DIFFERENTIAL GEOMETRY IN TANGENT BUNDLE

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The differential geometry of tangent bundle of a Riemannian manifold has been studied by Sasaki [4] and that of a Finslerian manifold by Yano and Davies [8].

It is now well known [1], [3], [5], [8] that the tangent bundle of a differentiable manifold with a linear connection admits an almost complex structure and that the integrability condition of the almost complex structure is the vanishing of torsion and curvature tensors of the linear connection. Yano and Davies [8] used this fact in their study of the tangent bundle of a Finslerian manifold.

A linear connection in an n-dimensional differentiable manifold may be defined as an n-dimensional distribution, transversal to the fibre and invariant by all right translations in the principal fibre bundle associated with the tangent bundle.

On the other hand, what we call a non-linear connection is defined as an n-dimensional distribution, transversal to the fibre and invariant by all dilatations in the tangent bundle, and consequently a linear connection is of course a non-linear connection.

The main purpose of the present paper is to study the differential geometry of tangent bundle of a differentiable manifold with a non-linear connection.

In § 1, we define the non-linear connection as a distribution in the tangent bundle and in § 2 we introduce what we call adapted frame which is very suitable for the study of differential geometry of tangent bundle of a manifold with a non-linear connection.

§ 3 is devoted to the study of the three kinds of lifts, horizontal, vertical and complete.

In § 4, we show that the tangent bundle of a manifold with a non-linear connection admits an almost complex structure and study the integrability condition of the almost complex structure.

In § 5, we study what we call restricted tensor fields which played an important rôle in the classical theory of manifolds with a non-linear connection and in the theory of Finslerian manifolds.

Since the tangent bundle of a manifold with non-linear connection admits an almost complex structure, we can talk about almost analytic vector fields in the tangent bundle. We study in § 6 these vector fields which could be obtained as lifts of vector fields in the underlying manifold.

We then in § 7 introduce a linear connection in tangent bundle of the tangent bundle of the manifold which has special importance.

In the last § 8, we shall discuss properties of curves which are obtained as lifts

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from curves in the base manifold.

§1. Nonlinear connection in $T(M)$.

Let there be given an n-dimensional differentiable manifold M of class C^∞ and denote its tangent bundle by $T(M)$. The bundle $'T(M)$ consisting of all non-zero vectors tangent to M is a subbundle of $T(M)$. If, for a non-zero real number a, we define a bundle transformation R_a: $T(M)\to T(M)$ by $R_{a(\sigma)}=a\sigma$, σ being an arbitrary element of $T(M)$, then $\pi\circ R_a$ is the identity mapping of M, where π is the bundle projection π: $T(M)\to M$. The group of all such bundle transformations R_a will be denoted by D. The group D is a group of bundle transformations of the subbundle $'T(M)$.

Let $'F_{\mathrm{P}}={}'\pi^{-1}(\mathrm{P})$ be the fibre of $'T(M)$ over a point P of M, $'\pi$ denoting the bundle projection $'\pi$: $'T(M)\to M$. In the tangent bundle $T('T(M))$ of $'T(M)$, there exists always a differentiable distribution H of class C^∞ satisfying the following two conditions:

$$\text{(a)}\qquad T_\sigma('T(M))=T_\sigma('F_{\mathrm{P}})+H_\sigma \qquad \text{(direct sum)},$$

$$(1.1)\qquad\qquad '\pi(\sigma)=\mathrm{P},$$

$$\text{(b)}\qquad dRa(H_\sigma)=H_{Ra(\sigma)} \qquad \text{for any real number}\quad a\neq 0,$$

where $T_\sigma('T(M))$ and $T_\sigma('F_{\mathrm{P}})$ are the tangent spaces of $'T(M)$ and $'F_{\mathrm{P}}$ at σ respectively and H_σ is the value of the distribution H at σ. Such a distribution H is called a *non-linear connection* in $'T(M)$, or by the abuse of terminology, in the tangent bundle $T(M)$. H_σ is sometimes called the *horizontal plane* at σ and H the *horizontal plane field*. The horizontal plane field H is necessarily n-dimensional because of the condition (a) given in (1.1).

Let U be a coordinate neighborhood of M and (ξ^h) local coordinates[1] defined in U. The open set $\pi^{-1}(U)$ is a coordinate neighborhood of $T(M)$ and there exist local coordinates (ξ^h, η^h) in $\pi^{-1}(U)$ such that for a point σ with local coordinates (ξ^h, η^h) the point $\mathrm{P}=\pi(\sigma)$ has coordinates (ξ^h) in U and (η^h) is the system of cartesian coordinates in the fibre $F_{\mathrm{P}}=\pi^{-1}(\mathrm{P})$ with respect to the natural basis $\partial/\partial\xi^h$. We call such coordinates (ξ^h, η^h) *adapted coordinates* associated with (ξ^h) in $\pi^{-1}(U)$.

Let there be given a non-linear connection H in $'T(M)$. Then, H will be regarded as a distribution in $T(M)$ with singularities along the zero cross-section. Taking account of the condition (a) given in (1.1), we easily see that H is expressed in $\pi^{-1}(U)$ by pfaffian equations[2]

$$(1.2)\qquad \omega^{h^*}=\Gamma_j{}^h(\xi, \eta)d\xi^j+d\eta^h=0$$

1) The indices h, i, j, k, s, t run over the range $\{1, 2, \cdots, n\}$.

2) The indices $h^*, i^*, j^*, k^*, s^*, t^*$ run over the range $\{1^*, 2^*, \cdots, n^*\}$. The index h^* will be sometimes identified with the corresponding index h.

outside the zero cross-section, where the coefficients $\Gamma_j{}^h(\xi, \eta)$ are uniquely determined by giving adapted coordinates (ξ^h, η^h) in $\pi^{-1}(U)$ and are defined in the domain $(\eta^h) \neq (0, 0, \cdots, 0)$. As an immediate consequence of the condition (b) given in (1.1), the functions $\Gamma_j{}^h(\xi, \eta)$ are necessarily homogeneous of degree one with respect to the arguments η^h, i.e.

$$\Gamma_j{}^h(\xi, t\eta) = t\Gamma_j{}^h(\xi, \eta), \quad (t \neq 0), \tag{1.3}$$

which is equivalent to the fact that Euler's formula

$$\frac{\partial \Gamma_j{}^h}{\partial \eta^t} \eta^t = \Gamma_j{}^h \tag{1.4}$$

and

$$\Gamma_j{}^h(\xi, -\eta) = -\Gamma_j{}^h(\xi, \eta)$$

are valid. The set of functions $\Gamma_j{}^h$ are called the *components of the non-linear connection* H with respect to adapted coordinates (ξ^h, η^h).

The distribution consisting of all tangent planes to the fibres is expressed by pfaffian equations

$$\omega^h = d\xi^h = 0 \tag{1.5}$$

in $\pi^{-1}(U)$ with respect to adapted coordinates (ξ^h, η^h).

