ON REGULARLY BRANCHED THREE-SHEETED COVERING RIEMANN SURFACES

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§1. Let R be an open Riemann surface. Let $\mathfrak{M}(R)$ be the family of non-constant analytic functions meromorphic on R. Let f be a member of $\mathfrak{M}(R)$. Let $P(f)$ be the number of Picard's exceptional values of f, where we say α a Picard's exceptional value of f when α is not taken by f on R. Let $P(R)$ be a quantity defined by

$$P(R)=\sup_{f\in\mathfrak{M}(R)} P(f).$$

In general $P(R)\geqq 2$. It has been shown that $P(R)$ is an important quantity belonging to R for a criterion of non-existence of analytic mapping (cf. Ozawa [5, 6]).

From now on we shall confine ourselves to the following Riemann surfaces:

Let R be a regularly branched three-sheeted covering Riemann surface formed by elements $p=(z, y)$ for each z, y satisfying the equation

$$y^3=g(z), \tag{1.1}$$

where $g(z)$ is an entire function having no zero other than an infinite number of simple or double zeros. Then we have $P(R)\leqq 6$ from Selberg's theory [9].

Hiromi and the author [1] has given *a characterization of Riemann surfaces R with $P(R)=6$* and proved that *there is no regularly branched three-sheeted covering Riemann surface R with $P(R)=5$* and that *every Riemann surface R defined by the equation* (1.1) *with an entire function $g(z)$, which have no zero other than an infinite number of simple zeros or have no zero other than an infinite number of double zeros, always satisfies $P(R)\leqq 4$.*

Hence as an example of surface R with $P(R)=4$, we have a Riemann surface R defined by the equation (1.1) with $g(z)=e^z+1$.

As for surfaces with $P(R)\leqq 4$ nothing is known other than above facts. Therefore we wish to get a perfect characterization of surfaces with $P(R)=4$. The author regrets to say that he could not give any perfect characterization of surfaces with $P(R)=4$ till now. In the present paper, however, under a certain additional condition we shall give a characterization of surfaces with $P(R)=4$ in § 4 and a criterion for $P(R)\leqq 4$ in § 5.

Next let S be another surface of the same type as R. Then Mutō [3] has established a perfect condition for the existence of analytic mappings from R into S. If $P(R)=P(S)=6$, then the possibility on the existence of analytic mappings from R into S remains by means of Ozawa's criterion on non-existence of analytic

Received December 23, 1965.

mapping [5]. In the present paper we shall give a perfect condition of the existence of analytic mappings between the special surfaces with $P(R)=P(S)=6$ in § 6.

The author wishes to express his heartiest thanks to Professors Y. Komatu and M. Ozawa and Mr. G. Hiromi for their valuable advices and kind encouragements in preparing the paper.

§ 2. Let R be a regularly branched three-sheeted covering Riemann surface defined by the equation (1. 1). In the first place let us recall the following results which Hiromi and the author [1] established about regular functions on R:

Let f be a three-valued entire algebroid function of z which is single-valued and regular on R. Then there exist two entire functions $f_1(z)$, $f_2(z)$ and a meromorphic function $f_3(z)$ single-valued and regular with exception of all the double zeros of $g(z)$ at which $f_3(z)$ has simple poles, such that

$$f(p)=f_1(z)+f_2(z)y+f_3(z)y^2. \tag{2. 1}$$

Conversely the function $f(p)$ defined by (2. 1) with $f_1(z)$, $f_2(z)$ and $f_3(z)$ having the described properties is clearly regular on R. Let the defining equation of f be

$$F(z, f)\equiv f^3-S_1(z)f^2+S_2(z)f-S_3(z)=0, \tag{2. 2}$$

where $S_1(z)$, $S_2(z)$ and $S_3(z)$ are entire functions. Then from (2. 1) we have the following relations:

$$\begin{cases} S_1(z)=3f_1(z), \\ S_2(z)=3f_1(z)^2-3f_2(z)f_3(z)g(z), \\ S_3(z)=f_1(z)^3+f_2(z)^3g(z)+f_3(z)^3g(z)^2-3f_1(z)f_2(z)f_3(z)g(z). \end{cases} \tag{2. 3}$$

Let $D(z)$ be the discriminant of the cubic equation (2. 2). Then we have

$$D(z)=-27g(z)^2(f_2(z)^3-f_3(z)^3g(z))^2, \tag{2. 4}$$

and from (2. 2)

$$D(z)=-4S_1(z)^3S_3(z)+S_1(z)^2S_2(z)^2+18S_1(z)S_2(z)S_3(z)-4S_2(z)^3-27S_3(z)^2. \tag{2. 5}$$

Eliminating f_1 and f_3 or f_1 and f_2 from (2. 3) we see that $f_2{}^3g$ and $f_3{}^3g^2$ are two roots of a quadratic equation

$$X^2-\left(\frac{2}{27}S_1(z)^3-\frac{1}{3}S_1(z)S_2(z)+S_3(z)\right)X+\frac{1}{27}\left(\frac{1}{3}S_1(z)^2-S_2(z)\right)^3=0. \tag{2. 6}$$

Let $D_1(z)$ be the discriminant of the quadratic equation (2. 6). The from (2. 5) we get

$$D_1(z)=-\frac{1}{27}D(z). \tag{2. 7}$$

§ 3. Lemmas. For our purpose we need some preparatory lemmas. The notations T, m, N, N_1 and $\bar{N}$ on meromorphic functions are used in the sense of Nevanlinna [4]. Hiromi and Ozawa [2] proved the following lemma A and lemma B:

LEMMA A. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *be meromorphic functions and let* $g_1(z), \cdots, g_n(z)$ *be entire functions. Further suppose that*

$$T(r, a_j)=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right), \qquad j=0, 1, \cdots, n,$$

holds outside a set of finite measure. If the identity

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}=a_0(z)$$

holds, then we have an identity

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z)e^{g_\nu(z)}=0,$$

where c_ν, $\nu=1, \cdots, n$, *are constants which are not all zero.*

LEMMA B. *Let* $a_1(z), \cdots, a_n(z)$ *be meromorphic functions and let* $g(z)$ *be an entire function. Further suppose that*

$$T(r, a_j)=o(m(r, e^{g})), \qquad j=1, 2, \cdots, n,$$

holds outside a set of finite measure. Then the identity

$$\sum_{\nu=1}^{n} a_\nu(z)e^{\nu g(z)}=0$$

is impossible unless all $a_1(z), \cdots, a_n(z)$ *are identically zero.*

Now we shall prove

LEMMA 1. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *be meromorphic functions and let* $g_1(z), \cdots, g_n(z)$ *be entire functions. Further suppose that*

$$T(r, a_j)=o(m(r, e^{g_\nu}))$$

and

$$T(r, a_j)=o(m(r, e^{g_1-g_\nu})), \qquad j=0, 1, \cdots, n; \quad \nu=k, k+1, \cdots, n,$$

outside a set of finite measure. If $a_1(z)\not\equiv 0$ *and the identity*

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}=a_0(z) \tag{3. 1}$$

halds, then we have

$$\sum_{\nu=1}^{k-1} c_\nu a_\nu(z)e^{g_\nu(z)}+c_0a_0(z)=0,$$

where $c_1=1$ *and* c_ν, $\nu=0, 2, 3, \cdots, k-1$, *are suitable constants.*

Proof. Suppose that the identity (3. 1) holds. Then, by virtue of lemma A we get

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z)e^{g_\nu(z)}=0, \tag{3. 2}$$

where c_ν, $\nu=1, \cdots, n$, are constants which are not all zero.

