# A LIMIT THEOREM ON (*J*, *X*)-PROCESSES

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**1.** Let $\{(J_n, X_n);\ n=0, 1, 2, \cdots\}$ be a two-dimensional stochastic process with the state space $\boldsymbol{I}_r \times R$, where $\boldsymbol{I}_r=\{1, 2, \cdots, r\}$ and $R=(-\infty, \infty)$, and let $\{Q_{jk}(\cdot);\ j, k=1, 2, \cdots, r\}$ be a family of non-decreasing functions defined on $R$, where $Q_{jk}(-\infty)=0$ for $j, k=1, 2, \cdots, r$ and $\sum_{k=1}^{\infty} Q_{jk}(+\infty)=1$ for $j=1, 2, \cdots, r$. If $X_0\equiv 0$ and

$$(1) \qquad P\{J_n=k,\ X_n\leqq x|(J_0, X_0), \cdots, (J_{n-1}, X_{n-1})\}=Q_{J_{n-1},k}(x) \qquad \text{(a. s.)}$$

for all $(k, x)\in \boldsymbol{I}_r\times R$, then $\{(J_n, X_n);\ n=0, 1, 2, \cdots\}$ is called a $(J, X)$-process, which has been introduced by Pyke [2]. $f$ being a real-valued Baire function defined on $\boldsymbol{I}_r\times R$, the random variable

$$\frac{1}{\sqrt{n}}\sum_{\nu=1}^{n}[f(J_\nu, X_\nu)-E\{f(J_\nu, X_\nu)\}]$$

is asymptotically normally distributed as $n\to\infty$. Taga [3] has proved this fact in the cases where $f(k, x)\equiv x$ and $f(k, x)\equiv \delta_{jk}x$, respectively. In this paper, we shall give an alternative proof of this fact in a general from, which is also regarded as an extension of the consequence in section 3 of [1].

**2.** Firstly, consider the $r\times r$ matrix $P=(p_{jk})$ where $p_{jk}\overset{\text{def}}{=}Q_{jk}(+\infty)$. When there exists a natural number $m$ such that every element of the matrix $P^m$ is strictly positive, then we have $z=1$ as a simple root of the equation $\det(I-zP)=0$, where $I$ is the $r\times r$ identity matrix, and it is known that $|\alpha_l|>1$ $(l=1, 2, \cdots, k)$, where $\alpha_1, \cdots, \alpha_k$ are the remainning roots of $\det(I-zP)=0$. In what follows, we assume that the equation $\det(I-zP)=0$ has the simple roots $1, \alpha_1, \cdots, \alpha_{r-1}$, where $|\alpha_l|>1$ $(l=1, 2, \cdots, r-1)$. Secondly, we introduce a family

$$\{R_n(j, k, t);\quad j\in \boldsymbol{I}_r,\quad k\in \boldsymbol{I}_r,\quad t\in R,\quad n=1, 2, \cdots\}$$

of real-valued random variables. And we set the following assumptions:

(i) the characteristic functions $\chi_{jkt}(\theta)\overset{\text{def}}{=}E\{e^{i\theta R_n(j,k,t)}\}$ of $R_n(j, k, t)$, where $i=\sqrt{-1}$, are independent of $n$ and $dQ_{jk}$-measurable on $t$ for any fixed $(j, k, \theta)$,

(ii) $\{R_n(j, k, t);\ n\geqq 1,\ j\in \boldsymbol{I}_r,\ k\in \boldsymbol{I}_r,\ t\in R\}$ and $\{(J_n, X_n);\ n\geqq 0\}$ are mutually independent,

(iii) $R_1(k, k_1, t_1)$, $R_2(k_1, k_2, t_2)$, $\cdots$ are mutually independent for every $(k, k_1, k_2, \cdots;\ t_1, t_2, \cdots)$,

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(iv) $Y_n \overset{\text{def}}{=} R_n(J_{n-1}, J_n, X_n)$ is a random variable or a measurable function on the probability space for every $n \geqq 1$, and