We shall now give a remark which is useful in later discussions. Let $\omega = \omega_h d\xi^h$ be a differential 1-form in M. Then ω is regarded as a function $\omega_h \eta^h$ in $T(M)$. If a vector field X in $T(M)$ satisfies $X(\omega) = 0$ for any 1-form ω in M, then X necessarily vanishes (cf. Yano and Ledger [9]). In fact, let X have components (X^h, Y^h) in adapted coordinates (ξ^h, η^h) and put $\omega = \omega_h(\xi) d\xi^h$. Then we have

$$X(\omega) = \frac{\partial \omega_h}{\partial \xi^i} X^i \eta^h + \omega_h Y^h = 0,$$

which implies $X^h = 0$, $Y^h = 0$, i.e. $X = 0$ because of the arbitrariness of ω. Taking account of this fact, we have

PROPOSITION 1.1. *The tangent bundle* $T(M)$ *admits a vector field* J *such that*

$$J \cdot \omega = \omega$$

for any 1*-form* ω *in* M. *The vector field* J *vanishes only along the zero cross-section of* $T(M)$ *and has components* $(0, \eta^h)$ *at point* (ξ^h, η^h) *with respect to adapted coordinates.*

It is easily verified that

$$\underset{J}{\mathcal{L}} \omega^{h*} = \frac{\partial \Gamma_j{}^h}{\partial \eta^t} \eta^t d\xi^j + d\eta^h = \omega^{h*} \tag{1.6}$$

where $\underset{J}{\mathcal{L}}$ denotes the Lie derivation in $T(M)$ with respect to the vector field J. Thus we have

PROPOSITION 1.2. *The vector field J generates the 1-parameter group $D^+=\{R_a|a>0\}$ of transformations in $T(M)$ and the Euler's formula* (1.4) *is equivalent to*

$$\underset{J}{\mathcal{L}}\omega^{h*}=\omega^{h*}.$$

Let there be given two intersecting coordinate neighborhoods U, U' in M, and let Σ_U and $\Sigma_{U'}$ be adapted coordinates in $\pi^{-1}(U)$ and $\pi^{-1}(U')$ respectively. Then we get the law of coordinate transformation[3)]

$$\xi^h=\xi^h(\xi^{h'}),\quad \eta^h=\frac{\partial\xi^h}{\partial\xi^{h'}}\eta^{h'} \tag{1.7}$$

in $\pi^{-1}(U)\cap\pi^{-1}(U')$, where (ξ^h, η^h) and $(\xi^{h'}, \eta^{h'})$ are coordinates of a point $\sigma\in\pi^{-1}(U)\cap\pi^{-1}(U')$ with respect to Σ_U and $\Sigma_{U'}$ respectively. Denoting by $\Gamma_j{}^h$ and $\Gamma_{j'}{}^{h'}$ the components of the given non-linear connection H with respect to Σ_U and $\Sigma_{U'}$ respectively, we find

$$\frac{\partial\xi^h}{\partial\xi^{h'}}\Gamma_{j'}{}^{h'}(\xi^{k'}, \eta^{k'})=\Gamma_j{}^h(\xi^k, \eta^k)\frac{\partial\xi^j}{\partial\xi^{j'}}+\frac{\partial^2\xi^h}{\partial\xi^{j'}\partial\xi^{i'}}\eta^{i'}, \tag{1.8}$$

because two systems of equations

$$\omega^{h*}=\Gamma_j{}^h d\xi^j+d\eta^h=0,\qquad \omega^{h*'}=\Gamma_{j'}{}^{h'}d\xi^{j'}+d\eta^{h'}=0$$

define the same distribution H. The relation (1.8) is the law of transformation of components of a non-linear connection.

The law (1.8) of transformation reduces to[4)]

$$\omega^{h*}=\frac{\partial\xi^h}{\partial\xi^{h'}}\omega^{h*'}. \tag{1.9}$$

On the other hand, we get

$$\omega^h=\frac{\partial\xi^h}{\partial\xi^{h'}}\omega^{h'} \tag{1.10}$$

directly from the definition (1.5) of ω^h.

Let $A=\{U\}$ be an open covering of M consisting of coordinate neighborhoods U and Σ_U adapted coordinates in $\pi^{-1}(U)$ where $U\in A$. If we now suppose that there are given n^2 functions $\Gamma_j{}^h(\xi, \eta)$, homogeneous of degree 1 with respect to η^h, in each $\pi^{-1}(U)$ and that the law (1.8) of transformation is valid in any intersection

3) The indices h', i', j', k', s', t' run over the range $\{1', 2', \cdots, n'\}$.

4) The indices $h^{*'}, i^{*'}, j^{*'}, k^{*'}, s^{*'}, t^{*'}$ run over the range $\{1^{*'}, 2^{*'}, \cdots, n^{*'}\}$.

$\pi^{-1}(U)\cap\pi^{-1}(U')$, U and U' belonging to A, then the distribution H_U defined by (1. 2) involving the given $\Gamma_j{}^h$ in each $\pi^{-1}(U)$ determines globally a distribution H in $'T(M)$, which is a non-linear connection having the given $\Gamma_j{}^h$ as its components in each $\pi^{-1}(U)$.

Differentiating the both sides of (1. 8) with respect to $\eta^{i'}$, we find

$$\frac{\partial\xi^h}{\partial\xi^{h'}}\Gamma_{j'}{}^{h'}{}_{i'}(\xi^{k'}, \eta^{k'})=\Gamma_j{}^h{}_i(\xi^k, \eta^k)\frac{\partial\xi^j}{\partial\xi^{j'}}\frac{\partial\xi^i}{\partial\xi^{i'}}+\frac{\partial^2\xi^h}{\partial\xi^{j'}\partial\xi^{i'}}, \tag{1. 11}$$

where

$$\Gamma_j{}^h{}_i=\frac{\partial}{\partial\eta^i}\Gamma_j{}^h, \qquad \Gamma_{j'}{}^{h'}{}_{i'}=\frac{\partial}{\partial\eta^{i'}}\Gamma_{j'}{}^{h'}. \tag{1. 12}$$

Thus the quantity $\Gamma_j{}^h{}_i$ defined above behaves just as a linear connection does.

As is well known, there exists always a linear connection in $T(M)$. Denoting by $\Gamma_j{}^h{}_i$ components of a linear connection in local coordinates (ξ^h) defined in a coordinate neighborhood U of M, and putting

$$\Gamma_j{}^h=\Gamma_j{}^h{}_i\eta^i \tag{1. 13}$$

in $\pi^{-1}(U)$, we see that $\Gamma_j{}^h$ thus defined determines a non-linear connection. A non-linear connection defined in terms of a linear connection by (1. 13) is briefly called a *linear connection.*

Let there be given a differentiable manifold in which a system of curves called *paths* is determined by a system of ordinary differential equations

$$\frac{d^2\xi^h}{dt^2}+\Gamma^h(\xi, \dot{\xi})=0, \qquad \dot{\xi}^h=\frac{d\xi^h}{dt}, \tag{1. 14}$$

where $\Gamma^h(\xi, \eta)$ are functions of $2n$ independent variables ξ^h and η^h, homogeneous of degree 2 with respect to η^h and t is a parameter determined up to an affine transformation. Such a space is called a general space of paths (cf. Douglas [2]). We can easily see that the quantity

$$\Gamma_j{}^h=\frac{1}{2}\frac{\partial\Gamma^h}{\partial\eta^j} \tag{1. 15}$$

determines a non-linear connection in the tangent bundle. Conversely, if there is given a non-linear connection $\Gamma_j{}^h$, putting

$$\Gamma^h(\xi, \eta)=\Gamma_j{}^h(\xi, \eta)\eta^j,$$

we see that the differential equations (1. 14) determine a system of paths.

§ 2. Adapted frame.

Let there be given a non-linear connection in $T(M)$. Then the $2n$ 1-forms ω^h and ω^{h*} appearing in (1. 2) and (1. 5) form in $\pi^{-1}(U)$ a coframe which is called the *adapted coframe* associated with coordinates (ξ^h) defined in U, U being a coordinate neighborhood of M. Putting[5)]

$$\omega^h = B^h{}_A d\xi^A, \qquad \omega^{h^*} = C^{h^*}{}_A d\xi^A, \tag{2.1}$$

we see that

$$(B^h{}_A) = (\delta_i^h, 0), \qquad (C^{h^*}{}_A) = (\Gamma_i{}^h, \delta_i^h). \tag{2.2}$$

If we put

$$A^h{}_A = B^h{}_A, \qquad A^{h^*}{}_A = C^{h^*}{}_A, \tag{2.3}$$

and define $2n$ 1-forms A^α by[6)]

$$A^\alpha = A^\alpha{}_A d\xi^A, \tag{2.4}$$

we have

$$A^h = \omega^h, \qquad A^{h^*} = \omega^{h^*}. \tag{2.5}$$

We now put

$$dA^\alpha = \Omega_{\gamma\beta}{}^\alpha A^\gamma \wedge A^\beta, \quad \Omega_{\gamma\beta}{}^\alpha + \Omega_{\beta\gamma}{}^\alpha = 0. \tag{2.6}$$