If $c_1=0$, then $a_\nu(z)e^{g_\nu(z)}$, $\nu=2, \cdots, n$, are linearly dependent. Hence by eliminating a suitable term, say $a_n(z)e^{g_n(z)}$, from (3. 1), we get

$$\sum_{\nu=1}^{n-1} d_\nu a_\nu(z)e^{g_\nu(z)}=a_0(z),$$

where $d_1=1$ and the other d_ν are suitable constants. Here if $d_\nu=0$, $\nu=k, k+1, \cdots, n-1$, then there is nothing to prove. If at least one of d_ν, $\nu=k, \cdots, n-1$, is not zero, then by virtue of lemma A we have

$$\sum_{\nu=1}^{n-1} d_\nu' d_\nu a_\nu(z)e^{g_\nu(z)}=0, \tag{3. 3}$$

where $d_1=1$ and d_ν' are constants which are not all zero.

If $c_1\neq 0$ and $c_\nu=0$, $\nu=k, \cdots, n$, then there is nothing to prove.

If $c_1\neq 0$ and at least one of c_ν, $\nu=k, \cdots, n$, say c_n, is not zero, then we have

$$\sum_{\nu=1}^{n-1} c_\nu a_\nu(z)e^{g_\nu(z)-g_n(z)}=-c_n a_n(z),$$

and by applying lemma A to this identity, we get

$$\sum_{\nu=1}^{n-1} c_\nu' c_\nu a_\nu(z)e^{g_\nu(z)-g_n(z)}=0,$$

where c_ν' are constants which are not all zero. Hence we obtain

$$\sum_{\nu=1}^{n-1} c_\nu' c_\nu a_\nu(z)e^{g_\nu(z)}=0. \tag{3. 4}$$

Thus (3. 2) implies (3. 3) or (3. 4). By the repetition of this process, we finally arrive at the desired result. Q.E.D.

The notations T, m, N on algebroid functions are used in the sense of Selberg [9]. Let $f(z)$ be an algebroid function. In a neighborhood of a zero z_0 of $f(z)$, let $f(z)$ be expanded:

$$f(z)=a_\tau(z-z_0)^{\tau/\lambda}+\cdots, \qquad (a_\tau\neq 0). \tag{3. 5}$$

Let $N_1^*(r, 0, f)$ and $N_2^*(r, 0, f)$ be the counting functions of zeros of $f(z)$ with $\tau>\lambda$ and $\tau\leqq\lambda$ in (3. 5), respectively. Let $N(r, \mathfrak{X}_f)$ and $N(r, \mathfrak{Z}_f)$ be the quantities defined by Selberg [9].

LEMMA 2. *Let $H(z)$ be an entire function and $h(z)$ be a k-valued entire algebroid function. If*

$$m(r, h)=o(m(r, e^H))$$

holds outside a set of finite measure, then we have

$$N_1^*(r, 0, e^H-h)=o(m(r, e^H)) \qquad \textit{and} \qquad N_2^*(r, 0, e^H-h)\sim m(r, e^H)$$

outside a set of finite measure.

Proof. We set

$$f=\frac{e^H-h}{-h}.$$

Then f is a k-valued algebroid function regular on $\mathfrak{X}_h$. Using ramification theorem (cf. Ullrich [10], Selberg [9]), we get

$$N(r,\infty,f)=N(r,0,h)\leqq m(r,h)+O(1)=o(m(r,e^H)),$$
$$N(r,1,f)=N(r,0,e^H)=0,$$
$$N(r,\infty,f')\leqq 2N(r,0,h)+N(r,\mathfrak{X}_h)$$
$$\leqq 2N(r,0,h)+(2k-2)T(r,h)+O(1)=o(m(r,e^H)),$$
$$T(r,f)\leqq T(r,e^H)+T(r,h)+T(r,1/h)+O(1)\leqq m(r,e^H)+o(m(r,e^H))$$

and

$$m(r,e^H)\leqq m(r,e^H-h)+m(r,h)+O(1)\leqq T(r,f)+o(m(r,e^H))$$

outside a set of finite measure. Nevanlinna-Selberg's second fundamental theorem applied to f gives

$$T(r,f)\leqq N(r,0,f)+N(r,\infty,f)+N(r,1,f)-N(r,\mathfrak{Z}_f)+N(r,\mathfrak{X}_f)+O(\log rT(r,f))$$

outside a set of finite measure. Since

$$N(r,\mathfrak{Z}_f)-N(r,\mathfrak{X}_f)=2N(r,\infty,f)+N(r,0,f')-N(r,\infty,f')$$
$$=N(r,0,f')+o(m(r,e^H))$$

and

$$N(r,\mathfrak{X}_f)=N(r,\mathfrak{X}_h)\leqq(2k-2)T(r,h)+O(1)=o(m(r,e^H))$$

outside a set of finite measure, we have

$$T(r,f)\leqq N(r,0,f)-N(r,0,f')+o(m(r,e^H))$$
$$\leqq N_2{}^*(r,0,f)+\bar{N}_1{}^*(r,0,f)+N(r,\mathfrak{X}_f)+o(m(r,e^H))$$
$$=N_2{}^*(r,0,e^H-h)+\bar{N}_1{}^*(r,0,e^H-h)+o(m(r,e^H))$$

outside a set of finite measure. On the other hand we have

$$N_2{}^*(r,0,e^H-h)+N_1{}^*(r,0,e^H-h)-\bar{N}_1{}^*(r,0,e^H-h)+\bar{N}_1{}^*(r,0,e^H-h)$$
$$=N(r,0,e^H-h)=N(r,0,f)\leqq T(r,f)+O(1)$$
$$\leqq N_2{}^*(r,0,e^H-h)+\bar{N}_1{}^*(r,0,e^H-h)+o(m(r,e^H))$$

outside a set of finite measure. Thus we obtain

$$\bar{N}_1{}^*(r,0,e^H-h)\leqq N_1{}^*(r,0,e^H-h)-\bar{N}_1{}^*(r,0,e^H-h)=o(m(r,e^H)),$$

and hence

$$N_1{}^*(r,0,e^H-h)=o(m(r,e^H)),$$

and by means of $T(r,f)=m(r,e^H)+o(m(r,e^H))$, we finally have

$$N_2^*(r, 0, e^H-h)=m(r, e^H)+o(m(r, e^H))$$

outside a set of finite measure. Thus lemma 2 has been proved.

Let $f(z)$ be an algebroid function. Let $\hat{N}_2^*(r, 0, f)$ be the counting function of zeros of $f(z)$ with $\tau=\lambda=1$ in (3. 5) whose projections do not coincide with projections of all the branch points of $\mathfrak{X}_f$.

LEMMA 3. *Under the hypotheses of lemma 2, we have*

$$\hat{N}_2^*(r, 0, e^H-h)\sim m(r, e^H)$$

outside a set of finite measure.

