ON FRAMED f-MANIFOLDS

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Introduction.

We have studied, in a previous paper [10][1], properties of submanifolds in a space, almost complex, complex, almost Hermitian or Kaehlerian, and introduced the concept of framed f-structure in a differentiable manifold. A framed f-structure is defined as an f-structure satisfying a particular condition. The purpose of the present paper is to study the almost complex structure determined canonically in the product manifold of two given framed f-manifolds.

We shall recall in § 1 the definitions of an f-structure and a framed f-structure, and their fundamental properties for the latter use.

In § 2, we shall show that, in the product manifold of two framed f-manifolds, i.e., of two differentiable manifolds V and $\bar{V}$ admitting framed f-structures, there exists canonically an almost complex structure determined by the framed f-structures on the given two framed f-manifolds. We prove that the almost complex structure on the product manifold $V\times\bar{V}$ is complex analytic if and only if the framed f-structures on the given two framed f-manifolds V and $\bar{V}$ are normal. In the last part of § 2, several properties of a framed f-manifold will be studied in metric cases.

§ 3 is devoted to the study of groups of automorphisms of framed f-manifolds and to the proof of a theorem that, in a compact differentiable manifold admitting a framed f-structure, the group of all automorphisms is a Lie transformation group, if the framed f-structure is normal.

§ 1. f-structures.

Let V be an n-dimensional connected differentiable manifold of class C^∞ and let there be given, in the manifold V, a non-null tensor field f of type (1, 1) and of class C^∞ satisfying the equation

$$f^3+f=0. \tag{1.1}$$

We call such a structure an *f-structure of rank r* when the rank of f is constant everywhere and is equal to r, where r is necessarily even [14]. A manifold is called an *f-manifold* when it admits an f-structure.

If we put

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1) Numbers in brackets refer to the bibliography at the end of the paper.

$$l=-f^2, \qquad m=f^2+1,$$

where 1 denotes the unit tensor, then we have

$$l+m=1, \qquad l^2=l, \qquad m^2=m, \qquad lm=ml=0.$$

These equations mean that the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators and there exist in the manifold complementary distributions L and M corresponding to the operators l and m respectively. Then the distribution L is r-dimensional and M is $(n-r)$-dimensional.

Concerning the relations between the structure f and the projection operators l and m, we have [14]

$$\begin{cases} fl=lf=f, & fm=mf=0, \\ f^2l=-l, & f^2m=0. \end{cases} \tag{1.2}$$

We denote by $f_i{}^j$, $l_i{}^j$ and $m_i{}^j$[2] components of an f-structure and the projection operators l and m, respectively. Let V be an n-dimensional manifold admitting an f-structure of rank r. Then, in the manifold V, there exists a positive definite Riemannian metric tensor g satisfying

$$g_{rs}f_j{}^rf_i{}^s+m_{ji}=g_{ji},$$

where g_{ji} are components of the Riemannian metric tensor g and we have put $m_{ji}=m_j{}^rg_{ri}$. We call a structure defined by such a pair (f, g) an *(f, g)-structure of rank r*. A Riemannian manifold admitting an (f, g)-structure is called an *(f, g)-manifold* (Yano [14]).

In an f-manifold V, the Nijenhuis tensor $N_{ji}{}^h$ of an f-structure of rank r is by definition

$$N_{ji}{}^h=f_j{}^r\nabla_rf_i{}^h-f_i{}^r\nabla_rf_j{}^h-(\nabla_jf_i{}^r-\nabla_if_j{}^r)f_r{}^h, \tag{1.3}$$

where ∇ denotes covariant derivation with respect to a symmetric linear connection. The Nijenhuis tensor $N_{ji}{}^h$ does not depend on the symmetric connection involved.

We suppose that there exists in each neighbourhood a coordinate system in which an f-structure f on an f-manifold has numerical components

$$(f_i{}^j)=\begin{pmatrix} 0 & -1_t & 0 \\ 1_t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 1_t denotes the $t\times t$ unit matrix, $r=2t$ being the rank of f. In this case, the f-structure is said to be *integrable*. Ishihara and Yano [6] have proved the following

2) The indices $i, j, \cdots$ run over the range $1, 2, \cdots, n$.

THEOREM A. *A necessary and sufficient condition for an f-structure to be integrable is that*

$$N_{ji}{}^{h}=0.$$

In a differentiable manifold V admitting an f-structure f of rank r the set of all tangent vectors belonging to the distribution M corresponding to the projection operator m has a bundle structure, which will be denoted by $M(V)$, where $M(V)$ is a vector bundle of $(n-r)$-dimension over the manifold V. We assume now that in the vector subbundle $M(V)$ of the tangent bundle there exist $n-r$ contravariant vector fields f_y[3] spanning the distribution M at each point in the manifold V and $n-r$ covariant vector fields f^x such that $f^x(f_y)=\delta_y^x$ and $f^2+1=f_y\otimes f^y$, where $\otimes$ denotes the tensor product. In such a case, the ordered set $\{f_y\}$ of contravariant vector fields f_y is called an $(n-r)$-frame. The set $(f, \{f_y\}, \{f^x\})$ of the structure f, an $(n-r)$-frame $\{f_y\}$ and an ordered set $\{f^x\}$ of covariant vector fields f^x is called an *f-structure with complementary frame* or briefly a *framed f-structure*. We denote it by (f, f_y, f^x). When a manifold V admits an f-structure with complementary frame, it is called an *f-manifold with complementary frame* or briefly a *framed f-manifold*.

