ON THE AUTOMORPHISM GROUPS OF f-MANIFOLDS

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Introduction.

In 1933, H. Cartan [4][1] proved that the group of all complex analytic transformations of a bounded domain in C^n is a Lie transformation group. As a matter of fact, the group of differentiable transformations on a differentiable manifold leaving a certain geometric structure invariant is often a Lie transformation group. The problem has been studied by many authors. Recently, Chu and Kobayashi [5] have summarized these known results in the chronological order and given systematic proofs. On the other hand, Ruh [11] has obtained a condition under which the group of differentiable transformations leaving a G-structure invariant on a compact differentiable manifold is a Lie transformation group.

The purpose of the present paper is to prove that the automorphism group of a compact f-manifold of some kind is a Lie transformation group (Theorem in § 2). We shall give the proof in § 4.

§ 1. (f, g)-manifolds.

Let V be an n-dimensional connected differentiable manifold of class C^∞. If there exists a non-null tensor field f of type (1, 1) and of class C^∞ satisfying

$$f^3+f=0, \tag{1.1}$$

and if the rank of f is constant everywhere and is equal to r, then we call such a structure an *f-structure of rank r* (Yano [14]). We call a differentiable manifold admitting an f-structure an *f-manifold*. We put

$$l=-f^2, \qquad m=f^2+1, \tag{1.2}$$

where 1 denotes the unit tensor, then we have

$$l+m=1, \qquad l^2=l, \qquad m^2=m, \qquad lm=ml=0. \tag{1.3}$$

These equations mean that the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators. Thus, there exist in the manifold complementary distributions L and M corresponding to the projection operators l and m respectively. When the rank of f is equal to r, L is r-dimensional and M is $(n-r)$-dimensional.

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1) Numbers in brackets refer to the bibliography at the end of the paper.

Let $f_i{}^j$ [2)] be components of an f-structure of rank r and $m_i{}^j$ those of the complementary projection operator m. Then it is known [14] that we can introduce, in a manifold admitting an f-structure of rank r, a positive definite Riemannian metric tensor g, say g_{ji}, satisfying

$$g_{rs}f_j{}^r f_i{}^s + m_{ji} = g_{ji}, \tag{1. 4}$$

where $m_{ji} = m_j{}^r g_{ri}$. If an f-manifold admits a positive definite Riemannian metric tensor satisfying (1. 4), then the structure is called an *(f, g)-structure* and the manifold an *(f, g)-manifold.*

In an (f, g)-manifold V, we put

$$\omega(X, Y) = g(mX, mY) \tag{1. 5}$$

for any vector fields X and Y in $\mathfrak{X}(V)$, where $\mathfrak{X}(V)$ is a Lie algebra of vector fields on V.

§ 2. Automorphisms of (f, g)-manifolds.

Let V and $\bar{V}$ be n-dimensional (f, g)-manifolds and let (f, g) and $(\bar{f}, \bar{g})$ be their (f, g)-structures respectively. A diffeomorphism h of V onto $\bar{V}$ is called an *isomorphism* of V onto $\bar{V}$ if the following conditions are satisfied:

$$dh \circ f = \bar{f} \circ dh \tag{2. 1}$$

and

$$\delta h \bar{\omega} = \omega, \tag{2. 2}$$

where dh and δh are respectively the differential mapping of h and the dual mapping of dh. Moreover, if $V = \bar{V}$ and $(f, g) = (\bar{f}, \bar{g})$, then an isomorphism h is called an *automorphism* of V. The set of all automorphisms of V forms a group of transformations on V, which will be denoted by $A(f, \omega)$. We state the main theorem, which will be proved in the last section.

THEOREM. *The automorphism group $A(f, \omega)$ of a compact (f, g)-manifold is a Lie transformation group with respect to the topology of uniform convergence of functions together with the partial derivations through the third order.*

Let there be given an f-manifold and denote by f its f-structure. Then the set of all tangent vectors belonging to the distribution M determined by the projection tensor m has a vector bundle structure, which will be denote by $M(V)$. As is well known, there exists a metric tensor $\tilde{\omega}$ in $M(V)$, that is, a real-valued bilinear mapping $\tilde{\omega}$ of $\mathfrak{X}(M(V)) \times \mathfrak{X}(M(V))$ such that

$$\tilde{\omega}(X, X) \geqq 0 \qquad \text{for } X \in \mathfrak{X}(M(V))$$

and $\tilde{\omega}(X, X) = 0$ for $X \in \mathfrak{X}(M(V))$ if and only if $X = 0$, where $\mathfrak{X}(M(V))$ denotes the vector space consisting of all cross-sections of $M(V)$ over the ring $\mathfrak{F}(M)$ of all

2) The indices $i, j, \cdots$ run over the range $1, 2, \cdots, n$.

differentiable functions on V.