Proof. By virtue of ramification theorem we have clearly

$$\begin{aligned} N_2^*(r, 0, e^H-h)-\hat{N}_2^*(r, 0, e^H-h) &\leqq N(r, \mathfrak{X}_h)+(k+1)\bar{N}(r, \mathfrak{X}_h) \\ &\leqq (k+2)(2k-2)T(r, h)+O(1)=o(m(r, e^H)) \end{aligned}$$

outside a set of finite measure. Therefore lemma 2 gives our desired result. Q.E.D.

Let $f(z)$ be a meromorphic function. Let $N_2(r, 0, f)$ be the counting function of simple zeros of $f(z)$.

LEMMA 4. *Let $H(z)$ and $\varphi_j(z)$ $(j=1, \cdots, \mu)$ be entire functions satisfying*

$$m(r, \varphi_j)=o(m(r, e^H)), \qquad j=1, \cdots, \mu,$$

outside a set of finite measure. If the algebraic equation

$$Q_\mu(h)\equiv h^\mu+\varphi_1(z)h^{\mu-1}+\cdots+\varphi_\mu(z)=0 \tag{3. 6}$$

is irreducible, then we have

$$N_2(r, 0, Q_\mu(e^H))\sim\mu m(r, e^H) \quad \textit{and} \quad N_1(r, 0, Q_\mu(e^H))=o(m(r, e^H))$$

outside a set of finite measure.

Proof. Let $h(z)$ be μ-valued entire algebroid function defined by the equation (3. 6) and $h_j(z)$ $(j=1, \cdots, \mu)$ its μ determinations. Then we have

$$\sum_{j=1}^{\mu}\hat{N}_2^*(r, 0, e^H-h_j)\leqq N_2(r, 0, Q(e^H))\leqq\sum_{j=1}^{\mu}N_2^*(r, 0, e^H-h_j)$$

and

$$N_1(r, 0, Q(e^H))\leqq\sum_{j=1}^{\mu}N_1^*(r, 0, e^H-h_j).$$

Therefore lemma 4 follows from lemma 2 and lemma 3.

REMARK. If $h(z)$ in lemma 2 reduces to an entire function or if $\mu=1$ in lemma 4, then these lemmas reduce to that of Hiromi and Ozawa [2], that is,

LEMMA C. *Let $H(z)$ be an entire function and let $g(z)$ be an entire function satisfying $m(r, g)=o(m(r, e^H))$ outside a set of finite measure. Then we have*

$$N_2(r, 0, e^H-g)\sim m(r, e^H) \quad \textit{and} \quad N_1(r, 0, e^H-g)=o(m(r, e^H))$$

outside a set of finite measure.

Let $f_1(z)$ and $f_2(z)$ be two meromorphic functions. Let $N_0(r, 0; f_1, f_2)$ be the counting function of common zeros of $f_1(z)$ and $f_2(z)$.

LEMMA 5. *Let $H(z)$, $\varphi_j(z)$ $(j=1, \cdots, \mu)$ and $\varphi_k^*(z)$ $(k=1, \cdots, \nu)$ be entire functions satisfying*

$$m(r, \varphi_j) = o(m(r, e^H)), \qquad j=1, 2, \cdots, \mu,$$

and

$$m(r, \varphi_k^*) = o(m(r, e^H)), \qquad k=1, 2, \cdots, \nu,$$

outside a set of finite measure. If the equations

$$Q_\mu(h) \equiv h^\mu + \varphi_1(z) h^{\mu-1} + \cdots + \varphi_\mu(z) = 0$$

and

$$Q_\nu^*(h) \equiv h^\nu + \varphi_1^*(z) h^{\nu-1} + \cdots + \varphi_\nu^*(z) = 0$$

are irreducible, respectively, and $Q_\mu(e^H) \not\equiv Q_\nu^*(e^H)$, then we have

$$N_0(r, 0; Q_\mu(e^H), Q_\nu^*(e^H)) = o(m(r, e^H))$$

outside a set of finite measure.

Proof. We denote the resultant of $Q_\mu(h)$ and $Q_\nu^*(h)$ by $J(z)$, that is,

$$J(z) \equiv \left|\begin{array}{ccccccc} 1 & \varphi_1 & \cdots & \varphi_\mu & & & \\ & 1 & \varphi_1 & \cdots & \varphi_\mu & & \\ & & \cdots & \cdots & \cdots & & \\ & & & 1 & \varphi_1 & \cdots & \varphi_\mu \\ 1 & \varphi_1^* & \cdots & \varphi_\nu^* & & & \\ & 1 & \varphi_1^* & \cdots & \varphi_\nu^* & & \\ & & \cdots & \cdots & \cdots & & \\ & & & 1 & \varphi_1^* & \cdots & \varphi_\nu^* \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\ \cdots\\1\end{array}}\right\}\nu \\ \left.\vphantom{\begin{array}{c}1\\1\\ \cdots\\1\end{array}}\right\}\mu \end{array}.$$

Then by means of hypotheses of the lemma, we have

$$N(r, 0, J) = o(m(r, e^H))$$

outside a set of finite measure. Hence we have

$$N_0(r, 0; Q_\mu(e^H), Q_\nu^*(e^H)) \leqq N(r, 0, J) + o(m(r, e^H)) = o(m(r, e^H))$$

outside a set of finite measure, which proves lemma 5.

Finally, we need

LEMMA D.[1] *Let $g(z)$ be a transcendental entire function and let $P(z)$ and $Q(z)$ be two polynomials. If the equation*

$$g\circ h(z)=P(z)g(z)+Q(z)$$

holds, then $h(z)$ must be of the form $az+b$.

§ 4. Now we shall consider to give a characterization of Riemann surfaces with $P(R)=4$ under an additional condition.

Let R be a regularly branched three-sheeted covering Riemann surface defined by the equation (1. 1) and suppose that $P(R)=4$. Then there exists a meromorphic function $f\in\mathfrak{M}(R)$ with $P(f)=4$. Further we may assume that its four Picard's exceptional values are $0, a_1, a_2$ and ∞. Then f becomes a three-valued entire algebroid function of z which is regular on R and satisfies (2. 2) and (2. 3). By Rémoundos' reasoning [8] of his celebrated generalization of Picard's theorem, it is sufficient to consider the following five cases:

$$\begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \end{pmatrix} = \quad \text{(i)} \begin{pmatrix} c_1 \\ c_2 \\ \beta e^H \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} \beta e^H \\ c_1 \\ c_2 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} c \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \end{pmatrix}, \quad \text{(iv)} \begin{pmatrix} \beta_1 e^{H_1} \\ c \\ \beta_2 e^{H_2} \end{pmatrix}, \quad \text{(v)} \begin{pmatrix} \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix},$$

where $c, c_1, c_2, \beta, \beta_1, \beta_2$ and β_3 are non-zero constants and $H(z), H_1(z), H_2(z)$ and $H_3(z)$ are non-constant entire functions satisfying $H(0)=H_1(0)=H_2(0)=H_3(0)=0$.