In a framed f-structure of rank r, the projection operator m is expressed as $m=f_y\otimes f^y$. We have easily the following equations

$$(1.4)\qquad f^2=-1+f_y\otimes f^y,\qquad ff_y=0,\qquad f^x(fX)=0,\qquad f^x(f_y)=\delta_y^x,$$

X being an arbitrary vector field in V.

Let V be an n-dimensional differentiable manifold admitting a framed f-structure of rank r. We have proved in [10] that the product manifold of V and an $(n-r)$-dimensional Euclidean space E^{n-r} admits an almost complex structure. The components of the Nijenhuis tensor of the induced almost complex structure defined on $V\times E^{n-r}$ are denoted by $S_{ji}{}^h$, $S_{ji}{}^x$, $S_{jy}{}^h$, $S_{jy}{}^x$ and $S_{zy}{}^h$. Then S's are the tensors defined on V and are given as follows:

$$(1.5)\qquad \begin{cases} S_{ji}{}^h=f_j{}^r\nabla_r f_i{}^h-f_i{}^r\nabla_r f_j{}^h-(\nabla_j f_i{}^r-\nabla_i f_j{}^r)f_r{}^h+(\nabla_j f^w{}_i-\nabla_i f^w{}_j)f_w{}^h,\\ S_{ji}{}^x=f_j{}^r\nabla_r f^x{}_i-f_i{}^r\nabla_r f^x{}_j-(\nabla_j f_i{}^r-\nabla_i f_j{}^r)f^x{}_r,\\ S_{jy}{}^h=-f_j{}^r\nabla_r f_y{}^h+f_y{}^r\nabla_r f_j{}^h+\nabla_j f_y{}^r\cdot f_r{}^h,\\ S_{jy}{}^x=f_y{}^r\nabla_r f^x{}_j+\nabla_j f_y{}^r\cdot f^x{}_r,\\ S_{zy}{}^h=f_z{}^r\nabla_r f_y{}^h-f_y{}^r\nabla_r f_z{}^h, \end{cases}$$

∇ denoting the convariant differentiation with respect to a symmetric linear connection, where $f_y{}^h$ and $f^x{}_i$ are components of the contravariant vector field f_y and the covariant vector field f^x, respectively. Concerning the tensor S's, we have

3) The indices $x, y, \cdots$ run over the range $1, 2, \cdots, n-r$.

THEOREM B. *If the tensor $S_{ji}{}^{h}$ vanishes identically, then so do the other S's.* (Nakawaga [10])

§ 2. Product of two framed *f*-manifolds.

Let V be an n-dimensional differentiable manifold admitting a framed f-structure $\Sigma=(f, f_y, f^x)$ of rank r and $\bar{V}$ an $\bar{n}$-dimensional diffrentiable manifold admitting a framed f-structure $\bar{\Sigma}=(\bar{f}, \bar{f}_w, \bar{f}^z)$ of rank $\bar{r}$, where we assume $n-r=\bar{n}-\bar{r}$ in this section.

We take sufficiently small open coverings $\mathfrak{U}$ of V and $\bar{\mathfrak{U}}$ of $\bar{V}$ by coordinate neighbourhoods. If we denote by $\{\eta^h\}$ and $\{\bar{\eta}^a\}$ systems of local coordinates of U_α in $\mathfrak{U}$ on V and of $\bar{U}_\lambda$ in $\bar{\mathfrak{U}}$ on $\bar{V}$ respectively, then $\{\eta^h, \bar{\eta}^a\}$ may be considered as a system of local coordinates of open sets $U_\alpha\times\bar{U}_\lambda$ of the product manifold $V\times\bar{V}$ and moreover the collection $\{U_\alpha\times\bar{U}_\lambda\}$ of all such open sets $U_\alpha\times\bar{U}_\lambda$ is an open covering of $V\times\bar{V}$. We denote this open covering of $V\times\bar{V}$ by $\mathfrak{U}\times\bar{\mathfrak{U}}$.

We take two intersecting coordinate neighbourhoods (U_α, η^h) and $(U_{\alpha'}, \eta^{h'})$ in V and two intersecting coordinate neighbourhoods $(\bar{U}_\lambda, \bar{\eta}^a)$ and $(\bar{U}_{\lambda'}, \bar{\eta}^{a'})$ in $\bar{V}$. We assume that the coordinate transformations are given by

$$\eta^{h'}=\eta^{h'}(\eta^j),\text{[4)]}$$

and

$$\bar{\eta}^{a'}=\bar{\eta}^{a'}(\bar{\eta}^b)$$

in $U_\alpha\cap U_{\alpha'}$, and $\bar{U}_\lambda\cap\bar{U}_{\lambda'}$, respectively. Then the coordinate transformation in $(U_\alpha\times\bar{U}_\lambda)\cap(U_{\alpha'}\times\bar{U}_{\lambda'})$ is given by

$$\begin{cases}\eta^{h'}=\eta^{h'}(\eta^i), \\ \bar{\eta}^{a'}=\bar{\eta}^{a'}(\bar{\eta}^b).\end{cases} \tag{2.1}$$

For the framed f-structures $\Sigma=(f, f_y, f^x)$ and $\bar{\Sigma}=(\bar{f}, \bar{f}_w, \bar{f}^z)$, we put

$$F_i{}^j=f_i{}^j, \qquad F_c{}^j=-f_y{}^j\bar{f}^y{}_c, \qquad F_i{}^b=\bar{f}_y{}^b f^y{}_i, \qquad F_c{}^b=\bar{f}_c{}^b \tag{2.2}$$

in every coordinate neighbourhood of $\mathfrak{U}\times\bar{\mathfrak{U}}$. Taking account of the coordinate transformation (2.1), we see easily that

$$F_C{}^B=\begin{pmatrix}F_i{}^j & F_c{}^j \\ F_i{}^b & F_c{}^b\end{pmatrix}$$

---

4) In this section, the indices run over the range as follows:
$i, j, \cdots$: $1, 2, \cdots, n$,
$x, y, \cdots$: $1, 2, n-r$,
$a, b, \cdots$: $n+1, \cdots, n+n$,
$A, B, \cdots$: $1, 2, \cdots, n, n+1, \cdots, n+n$.