We suppose now that there is given a metric tensor $\tilde{\omega}$ in $M(V)$. Then it is easily verified that there exists a Riemannian metric tensor g satisfying

$$g(mX, mY)=\tilde{\omega}(mX, mY)$$

for any elements X and Y of $\mathfrak{X}(V)$. If we denote now by $A(f, \tilde{\omega})$ the group of all transformations on V, which preserve the f-structure f and the metric tensor $\tilde{\omega}$ on $M(V)$, we have the following corollary to the main theorem.

COROLLARY. *If there is given a metric tensor $\tilde{w}$ on the vector bundle $M(V)$ over a compact f-manifold, then the group $A(f, \tilde{\omega})$ is a Lie transformation group with respect to the topology of uniform convergence of functions together with the partial derivations through the third order.*

An f-structure reduces to an almost complex structure, if its rank r is equal to the dimension n of the manifold V (Yano [14]). In such a case, the vector bundle $M(V)$ is trivially null. Thus, from the corollary above we have a theorem due to Boothby-Kobayashi-Wang [3], roughly speaking, that the automorphism group of a compact almost complex manifold is a Lie group.

An almost contact structure (Sasaki [12]) is defined by a triple (f, ξ, η) of a tensor field f of type $(1, 1)$, a vector field ξ and the covector field η such that

$$\begin{cases} f^2=-1+\xi\otimes\eta, \\ f(\xi)=0, \qquad \eta\circ f=0, \\ \eta(\xi)=1, \end{cases} \tag{2. 1}$$

where the first equation of (2. 1) means

$$f(fX)=-X+\eta(X)\xi$$

for any vector field X, which implies

$$f^3+f=0,$$

i.e. that f is an f-structure of rank $n-1$, the manifold being n-dimensional. In this case, the manifold is necessarily orientable. Conversely, it is well known that any orientable manifold with an f-structure of rank $n-1$ admits an almost contact structure (Yano [14]). If there is given an almost contact structure (f, ξ, η) in an n-dimensional manifold V, then we can define a metric tensor $\tilde{\omega}$ by

$$\tilde{\omega}(\xi, \xi)=1$$

in the vector bundle $M(V)$ consisting of all tangent vectors belonging to the distribution M determined by the projection operator $m=1+f^2=\xi\otimes\eta$. Thus, if we denote by $A(f, \xi)$ the automorphism group, i.e. the group of all transformations on V leaving f and ξ invariant, then $A(f, \xi)$ is nothing but the automorphism group $A(f, \tilde{w})$. Therefore, we have from the corollary above

COROLLARY. *The automorphism group $A(f, \xi)$ on a compact almost contact* *manifold is a Lie transformation group with respect to the topology of uniform convergence of functions together with the partial derivations through the third order.*

The torsion tensor N defined in [13] for the almost contact structure is given by

$$N(X, Y)=[X, Y]+f[fX, Y]+f[X, fY]-[fX, fY]$$
$$-\eta([X, Y])\xi-d\eta(X, Y)\xi.$$

When the torsion tensor vanishes identically, the almost contact structure is said to be *normal*. Morimoto [7] has proved the fact that the automorphism group $A(f, \xi)$ on a compact normal almost contact manifold is a Lie transformation group with respect to the compact open topology. Morimoto and Tanno [8] have also announced the corollary above without proof, however the present proof seems to be different from Morimoto and Tanno's.

§ 3. On elliptic differential equations.