If we take account of (1. 2), (1. 5) and (2. 5), we get

$$\begin{aligned} &\Omega_{\gamma\beta}{}^h = 0, \\ &\Omega_{ji}{}^{h^*} = K_{jit}{}^h \eta^t, \qquad \Omega_{j^*i}{}^{h^*} = \Gamma_i{}^h{}_j, \\ &\Omega_{j^*i^*}{}^{h^*} = 0, \end{aligned} \tag{2.7}$$

where we have put

$$K_{kji}{}^h = (\partial_k \Gamma_j{}^h{}_i - \Gamma_k{}^t \partial_{t^*} \Gamma_j{}^h{}_i) - (\partial_j \Gamma_k{}^h{}_i - \Gamma_j{}^t \partial_{t^*} \Gamma_k{}^h{}_i) + \Gamma_k{}^h{}_t \Gamma_j{}^t{}_i - \Gamma_j{}^h{}_t \Gamma_k{}^t{}_i, \tag{2.8}$$

$$\Gamma_j{}^h{}_i = \partial_{i^*} \Gamma_j{}^h, \tag{2.9}$$

the operators ∂_k and ∂_{k^*} being defined respectively by

$$\partial_k = \partial/\xi^k \qquad \text{and} \qquad \partial_{k^*} = \partial/\partial\eta^k.$$

Then the equations (2. 6) reduces to

$$d\omega^h = 0, \qquad d\omega^{h^*} = \frac{1}{2}(K_{jit}{}^h \eta^t)\omega^j \wedge \omega^i - \Gamma_j{}^h{}_i \omega^{j^*} \wedge \omega^i. \tag{2.10}$$

Let A_β be the $2n$ vector fields defined in $\pi^{-1}(U)$ by

$$A^\alpha(A_\beta) = \delta_\beta^\alpha \tag{2.11}$$

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5) Putting $\xi^{h^*} = \eta^h$, we have $(\xi^A) = (\xi^h, \eta^h)$, and in the sequel the indices A, B, C, D, E run over the range $\{1, 2, \cdots, n, 1^*, 2^*, \cdots, n^*\}$.

6) The indices $\alpha, \beta, \gamma, \delta, \varepsilon$ stand for h, i, j, k, s, t or $h^*, i^*, j^*, k^*, s^*, t^*$ and hence they run over the range $\{1, 2, \cdots, n, 1^*, 2^*, \cdots, n^*\}$.

and denote by $A_\beta{}^A$ the components of A_β with respect to adapted coordinates (ξ^h, η^h). We put

$$B_i = A_i, \qquad C_{i^*} = A_{i^*} \tag{2.12}$$

and

$$B_i{}^A = A_i{}^A, \qquad C_{i^*}{}^A = A_{i^*}{}^A. \tag{2.13}$$

Taking account of (2.2), (2.3) and (2.11), we see that the components $B_i{}^A$ of B_i and the components $C_{i^*}{}^A$ of C_{i^*} are given respectively by

$$(B_i{}^A) = \begin{pmatrix} \delta_i^h \\ -\Gamma_i{}^h \end{pmatrix}, \qquad (C_{i^*}{}^A) = \begin{pmatrix} 0 \\ \delta_i^h \end{pmatrix} \tag{2.14}$$

with respect to adapted coordinates.

As is well known, the equation (2.6) is equivalent to equations

$$[A_\gamma, A_\beta] = -\Omega_{\gamma\beta}{}^\alpha A_\alpha, \tag{2.15}$$

i.e. to equations

$$\begin{aligned} &[B_j, B_i] = -(K_{jit}{}^h \eta^t) C_{h^*}, \\ &[B_j, C_{i^*}] = \Gamma_j{}^h{}_i C_{h^*}, \\ &[C_{j^*}, C_{i^*}] = 0. \end{aligned} \tag{2.16}$$

The non-linear connection H, i.e. the horizontal plane field defined by $\omega^{h^*}=0$ is integrable if and only if we have

$$d\omega^{h^*} \equiv 0 \qquad (\mathrm{mod}\ \omega^{h^*}),$$

which is equivalent to the condition

$$\Omega_{ji}{}^{h^*} = 0, \qquad \text{i.e.}$$
$$K_{jit}{}^h \eta^t = 0.$$

Thus we have

THEOREM 2.1. *A necessary and sufficient condition for a non-linear connection is integrable is*

$$K_{jit}{}^h \eta^t = 0.$$

(Cf. Kandatu [3]).

Let V be an arbitrary vector on $T(M)$ at a point $\sigma \in T(M)$. Then there exists uniquely a vector $'V$ at σ such that $'V$ belongs to the horizontal plane H_σ and satisfies

$$\omega^{h^*}(V) = \omega^{h^*}('V).$$

There exists uniquely a tangent vector $''V$ at σ such that it belongs to the tangent plane $T_\sigma(F_{\pi(\sigma)})$ of the fibre $F_{\pi(\sigma)}$ and satisfies

$$\omega^h(V)=\omega^h(''V).$$

The two vectors $'V$ and $''V$ are called the *horizontal* and *vertical parts* of V respectively. When we have $'V=V$, V is said to be *horizontal*. When we have $''V=V$, V is said to be *vertical*.

If V has components

$$V=V^iB_i+V^{i^*}C_{i^*},$$

then we find

$$'V=V^iB_i, \qquad ''V=V^{i^*}C_{i^*}. \tag{2.17}$$

These terminologies will be applied also to vector fields on $T(M)$.

Let Π be an arbitrary 1-form on $T(M)$. Then there exists uniquely a 1-form $'\Pi$ such that

$$'\Pi('V)=\Pi('V), \qquad '\Pi(''V)=0$$

for any vector field V. There exists uniquely a 1-form $''\Pi$ such that

$$''\Pi('V)=0, \qquad ''\Pi(''V)=\Pi(''V)$$

for any vector field V. The two 1-forms $'\Pi$ and $''\Pi$ are called respectively the *horizontal* and *vertical parts* of Π. When we have $'\Pi=\Pi$, Π is said to be *horizontal*. When we have $''\Pi=\Pi$, Π is said to be *vertical*. If Π has components

$$\Pi=\Pi_i\omega^i+\Pi_{i^*}\omega^{i^*},$$

then we find

$$'\Pi=\Pi_i\omega^i, \qquad ''\Pi=\Pi_{i^*}\omega^{i^*}.$$

These terminologies will be applied also to covectors on $T(M)$.

§ 3. Lift of vector field.

Let v be a vector field in M. If we define a vertical vector field $''v$ in $T(M)$ by

$$''v\cdot(df)=df(v)\circ\pi \tag{3.1}$$

for any differentiable function f on M (Cf. [9]), then $''v$ has components

$$(''v^a)=\begin{pmatrix}0\\ v^h\end{pmatrix} \qquad \text{or} \qquad ''v=v^hC_{h^*} \tag{3.2}$$

with respect to adapted frame, where $v^h=v^h(\xi)$ are the components of v. $''v$ is called the *vertical lift* of v, which *is determined independently of non-linear connection.* The vertical lift $''v$ has components $''v^A = v^\alpha A_\alpha{}^A$, i.e.

$$(''v^A) = \begin{pmatrix} 0 \\ v^h \end{pmatrix} \tag{3.3}$$

with respect to adapted coordinates (ξ^h, η^h) because of (3. 2).

Let v be a vector field in M. We define a vector field $\bar{v}$ on $T(M)$ by

$$\bar{v}(\omega) = \underset{v}{\mathcal{L}}\, \omega \tag{3.4}$$

for any 1-form ω in M, $\underset{v}{\mathcal{L}}$ denoting the Lie derivation with respect to v (Cf. [9]). The vector field v is called the *complete lift* of v and has components

$$(\bar{v}^\alpha) = \begin{pmatrix} v^h \\ \eta^t \hat{\nabla}_t v^h \end{pmatrix} \quad \text{or} \quad \bar{v} = v^h B_h + (\eta^t \hat{\nabla}_t v^h) C_{h*} \tag{3.5}$$

with respect to adapted frame, where $\hat{\nabla}_i v^h$ is defined by

$$\hat{\nabla}_i v^h = \partial_i v^h + \Gamma_j{}^h{}_i v^j, \tag{3.6}$$

v^h being the components of v. The complete lift $\bar{v}$ of v *is determined independently of non-linear connection.* The complete lift $\bar{v}$ has components $\bar{v}^A = v^\alpha A_\alpha{}^A$, i.e.

$$(\bar{v}^A) = \begin{pmatrix} v^h \\ \eta^i \partial_i v^h \end{pmatrix} \tag{3.7}$$

with respect to adapted coordinates (ξ^h, η^h) because of (3. 5).