defines a tensor field on the product manifold $V\times\bar{V}$. Making use of all equations in (1. 4) and the definition (2. 2) of $F_C{}^B$, we have

$$F_E{}^B F_C{}^E = -\delta_C^B,$$

that is, the tensor field $F_C{}^B$ is an almost complex structure on $V\times\bar{V}$. We call such a structure F the almost complex structure induced on $V\times\bar{V}$ by framed f-structures Σ and $\bar{\Sigma}$. Thus we have

PROPOSITION 2. 1. *Let V be an n-dimensional differentiable manifold admitting a framed f-structure of rank r and $\bar{V}$ an $\bar{n}$-dimensional differentiable manifold admitting a framed f-structure of rank $\bar{r}$. Then the product manifold $V\times\bar{V}$ admits an almost complex structure induced by the framed f-structures of V and $\bar{V}$, if $n-r=\bar{n}-\bar{r}$.*

Let V be an n-dimensional differentiable manifold admitting a framed f-structure of rank r. We know that the product manifold $V\times E^{n-r}$ admits an almost complex structure induced by the framed f-structure. If the induced almost complex structure is complex analytic, then we say that the given framed f-structure is *normal.* (Cf. Ishihara [5], Ishihara-Yano [6] and Nakagawa [10].) In order to obtain the relation between the normal framed f-manifolds V, $\bar{V}$ and the induced almost complex manifold $V\times\bar{V}$, we shall prove the following

PROPOSITION 2. 2. *A necessary and sufficient condition for an framed f-structure to be normal is that the tensor $S_{ji}{}^h$ vanishes identically.*

Proof. We suppose that a framed f-structure is normal. From the definition of normality, the induced almost complex structure on $V\times E^{n-r}$ is complex analytic. The property is equivalent to the fact that the Nijenhuis tensor of the almost complex structure vanishes identically. Consequently all the tensor S's vanish identically. By making use of Theorem B, we get

$$S_{ji}{}^h=0.$$

Conversely, if we suppose that $S_{ji}{}^h=0$, then it follows from Theorem B that the Nijenhuis tensor vanishes identically. This implies that a given framed f-structure is normal.

We now calculate the Nijenhuis tensor of the almost complex structure F defined on $V\times\bar{V}$ induced by framed f-structures Σ on V and $\bar{\Sigma}$ on $\bar{V}$. Then the Nijenhuis tensor $\mathfrak{N}_{CB}{}^A$ is given by

$$\mathfrak{N}_{CB}{}^A = F_C{}^E\nabla_E F_B{}^A - F_B{}^E\nabla_E F_C{}^A - (\nabla_C F_B{}^E - \nabla_B F_C{}^E)F_E{}^A.$$

Making use of the definition (2. 2) of the tensor F, we calculate components of the Nijenhuis tensor by grouping the indices in two groups $(i, j, \cdots)$ and $(\bar{a}, \bar{b}, \cdots)$. Taking account of the definition (1. 9) of the tensor S's on V and $\bar{S}$'s on $\bar{V}$ induced by the framed f-structures Σ and $\bar{\Sigma}$ respectively, we get

$$(2.3)\quad \begin{cases} \mathfrak{N}_{ji}{}^{h}=S_{ji}{}^{h}, \\ \mathfrak{N}_{ci}{}^{h}=\bar{f}^{y}{}_{c}S_{yi}{}^{h}+f_{x}{}^{h}f^{y}{}_{i}\bar{S}_{cy}{}^{x}, \\ \mathfrak{N}_{cb}{}^{h}=-f_{x}{}^{h}\bar{S}_{cb}{}^{x}+\bar{f}^{z}{}_{c}\bar{f}^{y}{}_{b}S_{zy}{}^{h}, \\ \mathfrak{N}_{ji}{}^{a}=\bar{f}_{x}{}^{a}S_{ji}{}^{x}+f^{z}{}_{j}f^{y}{}_{i}\bar{S}_{zy}{}^{a}, \\ \mathfrak{N}_{ci}{}^{a}=-f^{y}{}_{i}\bar{S}_{cy}{}^{a}+\bar{f}^{z}{}_{c}\bar{f}_{x}{}^{a}S_{zi}{}^{x}, \\ \mathfrak{N}_{cb}{}^{a}=\bar{S}_{cb}{}^{a}. \end{cases}$$

We suppose that given framed f-structures on V and $\bar{V}$ are both normal. Since the tensor S's on V and $\bar{S}$'s on $\bar{V}$ vanish identically by virtue of the definition of normality, we get

$$\mathfrak{N}_{CB}{}^{A}=0.$$

Conversely, we suppose that the induced almost complex structure on $V\times\bar{V}$ is complex analytic. From the first and the last equations of (2. 3), we get

$$S_{ji}{}^{h}=0, \qquad \bar{S}_{cb}{}^{a}=0,$$

from which by Proposition 2. 2 the given framed f-structures are normal. Thus we have

THEOREM 2. 3. *Let Σ and $\bar{\Sigma}$ be framed f-structures on differentiable manifolds V and $\bar{V}$, respectively. Then a necessary and sufficient condition for Σ and $\bar{\Sigma}$ to be normal is that the almost complex structure on $V\times\bar{V}$ induced by Σ and $\bar{\Sigma}$ is complex analytic.*