(v) the functions

$$\eta_{jk}(\theta) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \chi_{jkt}(\theta) dQ_{jk}(t),$$

where $j \in \boldsymbol{I}_r$ and $k \in \boldsymbol{I}_r$, have continuous derivatives of the 2nd order in a neighborhood of $\theta=0$, respectively. Then, we have the following

THEOREM. *Under the assumptions mentioned above, $(Y_1+\cdots+Y_n-n\mu)/\sqrt{n}$ converges in distribution to a normal distribution, where $\mu$ is a constant.*

*Proof.* Let $\varphi_{kn}(\theta) \overset{\text{def}}{=} E\{e^{i\theta(Y_1+\cdots+Y_n)} \mid J_0=k\}$ be the characteristic function of $Y_1+\cdots+Y_n$ given that $J_0=k$, where $i=\sqrt{-1}$ and $\theta$ is a real-valued variable. Then, from (1), (i), (ii), (iii) and (iv), we have

$$\varphi_{kn}(\theta) = E\{e^{i\theta(Y_1+\cdots+Y_n)} \mid J_0=k\}$$

$$= \sum_{k_1,\cdots,k_n=1}^{r} \int_{t_1=-\infty}^{\infty} \cdots \int_{t_n=-\infty}^{\infty} P\{J_\nu=k_\nu,\ t_\nu \leqq X_\nu < t_\nu+dt_\nu\ (\nu=1,2,\cdots,n) \mid J_0=k\}$$

$$\times E\{e^{i\theta(R_1(k,k_1,t_1)+\cdots+R_n(k_{n-1},k_n,t_n))} \mid J_0=k, J_\nu=k_\nu, X_\nu=t_\nu\ (\nu=1,2,\cdots,n)\}$$

$$(2) \qquad = \sum_{k_1,\cdots,k_n} \int\cdots\int dQ_{kk_1}(t_1)dQ_{k_1k_2}(t_2)\cdots dQ_{k_{n-1}k_n}(t_n)\chi_{kk_1t_1}(\theta)\chi_{k_1k_2t_2}(\theta)\cdots\chi_{k_{n-1}k_nt_n}(\theta)$$

$$= \sum_{k_1,\cdots,k_n} \eta_{kk_1}(\theta)\eta_{k_1k_2}(\theta)\cdots\eta_{k_{n-1}k_n}(\theta).$$

Introducing the $r\times r$ matrix $H(\theta) \overset{\text{def}}{=} (\eta_{jk}(\theta))$, the $r$-dimensional vectors

$$\boldsymbol{\varphi}_n(\theta) \overset{\text{def}}{=} \begin{bmatrix} \varphi_{1n}(\theta) \\ \vdots \\ \varphi_{rn}(\theta) \end{bmatrix} \quad \text{and} \quad \boldsymbol{e} \overset{\text{def}}{=} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

(2) may be described as $\boldsymbol{\varphi}_n(\theta)=H(\theta)^n\boldsymbol{e}$ and so

$$(3) \qquad \boldsymbol{\varphi}_n(\theta)=H(\theta)\boldsymbol{\varphi}_{n-1}(\theta) \qquad (n=1, 2, \cdots),$$

where $\boldsymbol{\varphi}_0(\theta)=\boldsymbol{e}$. Since $\lim_{\theta\to 0} H(\theta)=H(0)=P$, the equation $\det(I-zH(\theta))=0$ has the $r$ simple roots $\zeta_0(\theta), \zeta_1(\theta), \cdots, \zeta_{r-1}(\theta)$, which satisfy that

$$\zeta_0(\theta)\to 1,\quad \zeta_1(\theta)\to\alpha_1,\ \cdots,\ \zeta_{r-1}(\theta)\to\alpha_{r-1} \qquad \text{as } \theta\to 0$$

and

(4) $$\left|\frac{\zeta_0(\theta)}{\zeta_l(\theta)}\right|<\rho<1 \quad \text{for} \quad |\theta|<\theta_0 \quad \text{and} \quad l=1, 2, \cdots, r-1,$$

where $\rho$ and $\theta_0$ are some positive constants. Then, applying the method used in section 3 of [1], we get