Let v be an arbitrary vector field in M. Then there exists on $T(M)$ uniquely a horizontal vector field $'v$ satisfying $\pi('v) = v$, which has the components

$$('v^\alpha) = \begin{pmatrix} v^h \\ 0 \end{pmatrix} \quad \text{or} \quad 'v = v^h B_h \tag{3.9}$$

with respect to adapted frame, v^h being the components of v. We call $'v$ the *horizontal lift* of v. The horizontal lift $'v$ has components $'v^A = 'v^\alpha A_\alpha{}^A$, i.e.

$$('v^A) = \begin{pmatrix} v^h \\ -\Gamma_i{}^h v^i \end{pmatrix} \tag{3.10}$$

with respect to adapted coordinates (ξ^h, η^h).

Taking account of (2. 16), we have from (3. 2) and (3. 9).

PROPOSITION 3. 1. *If u and v are vector fields in M, then*

$$\begin{aligned} ['u, 'v] &= '[u, v] - (u^j v^i K_{jit}{}^h \eta^t) C_{h*}, \\ [''u, 'v] &= (u^j \nabla_j v^h) C_{h*}, \\ [''u, ''v] &= 0, \end{aligned} \tag{3.11}$$

where u^h and v^h are components of u and v respectively and $'[u, v]$ is the horizontal lift of $[u, v]$, $\nabla_j v^h$ being defined by

$$\nabla_j v^h = \partial_j v^h + \Gamma_j{}^h{}_i v^i. \tag{3.12}$$

§ 4. Almost complex structure in $T(M)$.

Let there be given a non-linear connection in $T(M)$. We shall define in $T(M)$ a tensor field F of type (1, 1) by

$$F'v = ''v, \qquad F''v = -'v \tag{4.1}$$

for any vector field v in M. Then F has components of the form

$$(F_\beta{}^\alpha) = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \tag{4.2}$$

or

$$F = F_\beta{}^\alpha A_\alpha \otimes A^\beta = -\sum_{h=1}^n B_h \otimes \omega^{h*} + \sum_{h=1}^n C_{h*} \otimes \omega^h \tag{4.3}$$

with respect to adapted frame, E being the $n \times n$ identity matrix. The equation (4.2) implies immediately

$$F^2 = -I,$$

I being the Kronecker's identity tensor field in $T(M)$. This means that F is an almost complex structure in $T(M)$ (Cf. Yano [7]). Thus we have.

THEOREM 4.1. *If there is given a non-linear connection in $T(M)$, then there exists an almost complex structure corresponding to the given non-linear connection.* (Cf. [1], [3], [5], [8]).

The Nijenhuis tensor N of the almost complex structure F is by definition

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] - [X, Y] \tag{4.4}$$

X and Y being arbitrary vector fields in $T(M)$. Thus, taking account of (2.16) and (4.4), we find

$$\begin{aligned} N(B_j, B_i) &= -T_{ji}{}^h B_h + K_{ji}{}^h C_{h*}, \\ N(B_j, C_{i*}) &= -K_{ji}{}^h B_h - T_{ji}{}^h C_{h*}, \\ N(C_{j*}, C_{i*}) &= T_{ji}{}^h B_h - K_{ji}{}^h C_{h*}, \end{aligned} \tag{4.5}$$

where we have put

$$K_{ji}{}^h = K_{jit}{}^h \eta^t, \qquad T_{ji}{}^h = \Gamma_j{}^h{}_i - \Gamma_i{}^h{}_j. \tag{4.6}$$

Thus we have

THEOREM 4.2. *A necessary and sufficient condition for the almost complex structure corresponding to a non-linear connection in* $T(M)$ *to be complex analytic is that the non-linear connection satisfy the conditions*

$$K_{ji}{}^{h}=0, \qquad T_{ji}{}^{h}=0.$$

(Cf. Kandatu [3]).

§5. Restricted tensor field in $T(M)$.

Let (ξ^h) be coordinates defined in a coordinate neighborhood U of M. Then the n vertical vector fields C_{i*} and n horizontal 1-form ω^h, which are defined by (2.12) and (1.5) respectively, are determined in $\pi^{-1}(U)$ and they are independent of non-linear connection. Let $(\xi^{h'})$ be coordinates defined in another coordinate neighborhood U' of M, and $C_{i*'}$ and $\omega^{h'}$ respectively the n vertical vector fields and the n horizontal 1-forms associated with $(\xi^{h'})$ in $\pi^{-1}(U')$. Then, by means of (1.10), we obtain

$$C_{i*}=\frac{\partial\xi^{i'}}{\partial\xi^{i}}C_{i*'}, \qquad \omega^{h}=\frac{\partial\xi^{h}}{\partial\xi^{h'}}\omega^{h'} \tag{5.1}$$

in $\pi^{-1}(U)\cap\pi^{-1}(U')$.

Let V and Π be respectively a vertical vector field and a horizontal 1-form. Then we have

$$V=V^{i*}C_{i*}, \qquad \Pi=\Pi_{h}\omega^{h} \qquad \text{in } \pi^{-1}(U) \tag{5.2}$$

and

$$V=V^{i*'}C_{i*'}, \qquad \Pi=\Pi_{h'}\omega^{h'} \qquad \text{in } \pi^{-1}(U'). \tag{5.3}$$

Taking account of (5.1), (5.2) and (5.3), we find in $\pi^{-1}(U)\cap\pi^{-1}(U')$

$$V^{i*}=\frac{\partial\xi^{i}}{\partial\xi^{i'}}V^{i*'}, \qquad \Pi_{h}=\frac{\partial\xi^{h'}}{\partial\xi^{h}}\Pi_{h'}, \tag{5.4}$$

which are the laws of transformations of vertical vector and horizontal covector respectively. Thus we call any vertical vector field a *restricted tensor field of type* (1, 0) and any horizontal covector field *a restricted tensor field of type* (0, 1) *in proper sense.*

Let T be a tensor field of type (1, 2) in $T(M)$ such that

$$T=T_{ji}{}^{h*}\omega^{j}\otimes\omega^{i}\otimes C_{h*} \tag{5.5}$$

in each $\pi^{-1}(U)$. Then we find

$$T_{ji}{}^{h*}=\frac{\partial\xi^{j'}}{\partial\xi^{j}}\frac{\partial\xi^{i'}}{\partial\xi^{i}}\frac{\partial\xi^{h}}{\partial\xi^{h'}}T_{j'i'}{}^{h*'} \tag{5.6}$$

in $\pi^{-1}(U)\cap\pi^{-1}(U')$, where $T=T_{j'i'}{}^{h*'}\omega^{j'}\otimes\omega^{i'}\otimes C_{h*'}$

in $\pi^{-1}(U')$. We call thus such a tensor field T a *restricted tensor field of type* (1, 2). Similarly, we can define restricted tensor field of any type. The notion of restricted tensor field in the present sense does not depend on non-linear connection.