We suppose that $V=\bar{V}$ and $\Sigma=\bar{\Sigma}$. It follows that Σ is normal if and only if the almost complex structure on $V\times V$ induced by Σ is complex analytic. Combining this result with the property of normality stated previously, we get

COROLLARY. *In a differentiable manifold V admitting a framed structure Σ of rank r, the following three conditions are equivalent to each other:*

(1) *the tensor $S_{ji}{}^{h}$ vanishes identically,*

(2) *the almost complex structure on $V\times E^{n-r}$ induced by Σ is complex analytic,*

(3) *the almost complex structure on $V\times V$ induced by Σ is complex analytic.*

Next, let V and $\bar{V}$ be differentiable manifolds of dimension n and $\bar{n}$, respectively. Let (Σ, g) and $(\bar{\Sigma}, \bar{g})$ be framed (f, g)-structures of rank r on V and of rank $\bar{r}$ on $\bar{V}$, respectively. We put

$$(2.4)\qquad G_{ji}=g_{ji}, \qquad G_{cb}=\bar{g}_{cb}, \qquad G_{jb}=0, \qquad G_{ci}=0$$

in every coordinate neighbourhood of $\mathfrak{U}\times\bar{\mathfrak{U}}$. Then

$$G_{CB}=\begin{pmatrix} G_{ji} & G_{jb} \\ G_{ci} & G_{cb} \end{pmatrix}$$

defines a Riemannian metric tensor on the product manifold $V\times\bar{V}$. For the tensors F in Proposition 2. 1 and G induced on $V\times\bar{V}$ by the given framed (f, g)-structures (Σ, g) on V and $(\bar{\Sigma}, \bar{g})$ on $\bar{V}$, we get from (2. 2) and (2. 4)

$$G_{ED}F_C{}^E F_B{}^D=G_{CB}.$$

Since F is an almost complex structure, this shows that (F, G) is an almost Hermitian structure on $V\times\bar{V}$. We call such a pair the almost Hermitian structure on $V\times\bar{V}$ induced by (Σ, g) and $(\bar{\Sigma}, \bar{g})$. Thus we get

PROPOSITION 2. 4. *Let V be an n-dimensional differentiable manifold admitting a framed (f, g)-structure of rank r and $\bar{V}$ an $\bar{n}$-dimensional differentiable manifold admitting a framed (f, g)-structure of rank $\bar{r}$. Then the product manifold $V\times\bar{V}$ admits an almost Hermitian structure induced by the given framed (f, g)-structures on V and $\bar{V}$, if $n-r=\bar{n}-\bar{r}$.*

In the rest of this section, let V and $\bar{V}$ be differentiable manifolds admitting framed f-structures (Σ, g) and $(\bar{\Sigma}, \bar{g})$, respectively. Differentiating covariantly the both sides of (2. 2) with respect to the Riemannian connection $\{{}^{A}_{CB}\}$ determined by G defined by (2. 4), we get easily

$$(2.5)\qquad \begin{cases} \nabla_j F_i{}^h=\nabla_j f_i{}^h, \\ \nabla_c F_i{}^h=0, \\ \nabla_j F_b{}^h=-\bar{f}^y{}_b \nabla_j f_y{}^h, \\ \nabla_c F_b{}^h=-f_y{}^h \bar{\nabla}_c \bar{f}^y{}_b, \\ \nabla_j F_i{}^a=\bar{f}_y{}^a \nabla_j f^y{}_i, \\ \nabla_c F_i{}^a=f^y{}_i \bar{\nabla}_c \bar{f}_y{}^a, \\ \nabla_j F_b{}^a=0, \\ \nabla_c F_b{}^a=\bar{\nabla}_c \bar{f}_b{}^a, \end{cases}$$

where ∇ in the left hand sides of (2. 5) denotes the covariant derivation with respect to $\{{}^{A}_{CB}\}$ on $V\times\bar{V}$, ∇ and $\bar{\nabla}$ in the right hand sides denote the covariant derivations on V and $\bar{V}$ respectively. Making use of the first, the third, the sixth and the last equations of (2. 5), we get

$$(2.6)\qquad \begin{cases} \nabla_C F_j{}^C=\nabla_j f_i{}^j+f^y{}_i \bar{\nabla}_c \bar{f}_y{}^c, \\ \nabla_C F_b{}^C=\bar{\nabla}_c \bar{f}_b{}^c-\bar{f}^y{}_b \nabla_j f_y{}^j. \end{cases}$$

These equations imply that an induced almost Hermitian structure is almost semi-

Kaehlerian, i.e., $\nabla_C F_B{}^C=0$ ([1], see for example Yano [15],) if and only if

$$
\begin{aligned}
&\nabla_j f_i{}^j + f^y{}_i \bar{\nabla}_c \bar{f}_y{}^c = 0,\\
&\bar{\nabla}_c \bar{f}_b{}^c - \bar{f}^y{}_b \nabla_j f_y{}^j = 0.
\end{aligned}
\tag{2. 7}
$$

Thus we get

PROPOSITION 2. 5. *Let (Σ, g) and $(\bar{\Sigma}, \bar{g})$ be framed (f, g)-structures on V and $\bar{V}$, respectively. Then a necessary and sufficient condition for the almost Hermitian structure on $V\times\bar{V}$ induced by (Σ, g) and $(\bar{\Sigma}, \bar{g})$ to be almost semi-Kaehlerian is that the equations (2. 7) are valid.*