After calculation we obtain

$$\text{(4. 1)} \qquad \begin{cases} S_1=\dfrac{1}{a_2(a_1-a_2)}\beta e^H-\dfrac{1}{a_1a_2}c_1-\dfrac{1}{a_1(a_1-a_2)}c_2+a_1+a_2, \\ S_2=\dfrac{a_1}{a_2(a_1-a_2)}\beta e^H-\dfrac{a_1+a_2}{a_1a_2}c_1-\dfrac{a_2}{a_1(a_1-a_2)}c_2+a_1a_2, \\ S_3=-c_1 \end{cases}$$

in the case (i), and

$$\text{(4. 2)} \qquad \begin{cases} S_1=-\dfrac{1}{a_1a_2}\beta e^H-\dfrac{1}{a_1(a_1-a_2)}c_1+\dfrac{1}{a_2(a_1-a_2)}c_2+a_1+a_2, \\ S_2=-\dfrac{a_1+a_2}{a_1a_2}\beta e^H-\dfrac{a_2}{a_1(a_1-a_2)}c_1+\dfrac{a_1}{a_2(a_1-a_2)}c_2+a_1a_2, \\ S_3=-\beta e^H \end{cases}$$

in the case (ii), and

1) This lemma has been proved by considering the growth of $g\circ h(z)$ in contrast with that of $g(z)$ or $P(z)g(z)$ in Ozawa's note which is yet unpublished.

$$
(4.3)\qquad \begin{cases} S_1=-\dfrac{1}{a_1(a_1-a_2)}\beta_1e^{H_1}+\dfrac{1}{a_2(a_1-a_2)}\beta_2e^{H_2}-\dfrac{1}{a_1a_2}c+a_1+a_2, \\ S_2=-\dfrac{a_2}{a_1(a_1-a_2)}\beta_1e^{H_1}+\dfrac{a_1}{a_2(a_1-a_2)}\beta_2e^{H_2}-\dfrac{a_1+a_2}{a_1a_2}c+a_1a_2, \\ S_3=-c \end{cases}
$$

in the case (iii), and

$$
(4.4)\qquad \begin{cases} S_1=-\dfrac{1}{a_1a_2}\beta_1e^{H_1}+\dfrac{1}{a_2(a_1-a_2)}\beta_2e^{H_2}-\dfrac{1}{a_1(a_1-a_2)}c+a_1+a_2, \\ S_2=-\dfrac{a_1+a_2}{a_1a_2}\beta_1e^{H_1}+\dfrac{a_1}{a_2(a_1-a_2)}\beta_2e^{H_2}-\dfrac{a_2}{a_1(a_1-a_2)}c+a_1a_2, \\ S_3=-\beta_1e^{H_1} \end{cases}
$$

in the case (iv), and

$$
(4.5)\qquad \begin{cases} S_1=-\dfrac{1}{a_1a_2}\beta_1e^{H_1}-\dfrac{1}{a_1(a_1-a_2)}\beta_2e^{H_2}+\dfrac{1}{a_2(a_1-a_2)}\beta_3e^{H_3}+a_1+a_2, \\ S_2=-\dfrac{a_1+a_2}{a_1a_2}\beta_1e^{H_1}-\dfrac{a_2}{a_1(a_1-a_2)}\beta_2e^{H_2}+\dfrac{a_1}{a_2(a_1-a_2)}\beta_3e^{H_3}+a_1a_2, \\ S_3=-\beta_1e^{H_1} \end{cases}
$$

in the case (v).

Cases (i) and (ii). These cases are similar to the cases (i) and (ii) of §5 in Hiromi and the author [1]. Hence the discriminant $D(z)$ of the cubic equation (2.2) is a polynomial of degree 4 of e^H. From the same reasoning of §5 in [1] there exists a meromorphic function $f\in\mathfrak{M}(R)$ with $P(f)=6$, if the constant term[2)] of $D(z)$ does not vanish or if the constant term of $D(z)$ vanishes but the constant term of

$$
S(z)\equiv-\left(\frac{2}{27}S_1(z)^3-\frac{1}{3}S_1(z)S_2(z)+S_3(z)\right)
$$

in (2.6) does not vanish. Hence the constant terms of $D(z)$ and $S(z)$ must be zero. Therefore from the quadratic equation (2.6) we obtain the equations

$$
(4.6)\qquad f_2{}^3g=Ae^H(e^H-\gamma)(e^H-\delta),\qquad A\neq0,
$$

$$
(4.7)\qquad f_3{}^3g^2=Ae^H(e^H-\gamma')(e^H-\delta'),\qquad A\neq0.
$$

From (4.6) we see that γ and δ do not vanish simultaneously. Hence we may assume that γ is not zero.

First we assume that $\gamma\neq\delta$. Since a simple zero point z_1 of $e^H-\gamma$ is a simple zero point of the right hand term of (4.6), z_1 is a simple zero point of $g(z)$. Hence

2) Here we say "constant term" when we take $D(z)$ for a polinomial of $e^{H(z)}$. And we use the term "constant term" in this sense also for $S(z)$.

the equation (4. 7) gives $\gamma=\gamma'$ or $\gamma=\delta'$, say $\gamma=\gamma'$. Besides, since z_1 is a double zero point of $f_3{}^3g^2$, we get $\gamma=\gamma'=\delta'$. Therefore if $\delta\neq0$, then similarly we have $\delta=\gamma'=\delta'$, which contradicts $\gamma\neq\delta$. Thus we obtain $\gamma=\gamma'=\delta'$ and $\delta=0$, that is,

$$f_2{}^3g=Ae^{2H}(e^H-\gamma) \quad \text{and} \quad f_3{}^3g^2=Ae^H(e^H-\gamma)^2, \qquad A\gamma\neq0.$$

Next we assume that $\gamma=\delta$. Then similarly we get $\gamma=\gamma'$ or $\gamma=\delta'$, say $\gamma=\gamma'$. If $\gamma'=\delta'$, then from (4. 6) a simple zero point z_1 of $e^H-\gamma'$ is a double zero point of $g(z)$ and from (4. 7) z_1 is not a double zero point of $g(z)$, which is a contradiction. If $\gamma'\neq\delta'$ and $\delta'\neq0$, then, by virtue of (4. 7) and the properties of $f_3(z)$, a simple zero point z_2 of $e^H-\delta'$ is a double zero point of $g(z)$. Therefore we get $\gamma=\delta'$, which contradicts $\gamma=\gamma'\neq\delta'$. Thus we obtain $\gamma=\delta=\gamma'$ and $\delta'=0$, that is,

$$f_2{}^3g=Ae^H(e^H-\gamma)^2 \quad \text{and} \quad f_3{}^3g^2=Ae^{2H}(e^H-\gamma), \qquad A\gamma\neq0.$$

After all in the cases (i) and (ii), we have

$$f^*(z)^3g(z)=e^{H(z)}-\gamma \tag{4. 8}$$

or

$$\tilde{f}^*(z)^3g(z)=(e^{H(z)}-\gamma)^2, \tag{4. 9}$$

where $f^*(z)=f_2(z)e^{-2H(z)/3}/\sqrt[3]{A}$ and $\tilde{f}^*(z)=f_2(z)e^{-H(z)/3}/\sqrt[3]{A}$ are two entire functions and γ is a non-zero constant.