In the next step, we shall extend the notion of restricted tensor field by making use of a non-linear connection. Let there be given a non-linear connection in $T(M)$. Then, by means of (1. 9), we find

$$B_i = \frac{\partial \xi^{i'}}{\partial \xi^i} B_{i'}, \qquad \omega^{h^*} = \frac{\partial \xi^h}{\partial \xi^{h'}} \omega^{h^{*\prime}} \tag{5. 7}$$

in $\pi^{-1}(U) \cap \pi^{-1}(U')$. Let V and Π be a horizontal vector field and a vertical 1-form respectively. Then we have

$$V = V^i B_i, \qquad \Pi = \Pi_{h^*} \omega^{h^*} \qquad \text{in } \pi^{-1}(U) \tag{5. 8}$$

and

$$V = V^{i'} B_{i'}, \quad \Pi = \Pi_{h^{*\prime}} \omega^{h^{*\prime}} \qquad \text{in } \pi^{-1}(U'). \tag{5. 9}$$

Thus we have in $\pi^{-1}(U) \cap \pi^{-1}(U')$

$$V^i = \frac{\partial \xi^i}{\partial \xi^{i'}} V^{i'}, \qquad \Pi_{h^*} = \frac{\partial \xi^{h'}}{\partial \xi^h} \Pi_{h^{*\prime}}, \tag{5. 10}$$

which are the laws of transformations of horizontal vector and vertical covector respectively. We call now any horizontal vector field a *restricted tensor field of type* (1, 0) *in extended sense* and any vertical 1-form a *restricted tensor field of type* (0, 1) *in extended sense*. We can now define *restricted tensor fields of any type in extended sense*. For example, the tensor field having in $\pi^{-1}(U)$ the form

$$T = T_{ji^*}{}^h \omega^j \otimes \omega^{i^*} \otimes B_h$$

is a restricted tensor field of type (1, 2) in extended sense.

If there is given, for example, a tensor field T of type (1, 1), then T is decomposed in four parts as follows:

$$T = T_\beta{}^\alpha A^\beta \otimes A_\alpha = T_j{}^i \omega^j \otimes B_i + T_j{}^{i^*} \omega^j \otimes C_{i^*} + T_{j^*}{}^i \omega^{j^*} \otimes B_i + T_{j^*}{}^{i^*} \omega^{j^*} \otimes C_{i^*},$$

where each of the four parts determines globally a tensor field in $T(M)$. The second part $T_j{}^{i^*} \omega^j \otimes C_{i^*}$ is a restricted tensor field in proper sense and the other three parts are restricted tensor fields in extended sense. These four parts are called the *restricted parts* of the given T. In a similar way, we can define restricted parts of any tensor field.

The vector field J defined in Proposition 1. 1 is a restricted tensor field in proper sense. The quantity $K_{ji}{}^h$ defined by (4. 6) determines a restricted tensor field K in extended sense, where we have put

$$K = K_{ji}{}^h \omega^j \otimes \omega^i \otimes C_{h^*}.$$

In fact, we have from (2. 10)

$$K_{ji}{}^{h}\omega^{j}\wedge\omega^{i}=d\omega^{h^*}+\Gamma_{j}{}^{h}{}_{i}\omega^{j}\wedge\omega^{i^*}$$

and, taking account of (1. 9), (1. 10) and (1. 11),

$$K_{ji}{}^{h}=\frac{\partial\xi^{j'}}{\partial\xi^{j}}\frac{\partial\xi^{i'}}{\partial\xi^{i}}\frac{\partial\xi^{h}}{\partial\xi^{h'}}K_{j'i'}{}^{h'}$$

in $\pi^{-1}(U)\cap\pi^{-1}(U')$.

Let $V=V^{h^*}C_{h^*}$ and $\Pi=\Pi_i\omega^i$ be respectively restricted tensor fields of type (1, 0) and of type (0, 1) in proper sense. Then, if we put

$$\nabla_j V^{h^*}=\partial_j V^{h^*}-\Gamma_j{}^s\partial_{s^*}V^{h^*}+\Gamma_j{}^h{}_s V^{s^*},\qquad \hat{\nabla}_j V^{h^*}=\partial_j V^{h^*}-\Gamma_j{}^s\partial_{s^*}V^{h^*}+\Gamma_s{}^h{}_j V^{s^*}$$

(5. 11)

$$\nabla_j\Pi_i=\partial_j\Pi_i-\Gamma_j{}^s\partial_{s^*}\Pi_i-\Gamma_j{}^t{}_i\Pi_t,$$

we can easily verify, taking account of (1. 8), (1. 11) and (5. 4), that $\nabla_j V^{h^*}$ and $\hat{\nabla}_j V^{h^*}$ defined above are components of a restricted tensor field of type (1, 1), and that $\nabla_j\Pi_i$ are components of a restricted tensor fields of type (0, 2). Therefore, if we define

$$\nabla V=\nabla_j V^{h^*}\omega^j\otimes C_{h^*},\qquad \nabla\Pi=\nabla_j\Pi_i\omega^j\otimes\omega^i, \tag{5. 12}$$

they are restricted tensor fields and called respectively the *covariant derivatives* of V and Π *with respect to the given non-linear connection.* The operations ∇_j and $\hat{\nabla}_j$ will be defined by (5. 11) for restricted vector fields and covector fields of any sense and will be extended to restricted tensor fields of every type as a derivation.

In the next step, we shall introduce another derivation operating on restricted tensor fields. Let $V=V^{h^*}C_{h^*}$ be a restricted tensor field of type (1, 0) and put

$$\nabla_{j^*}V^{h^*}=\partial_{j^*}V^{h^*}. \tag{5. 13}$$

Then $\nabla_{j^*}V^{h^*}$ are components of a restricted tensor field of type (1, 1), which will be denoted by

$$\dot{\nabla}V=\nabla_{j^*}V^{h^*}\omega^{j^*}\otimes C_{h^*}. \tag{5. 14}$$

We define similarly $\dot{\nabla}$ applied to any restricted tensor fields of type (0, 1) and extend the operation $\dot{\nabla}$ to restricted tensor fields of any type as a derivation.

Let T be a restricted tensor field, say, $T=T_{ji}{}^{h^*}\omega^j\otimes\omega^i\otimes C_{h^*}$ and put

$$\nabla_X T=(X^t\nabla_t T_{ji}{}^{h^*})\omega^j\otimes\omega^i\otimes C_{h^*},\qquad \nabla_Y T=(Y^{t^*}\nabla_{t^*}T_{ji}{}^{h^*})\omega^j\otimes\omega^i\otimes C_{h^*}, \tag{5. 15}$$

$X=X^hB_h$ and $Y=Y^{h^*}C_{h^*}$ being respectively horizontal and vertical vector fields, local or global.

§ 6. Almost analytic vector field in $T(M)$.

Let there be given a non-linear connection and V be an arbitrary vector field in $T(M)$. If we put $V=V^{\alpha}A_{\alpha}$ in $\pi^{-1}(U)$, we find in $\pi^{-1}(U)$, taking account of the relations (2. 16),

$$\underset{V}{\mathcal{L}} A_{\beta}=-(A_{\beta}\cdot V^{\alpha}+V^{\gamma}\Omega_{\gamma\beta}{}^{\alpha})A_{\alpha}, \tag{6. 1}$$

because of the identity

$$\underset{X}{\mathcal{L}} Y=[X, Y],$$

where $\underset{X}{\mathcal{L}}$ denotes the Lie derivation with respect to a vector field X (Cf. Yano [6]). Furthermore, taking account of (6. 1), we have in $\pi^{-1}(U)$

$$\underset{V}{\mathcal{L}} A^{\alpha}=(A_{\beta}\cdot V^{\alpha}+V^{\gamma}\Omega_{\gamma\beta}{}^{\alpha})A^{\beta}, \tag{6. 2}$$

because of the identity

$$(\underset{V}{\mathcal{L}} A^{\alpha})(A_{\beta})=-A^{\alpha}(\underset{V}{\mathcal{L}} A_{\beta}).$$