In the framed (f, g)-manifold we put

$$
O_{ji}^{rs}=\frac{1}{2}(l_j{}^r l_i{}^s - f_j{}^r f_i{}^s), \qquad {}^*O_{ji}^{rs}=\frac{1}{2}(l_j{}^r l_i{}^s + f_j{}^r f_i{}^s), \qquad {}^{\#}O_{ji}^{rs}=\delta_j^r\delta_i^s - l_j{}^r l_i{}^s,
$$

where l is the projection operator. Making use of the properties of framed f-structure, we see that the operators O, ${}^*O$ and ${}^{\#}O$ satisfy the following relations:

$$
\begin{aligned}
&O+{}^*O+{}^{\#}O=1,\\
&O\cdot O=O, \qquad {}^*O\cdot{}^*O={}^*O, \qquad {}^{\#}O\cdot{}^{\#}O={}^{\#}O,\\
&O{}^*O={}^*OO=0, \qquad O{}^{\#}O={}^{\#}OO=0, \qquad {}^*O{}^{\#}O={}^{\#}O{}^*O=0.
\end{aligned}
$$

The equations show that the operators O, ${}^*O$ and ${}^{\#}O$ are complementary projection operators. We denote by $\bar{O}$, ${}^*\bar{O}$ and ${}^{\#}\bar{O}$ the similar complementary projection operators in the framed f-manifold $\bar{V}$. On the contrary, the operator ${}^*O$ with respect to the induced almost complex structure on $V\times\bar{V}$ is given by

$$
{}^*O_{CB}^{ED}=\frac{1}{2}(\delta_C^E\delta_B^D + F_C{}^E F_B{}^D).
$$

(See for example Yano [15].) Thus, we use the same letter ${}^*O$ in a framed f-manifold as in an almost complex manifold.

Calculating ${}^*O_{CB}^{ED}\nabla_E F_D{}^A$ in the almost complex manifold $V\times\bar{V}$ and making use of (2. 6), we get the following components:

$$
\left\{
\begin{aligned}
&{}^*O_{ji}^{ED}\nabla_E F_D{}^h=\frac{1}{2}[(2{}^*O+{}^{\#}O)_{ji}^{rs}\nabla_r f_s{}^h - f_j{}^r f^y{}_i \nabla_r f_y{}^h - f^y{}_j f^z{}_i f_x{}^h \bar{f}_y{}^e \bar{f}_z{}^d \bar{\nabla}_e \bar{f}^x{}_d],\\
&{}^*O_{ci}^{ED}\nabla_E F_D{}^h=\frac{1}{2}[-f_y{}^r \bar{f}^y{}_c f_i{}^s \nabla_r f_s{}^h + f_j{}^r \bar{f}^y{}_c f^z{}_i \nabla_r f_z{}^h + f^y{}_i f_x{}^h \bar{f}_c{}^e \bar{f}_y{}^d \bar{\nabla}_e \bar{f}^x{}_d],\\
&{}^*O_{jb}^{ED}\nabla_E F_D{}^h=\frac{1}{2}[-\bar{f}^x{}_b(20+{}^{\#}O)_{js}^{rh}\nabla_r f_y{}^s - f^y{}_j f_x{}^h \bar{f}_y{}^e \bar{f}_b{}^d \bar{\nabla}_e \bar{f}^x{}_d],\\
&{}^*O_{cb}^{ED}\nabla_E F_D{}^h=\frac{1}{2}[-f_x{}^h(2{}^*\bar{O}+{}^{\#}\bar{O})_{cb}^{ed}\bar{\nabla}_e \bar{f}^x{}_d + \bar{f}^y{}_c \bar{f}^z{}_b f_y{}^r f_z{}^s \nabla_r f_s{}^h],
\end{aligned}
\right.
\tag{2. 8}
$$

$$\left\{\begin{aligned}
&{}^{*}O_{ji}^{ED}\nabla_{E}F_{D}{}^{a}=\frac{1}{2}[f_{x}{}^{a}(2{}^{*}O+{}^{\#}O)_{ji}^{rs}\nabla_{r}f^{x}{}_{s}+f^{y}{}_{j}f^{z}{}_{i}\bar{f}_{y}{}^{c}\bar{f}_{z}{}^{d}\bar{\nabla}_{c}\bar{f}_{d}{}^{a}],\\
&{}^{*}O_{ci}^{ED}\nabla_{E}F_{D}{}^{a}=\frac{1}{2}[f^{y}{}_{i}(2\bar{O}+{}^{\#}\bar{O})_{cd}^{ea}\bar{\nabla}_{e}\bar{f}_{y}{}^{d}-\bar{f}^{y}{}_{c}\bar{f}_{x}{}^{a}f_{y}{}^{r}f_{i}{}^{s}\nabla_{r}f^{x}{}_{s}],\\
&{}^{*}O_{jb}^{ED}\nabla_{E}F_{D}{}^{a}=\frac{1}{2}[-\bar{f}^{y}{}_{b}\bar{f}_{x}{}^{a}f_{j}{}^{r}f_{y}{}^{s}\nabla_{r}f^{x}{}_{s}-f^{y}{}_{j}\bar{f}_{y}{}^{e}\bar{f}^{z}{}_{b}\bar{\nabla}_{e}\bar{f}_{z}{}^{a}+f^{y}{}_{j}\bar{f}_{y}{}^{e}\bar{f}_{b}{}^{d}\bar{\nabla}_{e}\bar{f}_{d}{}^{a}],\\
&{}^{*}O_{cb}^{ED}\nabla_{E}F_{D}{}^{a}=\frac{1}{2}[(2{}^{*}\bar{O}+{}^{\#}\bar{O})_{cb}^{ed}\bar{\nabla}_{e}\bar{f}_{d}{}^{a}-\bar{f}_{c}{}^{e}\bar{f}^{y}{}_{b}\bar{\nabla}_{e}\bar{f}_{y}{}^{a}+\bar{f}^{y}{}_{c}\bar{f}^{z}{}_{b}\bar{f}_{x}{}^{a}f_{y}{}^{r}f_{z}{}^{s}\nabla_{r}f^{x}{}_{s}].
\end{aligned}\right.$$