(5) $$\boldsymbol{\varphi}_n(\theta)=\sum_{l=0}^{r-1}\frac{\boldsymbol{\tau}_l(\theta)}{\zeta_l(\theta)^n}$$

and

(6) $$\boldsymbol{\tau}_0(0)=\boldsymbol{e},$$

where $\boldsymbol{\tau}_l(\theta)$ $(l=0, 1, \cdots, r-1)$ are $r$-dimensional vectors independent of $z$. Introducing the vector $\boldsymbol{\pi}=[\pi_1, \cdots, \pi_r]$ of initial probabilities, where $\pi_k=P\{J_0=k\}$ $(k=1, 2, \cdots, r)$, it follows that

(7) $$\varphi_n(\theta)\overset{\text{def}}{=}E\{e^{i\theta(Y_1+\cdots+Y_n)}\}=\sum_{k=1}^{r}\pi_k\varphi_{kn}(\theta)=\boldsymbol{\pi}\cdot\boldsymbol{\varphi}_n(\theta)=\sum_{l=0}^{r-1}\frac{\tau_l(\theta)}{\zeta_l(\theta)^n}$$

and

(8) $$\tau_0(0)=\boldsymbol{\pi}\cdot\boldsymbol{\tau}_0(0)=\boldsymbol{\pi}\cdot\boldsymbol{e}=1,$$

where $\tau_l(\theta)=\boldsymbol{\pi}\cdot\boldsymbol{\tau}_l(\theta)$ $(l=0, 1, \cdots, r-1)$. Now, from the assumption (v), we know that $\zeta_l(\theta)$ and $\tau_l(\theta)$ $(l=0, 1, \cdots, r-1)$ have continuous derivatives of the 2nd order in a neighborhood of $\theta=0$. Therefore, we have

(9) $$\varphi_n'(\theta)=\sum_{l=0}^{r-1}\left\{-\frac{n\zeta_l'(\theta)\tau_l(\theta)}{\zeta_l(\theta)^{n+1}}+\frac{\tau_l'(\theta)}{\zeta_l(\theta)^n}\right\},$$

which implies with (8) that

(10) $$\begin{aligned} i\cdot E\{Y_1+\cdots+Y_n\}&=\varphi_n'(0)\\ &=-n\zeta_0'(0)+\tau_0'(0)+\sum_{l=0}^{r-1}\left\{-\frac{n\zeta_l'(0)\tau_l(0)}{\alpha_l^{n+1}}+\frac{\tau_l'(0)}{\alpha_l^n}\right\}\\ &\fallingdotseq -n\zeta_0'(0)+\tau_0'(0) \qquad \text{as } n\to\infty. \end{aligned}$$

Hence we know that $\mu\overset{\text{def}}{=}i\cdot\zeta_0'(0)$ is a real number. Similarly, by deriving

(11) $$\begin{aligned} \mathrm{Var}(Y_1+\cdots+Y_n)&=E\{(Y_1+\cdots+Y_n)^2\}-(E\{Y_1+\cdots+Y_n\})^2\\ &=-\varphi_n''(0)-(-i\varphi_n'(0))^2\\ &\fallingdotseq n(\zeta_0''(0)-\zeta_0'(0)^2)+\tau_0'(0)^2-\tau_0''(0) \end{aligned}$$

as $n\to\infty$, we know that $\zeta_0''(0)+\mu^2$ is a non-negative number. Now, we shall consider the characteristic function $\psi_n(\theta)$ of the random variable $(Y_1+\cdots+Y_n-n\mu)/\sqrt{n}$. By (7), we have

$$\begin{aligned}\psi_n(\theta)&=e^{-i\sqrt{n}\mu\theta}\varphi_n\left(\frac{\theta}{\sqrt{n}}\right)\\ &=\frac{1}{\{e^{i\mu\theta/\sqrt{n}}\zeta_0(\theta/\sqrt{n})\}^n}\left\{\tau_0\left(\frac{\theta}{\sqrt{n}}\right)+\sum_{l=0}^{r-1}\left(\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_l(\theta/\sqrt{n})}\right)^n\tau_l\left(\frac{\theta}{\sqrt{n}}\right)\right\}\end{aligned}\tag{12}$$