The almost complex structure F associated with the given non-linear connection is expressed as

$$F=F_{\beta}{}^{\alpha}A_{\alpha}\otimes A^{\beta},$$

where $F_{\beta}{}^{\alpha}$ is defined by (4. 2). Taking Lie derivatives of the both sides and taking account of (6. 1) and (6. 2), we find

$$\underset{V}{\mathcal{L}} F=[V^{\varepsilon}(A_{\varepsilon}\cdot F_{\beta}{}^{\alpha})-F_{\beta}{}^{\varepsilon}(A_{\varepsilon}\cdot V^{\alpha}+V^{\gamma}\Omega_{\gamma\varepsilon}{}^{\alpha})+F_{\delta}{}^{\alpha}(A_{\beta}\cdot V^{\delta}+V^{\gamma}\Omega_{\gamma\beta}{}^{\delta})]A_{\alpha}\otimes A^{\beta}$$

for any vector field $V=V^{\alpha}A_{\alpha}$. Thus $\underset{V}{\mathcal{L}} F$ has the following components

$$(\underset{V}{\mathcal{L}} F)_{\beta}{}^{\alpha}=-F_{\beta}{}^{\varepsilon}(A_{\varepsilon}\cdot V^{\alpha}+V^{\gamma}\Omega_{\gamma\varepsilon}{}^{\alpha})+F_{\delta}{}^{\alpha}(A_{\beta}\cdot V^{\delta}+V^{\gamma}\Omega_{\gamma\beta}{}^{\delta}) \tag{6. 3}$$

with respect to adapted frame because of $A_{\varepsilon}\cdot F_{\beta}{}^{\alpha}=0$. By means of (2. 7) and (2. 14), the equation (6. 3) reduces to

$$\begin{aligned}
&(\underset{V}{\mathcal{L}} F)_{i}{}^{h}=-\nabla_{i}V^{h^*}-\nabla_{i}V^{h}-V^{s}K_{si}{}^{h},\\
&(\underset{V}{\mathcal{L}} F)_{i}{}^{h^*}=\hat{\nabla}_{i}V^{h}-\nabla_{i^*}V^{h^*},\\
&(\underset{V}{\mathcal{L}} F)_{i^*}{}^{h}=\hat{\nabla}_{i}V^{h}-\nabla_{i^*}V^{h^*},\\
&(\underset{V}{\mathcal{L}} F)_{i^*}{}^{h^*}=\nabla_{i}V^{h^*}+\nabla_{i^*}V^{h}+V^{s}K_{si}{}^{h},
\end{aligned} \tag{6. 4}$$

where ∇_{i} is defined by (5. 11) and ∇_{i^*} is defined by (5. 13).

When V is horizontal, V^{h^*} being zero, (6. 4) reduces to

$$(\underset{V}{\mathcal{L}}F)_i{}^h=-\nabla_{i^*}V^h-V^sK_{si}{}^h,\qquad (\underset{V}{\mathcal{L}}F)_i{}^{h^*}=\hat{\nabla}_iV^h,$$
(6. 5)
$$(\underset{V}{\mathcal{L}}F)_{i^*}{}^h=\hat{\nabla}_iV^h,\qquad (\underset{V}{\mathcal{L}}F)_{i^*}{}^{h^*}=\nabla_{i^*}V^h+V^sK_{si}{}^h.$$

When V is vertical, V^h being zero, (6. 4) reduces to

$$(\underset{V}{\mathcal{L}}F)_i{}^h=-\nabla_iV^{h^*},\qquad (\underset{V}{\mathcal{L}}F)_i{}^{h^*}=-\nabla_{i^*}V^{h^*},$$
(6. 6)
$$(\underset{V}{\mathcal{L}}F)_{i^*}{}^h=-\nabla_{i^*}V^{h^*},\qquad (\underset{V}{\mathcal{L}}F)_{i^*}{}^{h^*}=\nabla_iV^{h^*}.$$

A vector field V in $T(M)$ is said to be *almost analytic* (Cf. Yano [7]) when we have $\underset{V}{\mathcal{L}}F=0$. Thus we have from (6. 5).

THEOREM 6. 1. *In a tangent bundle with a non-linear connection, a horizontal vector field V is almost analytic if and only if*

$$\hat{\nabla}_iV^h=0,\qquad \nabla_{i^*}V^h+V^sK_{si}{}^h=0.$$

(Cf. Kandatu [3]).

We also have from (6. 6).

THEOREM 6. 2. *In a tangent bundle $T(M)$ with a non-linear connection, a vertical vector field V is almost analytic if and only if V is the vertical lift ${}^{v}v$ of a vector field v in M such that*

$$\partial_iv^h+\Gamma_i{}^h{}_tv^t=0.$$

(Cf. Kandatu [3]).

If V is the complete lift of a vector field v in M, then we have from (3. 5) and (6. 4)

$$(\underset{V}{\mathcal{L}}F)_i{}^h=-(\underset{V}{\mathcal{L}}F)_{i^*}{}^{h^*}=(\underset{v}{\mathcal{L}}\Gamma_i{}^h{}_s)\eta^s,$$
(6. 7)
$$(\underset{V}{\mathcal{L}}F)_{i^*}{}^h=(\underset{V}{\mathcal{L}}F)_i{}^{h^*}=0,$$

where $\underset{v}{\mathcal{L}}\Gamma_j{}^h{}_i$ denotes the Lie derivative of $\Gamma_j{}^h{}_i$ with respect to v and is by definition (Cf. Yano [6])

(6. 8)
$$\underset{v}{\mathcal{L}}\Gamma_j{}^h{}_i=\nabla_j\hat{\nabla}_iv^h+v^sK_{sji}{}^h+T_{sji}{}^h\eta^t\nabla_tv^s,$$

the tensor field $T_{kji}{}^h$ being defined by

(6. 9)
$$T_{kji}{}^h=\partial_{k^*}\Gamma_j{}^h{}_i=\partial_{k^*}\partial_{i^*}\Gamma_j{}^h$$

and $\hat{\nabla}_iv^h$ being defined by (3. 6). Thus we have

THEOREM 6. 3. *In a tangent bundle $T(M)$ with a non-linear connection, the complete lift of a vector field v in M is almost analytic if and only if*

$$(\underset{v}{\mathcal{L}}\Gamma_j{}^h{}_s)\eta^s=0.$$

(Cf. Kandatu [3]).

When the given non-linear connection reduces to a linear connection, we have from Theorem 6. 2

THEOREM 6. 4. *In a tangent bundle $T(M)$ with a linear connection, a vertical vector field V is almost analytic if and only if V is the vertical lift $''v$ of a parallel vector field v in M.* (Cf. Yano and Davies [8] for Riemannian case).

We have also from Theorem 6. 3

THEOREM 6. 5. *In a tangent bundle $T(M)$ with a linear connection, the complete lift of a vector field v in M is almost analytic if and only if v is an infinitesimal affine motion.* (Cf. Yano and Davies [8] for Riemannian case).

In the tangent bundle of a Riemannian manifold there exists the Riemannian connection. Thus we have from Theorem 6. 1

THEOREM 6. 6. *In the tangent bundle $T(M)$ of a Riemannian manifold M, the horizontal lift $'v$ of a vector field v in M is almost analytic if and only if v is parallel in M.* (Yano and Davies [8]).

§7. Linear connection in $\boldsymbol{T}(\boldsymbol{T}(\boldsymbol{M}))$.

Let there be given a non-linear connection H in $T(M)$ and a linear connection Λ in $T(T(M))$. If we denote by $\Lambda_\gamma{}^\alpha{}_\beta$ the components of Λ with respect to adapted frame constructed in $\pi^{-1}(U)$ by making use of the given non-linear connection H, then the covariant derivative of a vector field V in $T(M)$ is given by

$$D_\gamma V^\alpha = A_\gamma \cdot V^\alpha + \Lambda_\gamma{}^\alpha{}_\beta V^\beta \tag{7. 1}$$

in adapted frame, V^α being the components of V with respect to adapted frame. We have already obtained in (5. 1) and (5. 7)

$$A_\beta = a_\beta^{\beta'} A_{\beta'}, \qquad A^\alpha = a_{\alpha'}^\alpha A^{\alpha'} \tag{7. 2}$$

in $\pi^{-1}(U) \cap \pi^{-1}(U')$, where

$$(a_{\alpha'}^\alpha) = \begin{pmatrix} \dfrac{\partial \xi^h}{\partial \xi^{h'}} & 0 \\ 0 & \dfrac{\partial \xi^h}{\partial \xi^{h'}} \end{pmatrix} \tag{7. 3}$$

and

$$(a_\alpha^{\alpha'}) = (a_{\alpha'}^\alpha)^{-1}.$$

Thus, taking account of (7. 1) and (7. 2), we get

$$a_{\alpha'}^\alpha \Lambda_{\gamma'}{}^{\alpha'}{}_{\beta'} = \Lambda_\gamma{}^\alpha{}_\beta a_{\gamma'}^\gamma a_{\beta'}^\beta + A_{\gamma'} a_{\beta'}^\alpha \tag{7. 4}$$

in $\pi^{-1}(U)\cap\pi^{-1}(U')$, which is the law of transformation of $\Lambda_\gamma{}^\alpha{}_\beta$. Therefore, if we take account of (7.3) and (7.4), we have