In this section, we give the lemma concerning the elliptic partial differential equations, which is used in the proof of the main theorem. Let D be a bounded domain and let

$$a^{ji}(x)\frac{\partial^2 X^h}{\partial x^j \partial x^i}+F^h(x^j, X^i, \partial X^k/\partial x^l)=0 \tag{3. 1}$$

be a system of linear partial differential equations in n independent variables $x^1, \cdots, x^n$ and n unknown functions $X^1, \cdots, X^n$. In our case, the theorem due to Douglis-Nirenberg [6] reduces to

LEMMA 1. *In a system of partial differential equations* (3. 1), *we make the following assumptions*:

(1) *there exists a positive number* K *such that* $a^{ji}(x)\rho_j\rho_i \geqq K(\rho_1^2+\cdots+\rho_n^2)$ *for all* x *in* D *and all real numbers* $\rho_1, \cdots, \rho_n$,

(2) $a^{ji}(x)$ *is symmetric in* j *and* i *and differentiable in* D *and there exists a constant* C_1 *such that*

$$\left|\frac{\partial^k a^{ji}}{\partial x^{i_1}\cdots\partial x^{i_k}}\right| \leqq C_1 \qquad \textit{for } k=0, 1, 2;$$

(3) $X=(X^1, \cdots, X^n)$ *being a solution of* (3. 1) *in* D, *there exists a constant* C_2 *such that*

$$\left|\frac{\partial^k X^h}{\partial x^{i_1}\cdots\partial x^{i_k}}\right| \leqq C_2 \qquad \textit{for } k=0, 1, 2.$$

Then for any compact subset F *in* D *there exists a constant* C *depending only on* C_1, C_2 *and* K *such that*

$$\left|\frac{\partial^3 X^h(y)}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}}-\frac{\partial^3 X^h(z)}{\partial x^{i_1}\partial x^{i_2}\partial x^{i_3}}\right| \leqq C|y-z|^h,$$

where y *and* z *are arbitrary points in* F.

Let $\Omega=\{X(y)\}$ be a family of all solutions of (3. 1) satisfying the conditions in Lemma 1 with C fixed. Then it follows from Lemma 1 that the family Ω of functions and their partial derivatives through the third order is bounded and equicontinuous in F. Making use of Arzela's theorem, we see that every sequence in Ω has a subsequence which is convergent with respect to the topology of uniform convergence of functions together with their partial derivatives through the third order. This means that the family Ω is relatively compact in the space of all solutions in (3. 1) over D with respect to the topology above.

Now, let V be a compact differentiable manifold and let S be a vector space of infinitesimal transformations X such that, for every point in V there is a system (3. 1) of partial differential equations defined in a neighbourhood of that point and satisfied by all X in S. Moreover we assume that an infinitesimal transformation X in S satisfies the condition in Lemma 1. An infinitesimal transformation X in Ω is given by

$$X=X^j\partial/\partial x^j$$

in local coordinates. By choosing a Riemannian metric tensor, we define the norm of X to be

$$\|X\|=\operatorname*{Max}_{p\in V}|X|+\operatorname*{Max}_{p\in V}|\nabla X|+\operatorname*{Max}_{p\in V}|\nabla^2 X|+\operatorname*{Max}_{p\in V}|\nabla^3 X|,$$

where ∇ denotes the covariant derivative with respect to the Riemannian connection and $|\ \ |$ denotes the norm obtained by extending the Riemannian metric.

We see that Ω is a Banach space with the norm $\|\ \ \|$. The Banach space Ω is locally compact [2], since convergence in the norm $\|\ \ \|$ is equivalent to uniform convergence of functions together with their partial derivatives through the third order. As is well known [1], Ω is finite dimensional, because it is locally compact. Thus we find

LEMMA 2. *The vector space Ω is finite dimensional.* (See for example Ruh [11].)

§4. Proof of the main theorem.

We now state a well known theorem due to Palais concerning the Lie transformation group:

THEOREM (Palais [10]). *Let G be a certain group of differentiable transformations on a differentiable manifold V. Let $\mathfrak{G}'$ be the set of all vector fields X on V which generate a global 1-parameter group of transformations which belong to the given group G. Let $\mathfrak{G}$ be the Lie subalgebra of the Lie algebra $\mathfrak{X}(V)$ generated by $\mathfrak{G}'$. If $\mathfrak{G}$ is finite dimensional, then G is a Lie transformation group.*