We suppose that the almost Hermitian structure on $V\times\bar{V}$ induced by (Σ, g) and $(\bar{\Sigma}, \bar{g})$ is an almost *O-structure i.e., it satisfies ${}^{*}O_{CB}^{ED}\nabla_{E}F_{D}{}^{A}=0$. (Kotō [7]. See for example Yano [15].) Then, by definition, all the left hand sides of (2. 8) vanish identically. Transvecting the second equation of (2. 8) with $f_{z}{}^{i}\bar{f}_{y}{}^{c}$ and $f_{y}{}^{i}f^{x}{}_{h}\bar{f}_{b}{}^{c}$ respectively, we have

$$f_{y}{}^{r}\nabla_{r}f_{z}{}^{h}=0,\qquad \bar{f}_{y}{}^{d}\bar{\nabla}_{b}\bar{f}^{x}{}_{d}=\bar{f}^{z}{}_{b}\bar{f}_{z}{}^{e}\bar{f}_{y}{}^{d}\bar{\nabla}_{e}\bar{f}^{x}{}_{d}. \tag{2. 9}$$

Next, transvecting the seventh equation of (2. 8) with $f_{y}{}^{j}\bar{f}_{z}{}^{b}$ and $f_{i}{}^{j}\bar{f}^{x}{}_{a}\bar{f}_{z}{}^{b}$ respectively, we have

$$\bar{f}_{y}{}^{e}\bar{\nabla}_{e}\bar{f}_{z}{}^{a}=0,\qquad f_{y}{}^{s}\nabla_{j}f^{x}{}_{s}=f^{z}{}_{j}f_{z}{}^{r}f_{y}{}^{s}\nabla_{r}f^{x}{}_{s}. \tag{2. 10}$$

It follows from (2. 9) and (2. 10) that

$$f_{y}{}^{s}\nabla_{j}f^{x}{}_{s}=0,\qquad \bar{f}_{y}{}^{d}\bar{\nabla}_{b}\bar{f}^{x}{}_{d}=0. \tag{2. 11}$$

Substituting (2. 9), (2. 10) and (2. 11) into (2. 8), we get

$$\left\{\begin{aligned}
&(2{}^{*}O+{}^{\#}O)_{ji}^{rs}\nabla_{r}f_{s}{}^{h}=f_{j}{}^{r}f^{y}{}_{i}\nabla_{r}f_{y}{}^{h}, && (2{}^{*}\bar{O}+{}^{\#}\bar{O})_{cb}^{ed}\bar{\nabla}_{e}\bar{f}_{d}{}^{a}=\bar{f}_{c}{}^{e}\bar{f}^{y}{}_{b}\bar{\nabla}_{e}\bar{f}_{y}{}^{a},\\
&(2{}^{*}O+{}^{\#}O)_{ji}^{rs}\nabla_{r}f^{x}{}_{s}=0, && (2{}^{*}\bar{O}+{}^{\#}\bar{O})_{cb}^{ed}\bar{\nabla}_{e}\bar{f}^{y}{}_{d}=0
\end{aligned}\right. \tag{2. 12}$$

for any x and y.

Conversely we assume that framed (f, g)-structures (Σ, g) on V and $(\bar{\Sigma}, \bar{g})$ on $\bar{V}$ satisfy (2. 12). Transvecting the third and the last equations of (2. 12) with $f_{z}{}^{j}$ and $\bar{f}_{w}{}^{c}$ respectively, we get

$$f_{z}{}^{j}\nabla_{j}f^{x}{}_{i}=0,\qquad \bar{f}_{w}{}^{c}\bar{\nabla}_{c}\bar{f}^{y}{}_{b}=0.$$

Similarly, transvecting the first and the second equations of (2. 12) with $f_{z}{}^{j}$ and $\bar{f}_{w}{}^{c}$ respectively, we have

$$f_{z}{}^{j}\nabla_{j}f_{i}{}^{h}=0,\qquad \bar{f}_{w}{}^{c}\bar{\nabla}_{c}\bar{f}_{b}{}^{a}=0.$$

Substituting four equations above and the assumption (2. 12) into the second members of (2. 8), we see that all second members of (2. 8) vanish. Consequently we get

$$*O^{ED}_{CB}\nabla_E F_D{}^A=0.$$

Summing up, we have proved the following

PROPOSITION 2. 6. *Let* (Σ, g) *and* $(\bar{\Sigma}, \bar{g})$ *be framed* (f, g)*-structures on* V *and* $\bar{V}$, *respectively. Then a necessary and sufficient condition for the almost Hermitian structure on* $V\times\bar{V}$ *induced by* (Σ, g) *and* $(\bar{\Sigma}, \bar{g})$ *to be an almost *O-structure is that the expressions* (2. 12) *are valid.*

We suppose that a framed f-structure satisfies

$$(2^*O+{}^{\#}O)^{rs}_{ji}\nabla_r f_s{}^h=f_j{}^r f^y{}_i\nabla_r f_y{}^h, \qquad (2^*O+{}^{\#}O)^{rs}_{ji}\nabla_r f^x{}_s=0 \tag{2. 13}$$

for any x. Then we get

$$*O^{rs}_{ji}\nabla_r f_s{}^h=0, \qquad *O^{rs}_{ji}\nabla_r f^x{}_s=0 \tag{2.14}$$

for any x. In fact, transvecting the first equation of (2. 13) with $l_l{}^j l_k{}^i$ and taking account of properties of the projection operator l, we get

$$*O^{rs}_{lk}\nabla_r f_s{}^h=0.$$

The second equation of (2. 13) can be explicitely rewritten as follows:

$$\nabla_j f^x{}_i+f_j{}^r f_i{}^s\nabla_r f^x{}_s=0.$$

From this equation the second of (2. 14) is easily obtained.