PROPOSITION 7.1. *The following subsets of the components $(\Lambda_\gamma{}^\alpha{}_\beta)$ of a connection Λ given in $T(T(M))$ have the tensor property:*

$$(\Lambda_{j^*}{}^h{}_i),\quad (\Lambda_{j^*}{}^h{}_{i^*}),\quad (\Lambda_{j^*}{}^{h^*}{}_i),\quad (\Lambda_{j^*}{}^{h^*}{}_{i^*}),\quad (\Lambda_j{}^h{}_{i^*}),\quad (\Lambda_j{}^{h^*}{}_i).$$

The covariant derivative of the almost complex structure F associated with the given non-linear connection is given by

$$D_\gamma F_\beta{}^\alpha = A_\gamma\cdot F_\beta{}^\alpha - \Lambda_\gamma{}^\alpha{}_\delta F_\beta{}^\delta + \Lambda_\gamma{}^\varepsilon{}_\beta F_\varepsilon{}^\alpha,$$

which reduces to

$$D_\gamma F_\beta{}^\alpha = -\Lambda_\gamma{}^\alpha{}_\delta F_\beta{}^\delta + \Lambda_\gamma{}^\varepsilon{}_\beta F_\varepsilon{}^\alpha \tag{7.5}$$

because $F_\beta{}^\alpha$ are constants given by (4.2). When the connection Λ has the property $D_\gamma F_\beta{}^\alpha=0$, Λ is called an *F-connection.* Thus we have from (7.5)

PROPOSITION 7.2. *A connection Λ is an F-connection if and only if*

$$\begin{aligned}&\Lambda_\gamma{}^h{}_{i^*}+\Lambda_\gamma{}^{h^*}{}_i=0,\\ &\Lambda_\gamma{}^h{}_i-\Lambda_\gamma{}^{h^*}{}_{i^*}=0\end{aligned}\tag{7.6}$$

with respect to adapted frame.

Taking account of Propositions 7.1 and 7.2, we see that there exists an F-connection $\mathring{\Lambda}$ in $T(T(M))$, or, by the abuse of terminology, in $T(M)$ which has zero components except

$$\mathring{\Lambda}_j{}^h{}_i=\mathring{\Lambda}_j{}^{h^*}{}_{i^*}=\Gamma_j{}^h{}_i \tag{7.7}$$

with respect to adapted frame, where $\Gamma_j{}^h{}_i$ is defined by

$$\Gamma_j{}^h{}_i=\partial_{i^*}\Gamma_j{}^h,$$

$\Gamma_j{}^h$ being the components of a given non-linear connection H. Denoting by $\mathring{D}$ the covariant derivation with respect to $\mathring{\Lambda}$, we find

$$\mathring{D}_jU^h=\nabla_jU^h,\qquad \mathring{D}_jU^{h^*}=0,$$
$$\mathring{D}_{j^*}U^h=\nabla_{j^*}U^h,\qquad \mathring{D}_{j^*}U^{h^*}=0$$

for any horizontal vector field $U=U^hB_h$ and

$$\mathring{D}_jV^h=0,\qquad \mathring{D}_jV^{h^*}=\nabla_jV^{h^*},$$
$$\mathring{D}_{j^*}V^h=0,\qquad \mathring{D}_{j^*}V^{h^*}=\nabla_{j^*}V^{h^*}$$

for any vertical vector field $V=V^{h^*}C_{h^*}$, where ∇_j and ∇_{j^*} are defined respectively by (5.11) and (5.13). Thus we have

THEOREM 7.1. *If there is given a non-linear connection in $T(M)$, then there exists uniquely an F-connection $\mathring{\Lambda}$ in $T(T(M))$ satisfying*

$$\mathring{D}_X U=\nabla_X U, \qquad \mathring{D}_Y U=\nabla_Y U,$$

$$\mathring{D}_X V=\nabla_X V, \qquad \mathring{D}_Y V=\nabla_Y V$$

for any horizontal vector fields U, X and any vertical vector fields V, Y.

THEOREM 7.2. *A horizontal vector field is parallel with respect to the connection $\mathring{\Lambda}$ if and only if it is the horizontal lift of a vector field v^h in M such that $\partial_j v_h+\Gamma_j{}^h{}_i v^i=0$.*

A vertical vector field is parallel with respect to the connection $\mathring{\Lambda}$ if and only if it is the vertical lift of a vector field v^h in M such that $\partial_j v^h+\Gamma_j{}^h{}_i v^i=0$.

We have now from Theorems 6.2 and 7.2

THEOREM 7.3. *In a tangent bundle $T(M)$ with a non-linear connection, a vertical vector field is almost analytic if and only if it is parallel with respect to the connection $\mathring{\Lambda}$ and is the vertical lift of a vector field in M.*

On putting

$$\Theta_\beta{}^\alpha=\mathring{\Lambda}_\gamma{}^\alpha{}_\beta A^\gamma \tag{7.8}$$

we have

$$d\Theta_\beta{}^\alpha-\Theta_\beta{}^\varepsilon\Lambda\Theta_\varepsilon{}^\alpha=\mathring{K}_{\delta\gamma\beta}{}^\alpha A^\delta\wedge A^\gamma, \tag{7.9}$$

where $\mathring{K}_{\delta\gamma\beta}{}^\alpha$ are the components of the curvature tensor $\mathring{K}$ of the connection $\mathring{\Lambda}$ with respect to adapted frame. If we take account of (2.6) and (2.7), we see from (7.8) and (7.9) that $\mathring{K}$ has components all zero except

$$\begin{aligned}&\mathring{K}_{kji}{}^h=-\mathring{K}_{kji^*}{}^{h^*}=K_{kji}{}^h,\\ &\mathring{K}_{kj^*i}{}^h=\mathring{K}_{kj^*i^*}{}^{h^*}=-\partial_{j^*}\partial_{i^*}\Gamma_k{}^h,\end{aligned} \tag{7.10}$$

where $\mathring{K}_{kji}{}^h$ are defined by (2.8). Thus we have from (7.10)

THEOREM 7.4. *The connection $\mathring{\Lambda}$ is of zero curvature if and only if the given non-linear connection reduces to a linear connection of zero curvature.*

As is well known, we have

$$dA^\alpha-\Theta_\beta{}^\alpha\wedge A^\beta=\mathring{S}_{\delta\gamma}{}^\alpha A^\delta\wedge A^\gamma, \tag{7.11}$$

where $\mathring{S}_{\delta\gamma}{}^{\alpha}$ are the components of the torsion tensor $\mathring{S}$ of the connection $\mathring{\Lambda}$ with respect to adapted frame. Taking account of (2. 6) and (2. 7), we have from (7. 11)

$$\mathring{S}_{ji}{}^{h}=\mathring{S}_{ji^*}{}^{h^*}=\Gamma_j{}^h{}_i-\Gamma_i{}^h{}_j, \qquad \mathring{S}_{ji}{}^{h}=K_{ji}{}^{h}, \tag{7. 12}$$

the other $\mathring{S}_{\delta\gamma}{}^{\alpha}$ being all zero. Thus, taking account of Theorem 7. 4, we have

THEOREM 7. 5. *The connection* $\mathring{\Lambda}$ *is locally flat* ($K=0$, $S=0$) *if and only if the given non-linear connection reduces to a symmetric linear connection which is of zero curvature.*

Next, taking account of Theorem 4. 1, we have

THEOREM 7. 6. *The connection* $\mathring{\Lambda}$ *is symmetric if and only if the almost complex structure associated with the given non-linear connection is complex analytic.*

Let there be given a linear connection Λ in $T(T(M))$. Then, applying the formula (7. 11) to Λ, we have

$$\begin{aligned} &\Lambda_\delta{}^h{}_\gamma-\Lambda_\gamma{}^h{}_\delta=S_{\delta\gamma}{}^h, && \Lambda_j{}^{h^*}{}_i-\Lambda_i{}^{h^*}{}_j=-K_{ji}{}^h+S_{ji}{}^{h^*}, \\ &\Lambda_j{}^{h^*}{}_{i^*}-\Lambda_{i^*}{}^{h^*}{}_j=\Gamma_i{}^h{}_j+S_{ji^*}{}^{h^*}, && \Lambda_{j^*}{}^{h^*}{}_{i^*}-\Lambda_{i^*}{}^{h^*}{}_{j^*}=S_{j^*i^*}{}^{h^*}, \end{aligned} \tag{7. 12}$$

where $S_{\delta\gamma}{}^{\alpha}$ are components of the torsion tensor S of Λ. (Cf. Yano and Ledger [9]).