Making use of this theorem, we shall prove the main theorem. Let $\Phi(f,\omega)$ be the set of all infinitesimal transformations X on V such that

$$L_X f_i{}^j=0,\qquad L_X m_{ji}=0, \tag{4. 1}$$

where L_X denotes the Lie derivative with respect to X. The set $\Phi(f,\omega)$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(V)$. *Since V is compact, any infinitesimal transformation* X in $\Phi(f, \omega)$ is complete. Hence X generates a global 1-parameter group of transformations $\phi_t(-\infty<t<\infty)$ of V. Moreover, it follows from the definition of the Lie derivative that ϕ_t is an automorphism in $A(f, \omega)$. Accordingly, by virtue of the theorem due to Palais, in order to prove our theorem stated in § 2 it suffices to show that the Lie subalgebra $\Phi(f, \omega)$ is finite dimensional. Subsequently we *proceed to show that $\Phi(f, \omega)$ is finite dimensional.*

For any infinitesimal transformation X in $\Phi(f, \omega)$, we get

$$0=g^{ir}L_X f_r{}^j=X^r\nabla_r f^{ij}-f^{ir}\nabla_r X^j+f_r{}^j\nabla^i X^r,$$

where $f_i{}^j$ are components of an f-structure f and $f^{ji}=g^{jr}f_r{}^i$, $\nabla^i=g^{ir}\nabla_r$. Differentiating this equation covariantly, we get

$$\nabla_k X^s\cdot\nabla_s f^{ij}+X^s\nabla_k\nabla_s f^{ij}-\nabla_k f^{is}\cdot\nabla_s X^j-f^{is}\nabla_k\nabla_s X^j+\nabla_k f_s{}^j\cdot\nabla^i X^s+f_s{}^j\nabla_k\nabla^i X^s=0.$$

Operating $f_j{}^h$ to the equation above and then contracting with respect to k and i, we get

$$P^h-m_r{}^h P^r-f_t{}^h\nabla^r X^s(\nabla_r f_s{}^t+\nabla_s f_r{}^t)-f_t{}^h L_X f^t=0, \tag{4. 2}$$

where $P^h=g^{ji}L_X\{{}^{\,h}_{ji}\}$ and $f^j=\nabla_r f^{rj}$. On the other hand, we have $L_X m_{ji}=0$, from which by making use of the formula of the Lie derivative we get

$$L_X\nabla_j m_{ih}=-t_{ji}^r m_{rh}-t_{jh}^r m_{ir},$$

where $t_{ji}^h=L_X\{{}^{\,h}_{ji}\}$. Taking account of the equation above and of the fact that m_{ji} is symmetric in j and i, we have

$$L_X\nabla_j m_{ih}-L_X\nabla_i m_{hj}+L_X\nabla_h m_{ji}=-2t_{jh}^r m_{ir}.$$

Transvecting this with g^{jh}, we get

$$P^r m_{ir}=-\frac{1}{2}g^{jh}L_X m_{jih}, \tag{4. 3}$$

where $m_{jih}=\nabla_j m_{ih}-\nabla_i m_{hj}+\nabla_h m_{ji}$. Substituting (4. 3) into (4. 2), we get

$$H^{rs}\nabla_r\nabla_s X^h+H_{rs}{}^h\nabla^r X^s+H_r{}^h X^r=0, \tag{4. 4}$$

where

$$\begin{cases} H^{rs}=g^{rs}, \\ H_{rs}{}^h=g^{ht}m_{rts}+\dfrac{1}{2}\delta_r^h g^{tu}m_{tsu}-f_t{}^h(\nabla_r f_s{}^t+\nabla_s f_r{}^t)+f_s{}^h f_r, \\ H_r{}^h=K_r{}^h+\dfrac{1}{2}g^{hs}g^{tu}\nabla_r m_{tsu}-f_s{}^h\nabla_r f^s. \end{cases}$$

Thus we have a system of partial differential equations satisfied by all infinitesimal transformations X which leave an f-structure f and a tensor m_{ji} on V invariant. *Since V is compact and the Riemannian metric tensor g is positive definite, the* system (4. 4) is elliptic and satisfies the assumption of Lemma 1 in § 3. Hence Lemma 2 in § 3 shows that the Lie subalgebra $\Phi(f, \omega)$ is finite dimensional. Thus the main theorem is proved completely.

The topology of the automorphism group $A(f, \omega)$ is stronger than the compact open topology. We do not know whether the automorphism group $A(f, \omega)$ is a Lie group with respect to the compact open topology or not.

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