However (2. 14) is not a sufficient condition.

Given attention to (2. 5), we can prove the following results:

PROPOSITION 2. 7. *Under the assumptions in Proposition* 2. 6, *a necessary and sufficient condition for the induced almost Hermitian structure to be almost Tachibana is that the equations*

$$\nabla_j f_i{}^h+\nabla_i f_j{}^h=0, \qquad \bar{\nabla}_c\bar{f}_b{}^a+\bar{\nabla}_b\bar{f}_c{}^a=0,$$

$$\nabla_j f_y{}^h=0, \qquad \bar{\nabla}_c\bar{f}_z{}^a=0 \qquad \textit{for any } y \textit{ and } z$$

are satisfied.

Proof. For an almost Tachibnana structure F, which satisfies $\nabla_C F_B{}^A+\nabla_B F_C{}^A=0$ (Tachibana [13], see for example Yano [15]), it follows from the first and the last equations of (2. 5) that

$$\nabla_j f_i{}^h+\nabla_i f_j{}^h=0, \qquad \bar{\nabla}_c\bar{f}_b{}^a+\bar{\nabla}_b\bar{f}_c{}^a=0.$$

Making use of the second and the third equations, we get

$$0=\nabla_c F_i^h+\nabla_i F_c^h=0-\bar{f}^y{}_c\nabla_j f_y{}^h,$$

from which we have

$$\nabla_j f_y{}^h=0 \qquad \text{for any } y.$$

By a similar method, the result required is obtained. By virtue of (2. 5), the converse is evident.

PROPOSITION 2. 8. *Under the assumptions in Proposition* 2. 6, *a necessary and sufficient condition for the induced almost Hermitian structure to be almost Kaehlerian is that we have*

$$\nabla_{[j} f_{ih]}=0, \qquad \bar{\nabla}_{[c} \bar{f}_{ba]}=0,$$
$$\nabla_{[j} f^x{}_{i]}=0, \qquad \bar{\nabla}_{[c} \bar{f}^y{}_{b]}=0$$

for any x and y.

Proof. If we put

$$\nabla_C F_B{}^E \cdot G_{EA}=\nabla_C F_{BA},$$

then we heve, from (2. 4) and (2. 5)

$$\nabla_j F_{jh}+\nabla_i F_{hj}+\nabla_h F_{ji}=\nabla_j f_{ih}+\nabla_i f_{hj}+\nabla_h f_{ji},$$
$$\nabla_c F_{ih}+\nabla_i F_{hc}+\nabla_h F_{ci}=\bar{f}_{yc}(\nabla_i f^y{}_h-\nabla_h f^y{}_i),$$
$$\nabla_c F_{bh}+\nabla_b F_{hc}+\nabla_h F_{cb}=-f_{yh}(\bar{\nabla}_c \bar{f}^y{}_b-\bar{\nabla}_b \bar{f}^y{}_c),$$
$$\nabla_c F_{ba}+\nabla_b F_{ac}+\nabla_a F_{cb}=\bar{\nabla}_c \bar{f}_{ba}+\bar{\nabla}_b \bar{f}_{ac}+\bar{\nabla}_a \bar{f}_{cb},$$

where $f_{yh}=f_y{}^j g_{jh}$ and $\bar{f}_{yc}=\bar{f}_y{}^b \bar{g}_{bc}$. These equations show that the fundamental 2-form on the almost Hermitian manifold $V\times\bar{V}$ is closed if and only if we have

$$\nabla_{[j} f_{ih]}=0, \qquad \bar{\nabla}_{[c} \bar{f}_{ba]}=0,$$
$$\nabla_{[j} f^x{}_{i]}=0, \qquad \bar{\nabla}_{[c} \bar{f}^y{}_{b]}=0.$$

This completes the proof.

Taking account of (2. 5), we easily get the following result about a Kaehlerian manifold (Yano and Mogi [16]).

PROPOSITION 2. 9. *Under the assumptions in Proposition* 2. 6, *a necessary and sufficient condition for the induced almost Hermitian structure to be Kaehlerian is that given framed f-structures Σ and $\bar{\Sigma}$ are both covariant constant.*

§ 3. Automorphisms of framed *f*-structures.

Let V and $\bar{V}$ be n-dimensional differentiable manifolds admitting framed f-structures $\Sigma=(f, f_y, f^x)$ and $\bar{\Sigma}=(\bar{f}, \bar{f}_w, \bar{f}^z)$ of rank r, respectively. A diffeomorphism of V onto $\bar{V}$ is called an *isomorphism* of V onto $\bar{V}$ if the following conditions are satisfied:

$$h \circ f = \bar{f} \circ h \tag{3.1}$$

and for arbitrary index y

$$h(f_y) = \bar{f}_y, \tag{3.2}$$

where we denote the differential of h by the same letter h.

If, moreover, $V=\bar{V}$ and $\Sigma=\bar{\Sigma}$, then h is called an *automorphism* of V. It is easily seen that the set of all automorphisms of V forms a group of transformations on V. We denote the group by $A(V, \Sigma)$, or briefly by $A(\Sigma)$.