We now proceed to the consideration of other particular cases.

PROPOSITION 7. 3. *A linear connection* Λ *in* $T(T(M))$ *satisfies the following two conditions* (a) *and* (b) *if and only if we have*

$$\Lambda_{j^*}{}^{h^*}{}_i=0, \qquad \Lambda_{j^*}{}^{h}{}_{i^*}=0, \qquad \Lambda_{j^*}{}^{h^*}{}_{i^*}=0. \tag{7. 13}$$

(a) *Each fibre is totally geodesic and every path in each fibre is expressed by linear equations* $\eta^h=a^ht+b^h$, a^h *and* b^h *being constant and* t *the affine parameter.*

(b) *The horizontal plane field is parallel along each fibre.*

Taking account of (7. 12), we have

PROPOSITION 7. 4. *A symmetric linear connection* Λ *in* $T(T(M))$ *satisfies the conditions* (a) *and* (b) *mentioned in Proposition* 7. 3 *if and only if we have*

$$\begin{aligned} &\Lambda_{j^*}{}^{h^*}{}_{i^*}=0, && \Lambda_{j^*}{}^{h}{}_{i^*}=0, && \Lambda_{j^*}{}^{h^*}{}_{i}=0, \\ & && \Lambda_j{}^{h^*}{}_{i^*}=\Gamma_i{}^h{}_j, && \\ &\Lambda_j{}^h{}_i-\Lambda_i{}^h{}_j=0, && \Lambda_j{}^{h^*}{}_i-\Lambda_i{}^{h^*}{}_j=0, && \Lambda_j{}^h{}_{i^*}-\Lambda_{i^*}{}^h{}_j=0. \end{aligned} \tag{7. 14}$$

The condition $\Lambda_j{}^{h^*}{}_{i^*}=\Gamma_j{}^h{}_i$ *appearing above is equivalent to the fact that*

$$D_X V=\hat{\nabla}_X V \tag{7.15}$$

for any vertical vector field V *and any horizontal vector field* X.

PROPOSITION 7.5. *A symmetric linear connection in* $T(T(M))$ *satisfies the following condition* (c) *if and only if we have*

$$\Lambda_j{}^h{}_i=\frac{1}{2}(\Gamma_j{}^h{}_i+\Gamma_i{}^h{}_j).$$

(c) *For any horizontal vector field* V *we have*

$$'(D_V V)=\nabla_V V, \tag{7.16}$$

where the left hand side denotes the horizontal part of $D_V V$.

§8. Lift of curve.

Let C be a curve defined in M by equations $\xi^h=\xi^h(t)$ and $\nu^h(t)$ a vector field along C. If there is given a non-linear connection in $T(M)$, then we get in $T(M)$ a curve $\bar{C}$ defined by equations $\xi^A=\xi^A(t)$ of the form

$$\xi^h=\xi^h(t), \qquad \xi^{h^*}=\nu^h(t), \tag{8.1}$$

which is called the *lift* of the curve C with *hights* $\nu^h(t)$. If a curve $\bar{C}$ defined by (8.1) satisfies, at all its points, the relation

$$\omega^{h^*}\left(\frac{d\xi^h}{dt}\right)=0, \tag{8.2}$$

the curve C is said to be *horizontal*. The equation (8.2) is written as follows:

$$\frac{\delta\nu^h}{dt}=0, \tag{8.3}$$

where we have put

$$\frac{\delta\nu^h}{dt}=\frac{d\nu^h}{dt}+\Gamma_j{}^h(\xi,\nu)\frac{d\xi^j}{dt}=0. \tag{8.4}$$

When a lift $\bar{C}$ of a curve C in M is horizontal, $\bar{C}$ is called the *horizontal lift* of C. When ν^h is the tangent vector $d\xi^h/dt$ to the given curve C in M, the lift is called the *natural lift* of C and denoted by $\tilde{C}$.

If there is given a symmetric linear connection Λ in $T(T(M))$, the differential equations of a path $\xi^A=\xi^A(t)$, i.e. of a path $\xi^h=\xi^h(t)$, $\xi^{h^*}=\nu^h(t)$ are given by

$$\frac{d}{dt}\omega^{\alpha}\left(\frac{d\xi}{dt}\right)+\Lambda_{\gamma}{}^{\alpha}{}_{\beta}\omega^{\gamma}\left(\frac{d\xi}{dt}\right)\omega^{\beta}\left(\frac{d\xi}{dt}\right)=0 \tag{8. 5}$$

with respect to adapted frame.

If the symmetric linear connection Λ is supposed to satisfy the conditions (a) and (b) mentioned in Proposition 7. 3, we find from (1. 2) and (1. 5), taking account of Proposition 7. 4, that the equation (8. 5) becomes

$$\begin{aligned}&\left\{\frac{d^2\xi^h}{dt^2}+\Lambda_j{}^h{}_i(\xi,\nu)\frac{d\xi^{j'}}{dt}\frac{d\xi^i}{dt}\right\}+2\Lambda_j{}^h{}_{i*}(\xi,\nu)\frac{d\xi^j}{dt}\frac{\delta\nu^i}{dt}=0,\\&\left\{\frac{d}{dt}\left(\frac{\delta\nu^h}{dt}\right)+\Gamma_j{}^h{}_i(\xi,\nu)\frac{d\xi^j}{dt}\frac{\delta\nu^i}{dt}\right\}+\Lambda_j{}^{h^*}{}_i(\xi,\nu)\frac{d\xi^j}{dt}\frac{d\xi^i}{dt}=0.\end{aligned} \tag{8. 6}$$

If a curve C defined in M by $\xi^h=\xi^h(t)$ satisfies the differential equation

$$\frac{\delta}{dt}\left(\frac{d\xi^h}{dt}\right)=\frac{d^2\xi^h}{dt^2}+\Gamma_j{}^h\left(\xi,\frac{d\xi}{dt}\right)\frac{d\xi^j}{dt}=0, \tag{8. 7}$$

the curve C is called a *generalized path* in M. Thus we have.

PROPOSITION 8. 1. *Let there be given a symmetric linear connection Λ in $T(T(M)$ satisfying the conditions* (a), (b), (c) *mentioned in Propositions* 7. 4 *and* 7. 5, *a non-linear connection being given in $T(M)$. The natural lift of a generalized path $\xi^h=\xi^h(t)$ in M is a path in $T(M)$ if and only if we have*

$$\Lambda_j{}^{h^*}{}_{i^*}\left(\xi,\frac{d\xi}{dt}\right)\frac{d\xi^j}{dt}\frac{d\xi^i}{dt}=0. \tag{8. 8}$$

If the natural lift of a generalized path having the properties (8. 8) *is horizontal, then the natural lift is a path in $T(M)$.*

We have finally

THEOREM 8. 1. *Let there be given a symmetric linear connection Λ in $T(T(M))$ satisfying the conditions* (a) *and* (b) *mentioned in Proposition* 7. 4, *a non-linear connection being given in $T(M)$. Any horizontal lift of an arbitrary generalized path in M is a path in $T(M)$ if and only if the given non-linear connection reduces to a linear connection and Λ satisfies the condition* (c) *mentioned in Proposition* 7. 5 *and the following condition:*

$$\Lambda_j{}^{h^*}{}_i=-\Lambda_i{}^{h^*}{}_j=-\frac{1}{2}K_{jit}{}^h\eta^t.$$

(Cf. Yano and Davies [8]).

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