Conversely, let R be a regularly branched three-sheeted covering Riemann surface defined by the equation (1. 1) with $g(z)$ satisfying (4. 8). Then the function $f_0=\sqrt[3]{e^H-\gamma}$ belong to $\mathfrak{M}(R)$ and $P(f_0)=4$. Hence we have $P(R)\geqq4$. In order to prove $P(R)=4$, by virtue of theorem 1 and theorem 2 in [1], it suffices to show the impossibility of an identity of the form

$$\tilde{f}(z)^3(e^{H(z)}-\gamma)=(e^{L(z)}-\alpha)\,(e^{L(z)}-\beta)^2, \qquad \alpha\beta(\alpha-\beta)\neq0, \tag{4. 10}$$

where $L(z)$ is a non-constant entire function with $L(0)=0$, α and β are two constants and $\tilde{f}(z)$ is a meromorphic function which has zeros and poles possibly at the zeros of order at least 3 of $(e^L-\alpha)\,(e^L-\beta)^2$ and $e^H-\gamma$, respectively.

Now we shall show the impossibility of the identity (4. 10) using lemma C. Let N_3 be the counting function of double zeros of the refered meromorphic function. Then we have

$$N_2(r, 0, e^H-\gamma)\leqq N_2(r, 0, \tilde{f}^3(e^H-\gamma))\leqq N_2(r, 0, e^H-\gamma)+N_1(r, 0, e^H-\gamma),$$

$$N_2(r, 0, (e^L-\alpha)\,(e^L-\beta)^2)=N_2(r, 0, e^L-\alpha),$$

and thus

$$m(r, e^L)\sim m(r, e^H)$$

outside a set of finite measure. On the other hand we have

$$N_3(r, 0, \tilde{f}^3(e^H-\gamma))\leqq 2N_1(r, 0, e^H-\gamma),$$

$$2N_2(r, 0, e^L-\beta)\leqq N_3(r, 0, (e^L-\alpha)\,(e^L-\beta)^2)\leqq 2N_2(r, 0, e^L-\beta)+2N_1(r, 0, e^L-\alpha),$$

and thus

$$2m(r, e^{L})=o(m(r, e^{H}))$$

outside a set of finite measure. This is a contradiction. Thus we have shown the impossibility of the identity (4. 10), that is, $P(R)=4$.

Secondly let R' be a regularly branched three-sheeted covering Riemann surface defined by the equation (1. 1) with $g(z)$ staisfying (4. 9). Then the function $f_0=1/\sqrt[3]{e^{H}-\gamma}$ belongs to $\mathfrak{M}(R')$ and $P(f_0)=4$. Hence $P(R')\geqq 4$. However we can similarly show the impossibility of the identity of the form

$$\tilde{f}(z)^{3}(e^{H(z)}-\gamma)^{2}=(e^{L(z)}-\alpha)(e^{L(z)}-\beta)^{2}, \qquad \alpha\beta(\alpha-\beta)\neq 0,$$

where $L(z)$ is a non-constant entire function with $L(0)=0$, α and β are two constants and $\tilde{f}(z)$ is a meromorphic function which has zeros and poles possibly at the zeros of order at least 3 of $(e^{L}-\alpha)(e^{L}-\beta)^{2}$ and $(e^{H}-\gamma)^{2}$, respectively. Hence we have also $P(R')=4$.

Therefore we obtain a perfect characterization of R with $P(R)=4$ in the cases (i) and (ii).

Now, we shall discuss the cases (iii), (iv) and (v). Let us suppose that one of the growth of e^{H_1}, e^{H_2} and e^{H_3} in (4. 3), (4. 4) and (4. 5) is more rapid than the others. We denote by e^{H} the function having the above property. Then substituting (4. 3), (4. 4) and (4. 5) into (2. 5), respectively, we get an equation of the form

(4. 11) $$D(z)=A(e^{4H(z)}+\zeta_1(z)e^{3H(z)}+\zeta_2(z)e^{2H(z)}+\zeta_3(z)e^{H(z)}+\zeta_4(z)),$$

where A is a non-zero constant and all $\zeta_j(z)\ (j=1,\cdots,4)$ are polynomials of e^{H_1} or e^{H_2} in the cases (iii) and (iv), or all are polynomials of two of e^{H_1}, e^{H_2} and e^{H_3} in the case (v). Hence we have, in these cases,

$$m(r, \zeta_j)=o(m(r, e^{H})), \qquad j=1, 2, 3, 4,$$

outside a set of finite measure. On the other hand from (2. 4) we have

(4. 12) $$-27g^{2}(f_2{}^{3}-f_3{}^{3}g)^{2}=A(e^{4H}+\zeta_1e^{3H}+\zeta_2e^{2H}+\zeta_3e^{H}+\zeta_4).$$

If the equation

$$Q_4(h)\equiv h^{4}+\zeta_1(z)h^{3}+\zeta_2(z)h^{2}+\zeta_3(z)h+\zeta_4(z)=0$$

is irreducible, then, by virtue of lemma 4, the right hand side of (4. 12) has simple zeros, while the left hand side has not any simple zero. This is a contradiction. Hence the equation, $Q_4(h)=0$, is not irreducible. According to the similar discussion as the above using lemma 4 and lemma 5, we finally get

(4. 13) $$D(z)=A(e^{2H}+\zeta_1^{*}(z)e^{H(z)}+\zeta_2^{*}(z))^{2}.$$

Then (4. 11) and (4. 13) yield

(4. 14) $$\zeta_1(z)=2\zeta_1^{*}(z), \quad \zeta_2(z)=\zeta_1^{*}(z)^{2}+2\zeta_2^{*}(z), \quad \zeta_3(z)=2\zeta_1^{*}(z)\zeta_2^{*}(z), \quad \zeta_4(z)=\zeta_2^{*}(z)^{2}.$$

Case (iii). First let us suppose that in (4. 3)

$$m(r, e^{H_1})=o(m(r, e^{H_2}))$$

outside a set of finite measure. Then by substituting (4. 3) into (2. 5) and taking

$e^H \equiv \beta_2 e^{H_2}$ in (4.11) and (4.13) into account, we have

$$\zeta_1^* = -\frac{1}{a_1^2}[a_2(a_1+a_2)\beta_1 e^{H_1}+(a_1-a_2)(2a_1-a_2)c+a_1^2a_2(a_1-a_2)(a_1-2a_2)],$$

$$\zeta_2^* = (\zeta_2-\zeta_1^{*2})/2 = \frac{1}{a_1^4}[a_1a_2^3\beta_1^2e^{2H_1}+a_2(a_1-a_2)\{(a_1^2-2a_1a_2+2a_2^2)c - a_1^2a_2(2a_1^2-2a_1a_2+a_2^2)\}\beta_1e^{H_1}+a_1(a_1-a_2)^2\{(a_1-a_2)c^2 + a_1a_2(a_1^2+a_2^2)c-a_1^3a_2^3(a_1-a_2)\}],$$

and

$$\zeta_4-\zeta_2^{*2} = \frac{-4}{a_1^8}[a_1a_2^4(a_1-a_2)^3(c-a_1^2a_2)\beta_1^3e^{3H_1}+a_2^3(a_1-a_2)^3\{(a_1^2+a_1a_2-a_2^2)c^2 - a_1^2(a_1-a_2)(a_1^2+a_2^2)c+a_1^5a_2(a_1^2-a_1a_2-a_2^2)\}\beta_1^2e^{2H_1}+a_1a_2^3(a_1 - a_2)^3\{(a_1-a_2)c^3+a_1a_2(2a_1^2-2a_1a_2+a_2^2)c^2-a_1^5(a_1^2-2a_1a_2 + 2a_2^2)c+a_1^7a_2^2(a_1-a_2)\}\beta_1e^{H_1}+a_1^3a_2^3(a_1-a_2)^4c\{(a_1-a_2)c^2 - a_1^2(a_1^2-3a_1a_2+a_2^2)c-a_1^5a_2(a_1-a_2)\}]=0.$$

From lemma B we have

$c-a_1^2a_2=0$ and $A' \equiv (a_1^2+a_1a_2-a_2^2)c^2-a_1^2(a_1-a_2)(a_1^2+a_2^2)c+a_1^5a_2(a_1^2-a_1a_2-a_2^2)=0.$

By substituting $c=a_1^2a_2$ into A', however, we have $A'=a_1^5a_2^2(a_1-a_2)\neq 0$, which contradicts $A'=0$.