For any fixed $\theta$, we have $|\theta/\sqrt{n}|<\theta_0$ for all sufficiently large $n$, so that it follows from (5) that

$$\left|\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_l(\theta/\sqrt{n})}\right|<\rho<1 \qquad (l=1,2,\cdots,r-1)$$

and so

$$\left(\frac{\zeta_0(\theta/\sqrt{n})}{\zeta_l(\theta/\sqrt{n})}\right)^n\to 0 \qquad \text{as } n\to\infty.\tag{13}$$

On the other hand, we have

$$\begin{aligned}\zeta_0(\theta)&=\zeta_0(0)+\zeta_0'(0)\theta+\frac{\zeta_0''(0)}{2}\theta^2+o(\theta^2)\\ &=1-i\mu\theta+\frac{\zeta_0''(0)}{2}\theta^2+o(\theta^2) \qquad \text{as } \theta\to 0,\end{aligned}$$

which implies for any fixed $\theta$ that

$$\begin{aligned}&e^{i\mu\theta/\sqrt{n}}\zeta_0\left(\frac{\theta}{\sqrt{n}}\right)\\ &=\left(1+i\mu\frac{\theta}{\sqrt{n}}-\frac{\mu^2}{2}\frac{\theta^2}{n}+o\left(\frac{1}{n}\right)\right)\cdot\left(1-i\mu\frac{\theta}{\sqrt{n}}+\frac{\zeta_0''(0)}{2}\frac{\theta^2}{n}+o\left(\frac{1}{n}\right)\right)\\ &=1+\frac{\theta^2}{2n}(\zeta_0''(0)+\mu^2)+o\left(\frac{1}{n}\right)\end{aligned}$$

and

$$\left\{e^{i\mu\theta/\sqrt{n}}\zeta_0\left(\frac{\theta}{\sqrt{n}}\right)\right\}^n\to e^{(\zeta_0''(0)+\mu^2)\theta^2/2} \qquad \text{as } n\to\infty.\tag{14}$$

Hence we have by (8), (12), (13) and (14) that

$$\psi_n(\theta)\to e^{-(\zeta_0''(0)+\mu^2)\theta^2/2} \qquad \text{as } n\to\infty,\tag{15}$$

which proves our theorem.

**3.** If it holds that $Y_n \geqq 0$ (a. s.) and $Y_1+Y_2+\cdots+Y_n \to \infty$ (a. s.), we can defined a random variable $N(t)$ for every positive number $t$ such that

$$Y_1+\cdots+Y_{N(t)} < t \leqq Y_1+\cdots+Y_{N(t)+1}. \tag{16}$$

Noticing that $P\{N(t)<n\}=P\{S_n>t\}$, we have immediately the following corollary, which gives a property of renewal type. (See the proof of Theorem 4. 2. in [3])

COROLLARY. *Under the same assumptions of the foregoing theorem,* $(N(t)-t/\mu)/\sqrt{t}$ *converges in distribution to a normal distribution as* $t\to\infty$.

## REFERENCES

[1] HATORI, H., On Markov chains with rewards. Kōdai Math. Sem. Rep. **18** (1966), 184–192.
[2] PYKE, R., Markov renewal process; definitions and preliminary properties. Ann. Math. Stat. **32** (1961), 1231–1242.
[3] TAGA, Y., On the limiting distributions in Markov renewal processes with finitely many states, Ann. Inst. Statist. Math. **15** (1963), 1–10.

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