It is easily known that a framed f-structure can be regarded as a generalization of an almost contact structure. (Concerning the almost contact structure, see for example Sasaki [12].) Accordingly $A(\Sigma)$ may be considered as a generalization of an automorphism group of an almost contact structure, which is studied by Morimoto [8]. (Cf. Morimoto and Tanno [9].) Furthermore the properties of $A(\Sigma)$ stated here can be proved by the same method as that for the proof of the theorem on the automorphism group of an almost contact structure obtained by Morimoto [8]. On that account, we state the properties without proof.

First, concerning the covariant vector fields in a given framed f-structure, we get easily the following

LEMMA. *An automorphism h in $A(\Sigma)$ leaves any vector field f^x invariant.*

Let V_i be n-dimensional differentiable manifolds admitting framed f-structures Σ_i of rank r for $i=1, 2$ and $\bar{V}_j$ be $\bar{n}$-dimensional differentiable manifolds admitting framed f-structures $\bar{\Sigma}_j$ of rank $\bar{r}$ such that $n-r=\bar{n}-\bar{r}$. Let h be an isomorphism of V_1 onto V_2 and $\bar{h}$ an isomorphism of $\bar{V}_1$ onto $\bar{V}_2$. For the isomorphisms h and $\bar{h}$, we can define a transformation $h\times\bar{h}$ of $V_1\times\bar{V}_1$ onto $V_2\times\bar{V}_2$ such that

$$(h\times\bar{h})(p, q)=(h(p), \bar{h}(q))$$

for any point p in V_1 and any point q in $\bar{V}_1$. Then the transformation $h\times\bar{h}$ is a diffeomorphism.

THEOREM 3.1. *A diffeomorphism $h\times\bar{h}$ is an isomorphism of an almost complex manifold $V_1\times\bar{V}_1$ onto an almost complex manifold $V_2\times\bar{V}_2$.*

In fact, let F_1 and F_2 be induced almost complex structures on $V_1\times\bar{V}_1$ and $V_2\times\bar{V}_2$, respectively. Denoting the differential of $h\times\bar{h}$ by the same letter $h\times\bar{h}$

and making use of the lemma above, we get

$$F_2\circ(h\times\bar{h})\begin{pmatrix}X_p\\ \bar{X}_q\end{pmatrix}=(h\times\bar{h})\circ F_1\begin{pmatrix}X_p\\ \bar{X}_q\end{pmatrix}$$

for any tangent vector X_p in $T_p(V_1)$ and for any tangent vector $\bar{X}_q$ in $T_q(\bar{V}_1)$, from which we have

$$F_2\circ(h\times\bar{h})=(h\times\bar{h})\circ F_1.$$

This shows that $h\times\bar{h}$ is an isomorphism of $V_1\times\bar{V}_1$ onto $V_2\times\bar{V}_2$.

In Theorem 3.1, we assume that $V=V_1=V_2$, $\bar{V}=\bar{V}_1=\bar{V}_2$, $\Sigma=\Sigma_1=\Sigma_2$ and $\bar{\Sigma}=\bar{\Sigma}_1=\bar{\Sigma}_2$. If $h\in A(\Sigma)$ and $\bar{h}\in A(\bar{\Sigma})$, then a transformation $h\times\bar{h}$ is an automorphism of $V\times\bar{V}$ onto itself. Consequently, we can define a mapping μ of $A(\Sigma)\times A(\bar{\Sigma})$ into $A(F)$ by

$$\mu(h,\bar{h})=h\times\bar{h},$$

where F is an induced almost complex structure on $V\times\bar{V}$ by Σ and $\bar{\Sigma}$. Thus we find

COROLLARY 1. *A mapping μ of $A(\Sigma)\times A(\bar{\Sigma})$ into $A(F)$ defined by*

$$\mu(h,\bar{h})=h\times\bar{h}$$

is a homomorphism.

The following corollaries will be easily proved.

COROLLARY 2. *If $A(\Sigma)$ and $A(\bar{\Sigma})$ operate transitively on V and $\bar{V}$ respectively, then $A(F)$ operates transitively on $V\times\bar{V}$.*

If Σ and $\bar{\Sigma}$ are normal, then Theorem 2.4 shows that $V\times\bar{V}$ is a complex manifold. As is well known [4], if the group of all holomorphic diffeomorphism of a complex manifold onto itself operates transitively, then the manifold is said to be homogeneous. Combining these matters and Corollary 2, we get

COROLLARY 3. *If Σ and $\bar{\Sigma}$ are normal and moreover $A(\Sigma)$ and $A(\bar{\Sigma})$ operate transitively on V and on $\bar{V}$ respectively, then $V\times\bar{V}$ is a homogeneous complex manifold.*

Finally, we state

THEOREM 3.2. *In a compact differentiable manifold V admitting a normal framed f-structure Σ, the automorphism group $A(\Sigma)$ is a Lie transformation group with respect to the compact open topology.*

We sketch only the proof. Since V is compact, $V\times V$ is compact and because Σ is normal, Theorem 2.4 shows that $V\times V$ admits a complex structure F induced by Σ. From properties of holomorphic functions, it is seen that the automorphism group $A(F)$ is locally compact and acts effectively on $V\times V$. (See for example Bochner and Montgomery [2].) Thus we can deduce that the automorphism group $A(\Sigma)$ on V is locally compact with respect to the compact open topology and acts effectively on V. Hence a theorem of Bochner and Montgomery [2] proves Theorem 3. 2.

Taking account of the result obtained by Boothby-Kobayashi-Wang [3], we can prove Theorem 3. 2 without the assumption of normality, but in this case the topology of the Lie group is stronger than the compact open topology. However, the author proves the more generalized result than Theorem 3. 2 with respect to the stronger compact open topology, which is stated in the forthcoming paper [11].

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