Next let us suppose that in (4.3)

$$m(r, e^{H_2})=o(m(r, e^{H_1}))$$

outside a set of finite measure. Then we have similarly a contradiction.

Therefore the case (iii) does not occur under a condition that one of the growth of e^{H_1} and e^{H_2} is more rapid than the other.

Case (iv). First let us suppose that in (4.4)

$$m(r, e^{H_1})=o(m(r, e^{H_2}))$$

outside a set of finite measure. Then by substituting (4.4) into (2.5) and taking $e^H \equiv \beta_2 e^{H_2}$ in (4.11) and (4.13) into account, we have, similarly as in the case (iii),

$$\zeta_4-\zeta_2^{*2} = -\frac{4}{a_1^8}[a_1a_2^3(a_1-a_2)^4\{c+a_1^2(a_1-a_2)\}\beta_1^3e^{3H_1}+a_2^3(a_1-a_2)^3\{(a_1^2+a_1a_2 - a_2^2)c^2+a_1^2a_2(2a_1^2-2a_1a_2+a_2^2)c-a_1^5(a_1-a_2)(a_1^2-3a_1a_2 + a_2^2)\}\beta_1^2e^{2H_1}+a_1a_2^3(a_1-a_2)^3\{a_2c^3-a_1(a_1-a_2)(a_1^2+a_2^2)c^2 - a_1^5(a_1^2-2a_1a_2+2a_2^2)c-a_1^7a_2(a_1-a_2)^2\}\beta_1e^{H_1}-a_1^3a_2^4(a_1 - a_2)^3c\{a_2c^2-a_1^2(a_1^2-a_1a_2-a_2^2)c-a_1^5a_2(a_1-a_2)\}]=0.$$

From lemma B we have

$$c=-a_1^2(a_1-a_2)$$

and

$$B' \equiv (a_1^2+a_1a_2-a_2^2)c^2+a_1^2a_2(2a_1^2-2a_1a_2+a_2^2)c-a_1^5(a_1-a_2)(a_1^2-3a_1a_2+a_2^2)=0.$$

By substituting $c=-a_1^2(a_1-a_2)$ into B', however, we have $B'=a_1^5a_2(a_1-a_2)\neq 0$, which contradicts $B'=0$.

Next let us suppose that in (4.4)

$$m(r, e^{H_2})=o(m(r, e^{H_1}))$$

outside a set of finite measure. Then by substituting (4.4) into (2.5) and taking $e^H \equiv \beta_1 e^{H_1}$ in (4.11) and (4.13) into account, we have

$$\zeta_4=\frac{1}{(a_1-a_2)^6}\{a_1^2\beta_2e^{H_2}-a_2^2c+a_1^2a_2^2(a_1-a_2)\}^2[a_1^2\beta_2^2e^{2H_2}-2\{a_1a_2c+a_1^2a_2(a_1 -a_2)^2\}\beta_2e^{H_2}+a_2^2c^2-2a_1a_2^2(a_1-a_2)^2c+a_1^2a_2^2(a_1-a_2)^4]=\zeta_2^{*2}.$$

Hence by means of lemma C, we see that the quantity in the brackets [ ] must be of the form $a_1^2(\beta_2e^{H_2}-\gamma)^2$. However, the discriminant of the quadratic equation

$$a_1^2X^2-2\{a_1a_2c+a_1^2a_2(a_1-a_2)^2\}X+a_2^2c^2-2a_1a_2^2(a_1-a_2)^2c+a_1^2a_2^2(a_1-a_2)^4=0,$$

is equal to $16a_1^3a_2^2(a_1-a_2)^2c\neq 0$. This is a contradiction.

Therefore the case (iv) does not occur under a condition that one of the growth of e^{H_1} and e^{H_2} is more rapid than the other.

Case (v). First let us suppose that in (4.5)

$$m(r, e^{H_2})=o(m(r, e^{H_1})) \qquad \text{and} \qquad m(r, e^{H_3})=o(m(r, e^{H_1}))$$

outside a set of finite measure. Then by substituting (4.5) into (2.5) and taking $e^H\equiv\beta_1e^{H_1}$ in (4.11) and (4.13) into account, we have, similarly as in the case (iii),

$$\begin{aligned}\zeta_4-\zeta_2^{*2}=\frac{-4a_1^3a_2^3}{(a_1-a_2)^8}&[a_2(a_1-a_2)\beta_2^3\beta_3e^{3H_2+H_3}-(a_1^2-3a_1a_2+a_2^2)\beta_2^2\beta_3^2e^{2H_2+2H_3}\\&-a_1(a_1-a_2)\beta_2\beta_3^3e^{H_2+3H_3}+a_2^2(a_1-a_2)^3\beta_2^3e^{3H_2}-a_1^2(a_1-a_2)^3\beta_3^3e^{3H_3}\\&+a_1(a_1-a_2)^2(a_1^2-2a_1a_2+2a_2^2)\beta_2^2\beta_3e^{2H_2+H_3}+a_2(a_1-a_2)^2(2a_1^2\\&-2a_1a_2+a_2^2)\beta_2\beta_3^2e^{H_2+2H_3}+a_2(a_1-a_2)^5(a_1^2-a_1a_2-a_2^2)\beta_2^2e^{2H_2}\\&+a_1(a_1-a_2)^5(a_1^2+a_1a_2-a_2^2)\beta_3^2e^{2H_3}+(a_1-a_2)^6(a_1^2+a_2^2)\beta_2\beta_3e^{H_2+H_3}\\&-a_1a_2^2(a_1-a_2)^8\beta_2e^{H_2}-a_1^2a_2(a_1-a_2)^8\beta_3e^{H_3}]=0.\end{aligned} \tag{4.15}$$

