for every θ and sufficiently large n. Applying the method used in [1], we have from (15) and (16) that

$$\text{(17)} \qquad \psi_n(\theta)\to e^{-(\zeta_0''(0)+\mu^2)\theta^2/2} \qquad \text{as} \quad n\to\infty.$$

We can deduce, by the method used above, that $e^{-(\zeta_0''(0)+\mu^2)\theta^2/2}$ is a characteristic function. Hence we have that $\zeta_0''(0)+\mu^2$ is a non-negative number. Consequently $e^{-(\zeta_0''(0)+\mu^2)\theta^2/2}$ is the characteristic function of the normal distribution $N(0, \zeta_0''(0)+\mu^2)$, and our theorem was completely proved.

Reference

[1] Hatori, H., A limit theorem on (J, X)-processes., Kōdai Math. Sem. Rep. **18** (1966), 317–321.

Tokyo College of Science, and
Chūbu Institute of Technology.