Now we shall show the impossibility of the identity (4.15). If $3H_2+H_3\neq 0$, $H_2+3H_3\neq 0$, $2H_2+H_3\neq 0$, $H_2+2H_3\neq 0$, $H_2+H_3\neq 0$, $3H_3-H_2\neq 0$ and $2H_3-H_2\neq 0$, then, by virtue of lemma 1, we have

$$a_1a_2^2(a_1-a_2)^8\beta_2e^{H_2}+da_1^2a_2(a_1-a_2)^8\beta_3e^{H_3}=0,$$

where d is a constant. Since $H_2\not\equiv$const., we get $d\neq 0$ and $H_2-H_3\equiv$const. $(=0)$. Then, writing the identity (4.15) in the form

$$\begin{aligned}&\beta_2\beta_3\{a_2(a_1-a_2)\beta_2^2-(a_1^2-3a_1a_2+a_2^2)\beta_2\beta_3-a_1(a_1-a_2)\beta_3^2\}e^{4H_2}+(a_1-a_2)^2\{a_2^2(a_1\\&-a_2)\beta_2^3+a_1(a_1^2-2a_1a_2+2a_2^2)\beta_2^2\beta_3+a_2(2a_1^2-2a_1a_2+a_2^2)\beta_2\beta_3^2-a_1^2(a_1-a_2)\beta_3^3\}e^{3H_2}\\&+(a_1-a_2)^5\{a_2(a_1^2-a_1a_2-a_2^2)\beta_2^2+(a_1-a_2)(a_1^2+a_2^2)\beta_2\beta_3+a_1(a_1^2+a_1a_2-a_2^2)\beta_3^2\}e^{2H_2}\\&-a_1a_2(a_1-a_2)^8(a_2\beta_2+a_1\beta_3)e^{H_2}=0,\end{aligned}$$

we obtain, by means of lemma B,

$$a_2\beta_2+a_1\beta_3=0$$

and

$$C'\equiv a_2(a_1^2-a_1a_2-a_2^2)\beta_2^2+(a_1-a_2)(a_1^2+a_2^2)\beta_2\beta_3+a_1(a_1^2+a_1a_2-a_2^2)\beta_3^2=0.$$

By substituting $\beta_3=-a_2\beta_2/a_1$ into C', however, we have $C'=a_2^2(a_1-a_2)\beta_2^2\neq 0$. This is a contradiction.

If $3H_2+H_3\equiv 0$ or $H_2+3H_3\equiv 0$ or $2H_2+H_3\equiv 0$ or $H_2+2H_3\equiv 0$ or $H_2+H_3\equiv 0$ or $3H_3-H_2\equiv 0$ or $2H_3-H_2\equiv 0$, then the identity (4.15) is impossible by virtue of lemma B because of $a_2^2(a_1-a_2)^3\beta_2^3\neq 0$, which is a coefficient of e^{3H_2} in (4.15).

If $2H_3-H_2\equiv 0$, then the identity (4.15) is also impossible because of $a_2(a_1-a_2)\beta_2^3\beta_3\neq 0$, which is a coefficient of $e^{3H_2+H_3}$ in (4.15).

Secondly let us suppose that in (4.5)

$$m(r, e^{H_1})=o(m(r, e^{H_3})) \quad \text{and} \quad m(r, e^{H_2})=o(m(r, e^{H_3}))$$

outside a set of finite measure. Then by substituting (4.5) into (2.5) and taking $e^{H}\equiv\beta_3e^{H_3}$ in (4.11) and (4.13) into account, we have, similarly as in the case (iii),

$$\begin{aligned}\zeta_4-\zeta_2^{*2}=-\frac{a_2^3}{a_1^8}(a_1-a_2)^3\,[&a_1(a_1-a_2)\beta_1^3\beta_2e^{3H_1+H_2}+(a_1^2+a_1a_2-a_2^2)\beta_1^2\beta_2^2e^{2H_1+2H_2}\\ &+a_1a_2\beta_1\beta_2^3e^{H_1+3H_2}+a_1^3(a_1-a_2)^2\beta_1^3e^{3H_1}-a_1^3a_2^2\beta_2^3e^{3H_2}\\ &+a_1^2a_2(2a_1^2-2a_1a_2+a_2^2)\beta_1^2\beta_2e^{2H_1+H_2}-a_1^2(a_1-a_2)(a_1^2\\ &+a_2^2)\beta_1\beta_2^2e^{H_1+2H_2}-a_1^5(a_1-a_2)(a_1^2-3a_1a_2+a_2^2)\beta_1^2e^{2H_1}\\ &+a_1^5a_2(a_1^2-a_1a_2-a_2^2)\beta_2^2e^{2H_2}-a_1^6(a_1^2-2a_1a_2+2a_2^2)\beta_1\beta_2e^{H_1+H_2}\\ &-a_1^8a_2(a_1-a_2)^2\beta_1e^{H_1}+a_1^8a_2^2(a_1-a_2)\beta_2e^{H_2}]=0.\end{aligned} \tag{4.16}$$

Now we shall show the impossibility of the identity (4.16). By virtue of the above reasoning, it is sufficient to consider the case $H_1(z)\equiv H_2(z)$, because two coefficients of $e^{3H_1+H_2}$ and e^{3H_1} in (4.16) are not zero. Then, writing the identity (4.16) in the form

$$\begin{aligned}&\beta_1\beta_2\{a_1(a_1-a_2)\beta_1^2+(a_1^2+a_1a_2-a_2^2)\beta_1\beta_2+a_1a_2\beta_2^2\}e^{4H_1}+a_1^2\{a_1(a_1-a_2)^2\beta_1^3\\ &+a_2(2a_1^2-2a_1a_2+a_2^2)\beta_1^2\beta_2-(a_1-a_2)(a_1^2+a_2^2)\beta_1\beta_2^2-a_1a_2^2\beta_2^3\}e^{3H_1}\\ &-a_1^5\{(a_1-a_2)(a_1^2-3a_1a_2+a_2^2)\beta_1^2+a_1(a_1^2-2a_1a_2+2a_2^2)\beta_1\beta_2\\ &-a_2(a_1^2-a_1a_2-a_2^2)\beta_2^2\}e^{2H_1}-a_1^8a_2(a_1-a_2)\{(a_1-a_2)\beta_1-a_2\beta_2\}e^{H_1}=0,\end{aligned}$$

we obtain, by means of lemma B,

$$(a_1-a_2)\beta_1-a_2\beta_2=0$$

and

$$D'\equiv(a_1-a_2)^2\beta_1^2+a_1(a_1^2-2a_1a_2+2a_2^2)\beta_1\beta_2-a_2(a_1^2-a_1a_2-a_2^2)\beta_2^2=0.$$

By substituting $\beta_2=(a_1-a_2)\beta_1/a_2$ into D', however, we have $D'=a_1(a_1-a_2)^2\beta_1^2\neq 0$. This is a contradiction.

Finally let us suppose that in (4.5)

$$m(r, e^{H_1})=o(m(r, e^{H_2})) \quad \text{and} \quad m(r, e^{H_3})=o(m(r, e^{H_2}))$$

outside a set of finite measure. Then the case is analogous to the last case. Hence we have similarly a contradiction.

Therefore the case (v) does not occur under a condition that one of the growth of e^{H_1}, e^{H_2} and e^{H_3} is more rapid than the others.

By virtue of the above discussion in the cases (i), (ii), (iii), (iv) and (v), we conclude

THEOREM 1. *Let R be a regularly branched three-sheeted covering Riemann surface defined by the equation* (1.1). *If $P(R)=4$, then there exist an entire function $f_2(z)$ and an analytic function $f_3(z)$ single-valued and regular with the exception of all the double zeros of $g(z)$ at which $f_3(z)$ has simple poles, such that $f_2(z)^3g(z)$ and $f_3(z)^3g(z)^2$ are two roots of one among the five quadratic equations* (2.6) *with the coefficients* (4.1), (4.2), (4.3), (4.4) *and* (4.5), *respectively.*

Further in the cases (i) *and* (ii) *we have*

$$f^*(z)^3g(z)=e^{H(z)}-\gamma \tag{4.8}$$

or

$$\tilde{f}^*(z)^3g(z)=(e^{H(z)}-\gamma)^2, \tag{4.9}$$

where $f^*(z)$ and $\tilde{f}^*(z)$ are two entire functions and γ is a non-zero constant. Conversely if $g(z)$ satisfies the equation (4.8) *or* (4.9), *then we have $P(R)=4$.*

And the cases (iii), (iv) *and* (v) *do not occur under a condition that one of the growth of e^{H_1}, e^{H_2} and e^{H_3} in* (4.3), (4.4) *and* (4.5) *is more rapid than the others.*

§5. Now we shall give a criterion for $P(R)\leqq 4$, that is,

THEOREM 2. *Let R be a regularly branched three-sheeted covering Riemann surface defined by the equation* (1.1) *with an entire function $g(z)$ satisfying*

$$\tilde{f}(z)^3g(z)=e^{3H(z)}+\varphi_1(z)e^{2H(z)}+\varphi_2(z)e^{H(z)}+\varphi_3(z), \qquad H(z)\not\equiv\text{const.},\ H(0)=0,$$

where $\tilde{f}(z)$, $H(z)$, $\varphi_1(z)$, $\varphi_2(z)$ and $\varphi_3(z)$ are entire functions satisfying

$$m(r, \varphi_j)=o(m(r, e^H)) \qquad (j=1, 2, 3)$$

outside a set of finite measure, and $\varphi_3(z)$ has at least one zero. Then we have $P(R)\leqq 4$.

Proof. In order to prove $P(R)\leqq 4$, from theorem 1 and theorem 2 in [1], it is sufficient to show the impossibility of an identity of the form

$$f^3(e^{3H}+\varphi_1e^{2H}+\varphi_2e^H+\varphi_3)=(e^L-\alpha)(e^L-\beta)^2, \qquad \alpha\beta(\alpha-\beta)\neq 0, \tag{5.1}$$

where $L(z)$ is a non-constant entire function with $L(0)=0$, α and β are two constants and $f(z)$ is a meromorphic function which has zeros and poles possibly at the zeros of order at least 3 of $(e^L-\alpha)(e^L-\beta)^2$ and $e^{3H}+\varphi_1e^{2H}+\varphi_2e^H+\varphi_3$, respectively. If the equation

$$Q_3(h) \equiv h^3 + \varphi_1(z)h^2 + \varphi_2(z)h + \varphi_3(z) = 0$$

is irreducible, then, from lemma 4, we have

$$(5.2) \qquad N_2(r, 0, f^3Q_3(e^H)) \sim 3m(r, e^H) \quad \text{and} \quad N_3(r, 0, f^3Q_3(e^H)) = o(m(r, e^H))$$

outside a set of finite measure. On the other hand we have

$$(5.3) \quad N_2(r, 0, (e^L-\alpha)(e^L-\beta)^2) \sim m(r, e^L) \text{ and } N_3(r, 0, (e^L-\alpha)(e^L-\beta)^2) \sim 2m(r, e^L)$$

outside a set of finite measure. By virtue of the identity (5.1), the comparison of (5.2) and (5.3) yields two contradictory facts each other, that is,

$$m(r, e^L) \sim 3m(r, e^H) \quad \text{and} \quad m(r, e^L) = o(m(r, e^H))$$

outside a set of finite measure. Hence the equation $Q_3(h)=0$ is not irreducible. By the similar discussion as above using lemma 4 and lemma 5, the identity (5.1) must reduce to an identity

$$f^3(e^H-\varphi_1^*)(e^H-\varphi_2^*)^2 = (e^L-\alpha)(e^L-\beta)^2,$$

where $\varphi_1^*(z)$ and $\varphi_2^*(z)$ are two entire functions satisfying $m(r, \varphi_j^*) = o(m(r, e^H))$ $(j=1, 2)$ outside a set of finite measure. Hence by means of lemma C we have

$$(5.4) \qquad m(r, e^H) \sim m(r, e^L)$$

outside a set of finite measure. Further we have

$$T(r, f) = O(T(r, e^H) + T(r, e^L))$$

and

$$\begin{aligned} &N(r, \infty, f'/f) \leqq N(r, 0, f) + N(r, \infty, f) \\ &\leqq N_1(r, 0, e^L-\alpha) + N_1(r, 0, e^L-\beta) + N_1(r, 0, e^H-\varphi_1^*) + N_1(r, 0, e^H-\varphi_2^*) \\ &= o(m(r, e^H) + m(r, e^L)) \end{aligned}$$

outside a set of finite measure. Thus we obtain

$$\begin{aligned} T(r, f'/f) &= m(r, f'/f) + N(r, \infty, f'/f) \leqq O(\log rT(r, f)) + N(r, \infty, f'/f) \\ &= o(m(r, e^H) + m(r, e^L)), \end{aligned}$$

so that

$$(5.5) \qquad T(r, f'/f) = o(m(r, e^H) + m(r, e^L))$$

outside a set of finite measure.

Now we shall prove the impossibility of the identity (5.1) under (5.4) and (5.5). By differentiating both sides of (5.1) and setting $\eta_1 = -(\alpha+2\beta)$, $\eta_2 = 2\alpha\beta+\beta^2$ and $\eta_3 = -\alpha\beta^2$, we obtain

$$\begin{aligned} &f^3[3(f'/f+H')e^{3H} + (3\varphi_1 f'/f + 2\varphi_1 H' + \varphi_1')e^{2H} + (3\varphi_2 f'/f + \varphi_2 H' + \varphi_2')e^H + 3\varphi_3 f'/f + \varphi_3'] \\ &\qquad = L'(3e^{3L} + 2\eta_1 e^{2L} + \eta_2 e^L), \end{aligned}$$

and again by using the identity (5.1), we get

$$\begin{aligned}(5.6)\quad &3(f'/f+H'-L')e^{3H+3L}+\eta_1(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1'-2\varphi_1 L')e^{2H+2L}\\ &+\eta_2(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-\varphi_2 L')e^{H+L}+\eta_1(3f'/f+3H'-2L')e^{3H+2L}\\ &+(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1'-3\varphi_1 L')e^{2H+3L}+\eta_2(3f'/f+3H'-L')e^{3H+L}\\ &+(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-3\varphi_2 L')e^{H+3L}+\eta_2(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1'-\varphi_1 L')e^{2H+L}\\ &+\eta_1(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-2\varphi_2 L')e^{H+2L}+3\eta_3(f'/f+H')e^{3H}\\ &+\eta_3(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1')e^{2H}+\eta_3(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2')e^{H}\\ &+(3\varphi_3 f'/f+\varphi_3'-3\varphi_3 L')e^{3L}+\eta_1(3\varphi_3 f'/f+\varphi_3'-2\varphi_3 L')e^{2L}\\ &+\eta_2(3\varphi_3 f'/f+\varphi_3'-\varphi_3 L')e^{L}+\eta_3(3\varphi_3 f'/f+\varphi_3')=0.\end{aligned}$$

Here we note from (5.4) that all the functions $\nu H(z)+\mu L(z)$, $|\nu|\neq|\mu|$; $\nu, \mu =\pm 1, \pm 2, \pm 3$, are not constants and further satisfy

$$(5.7)\qquad T(r, a)=o(m(r, e^{\nu H+\mu L}))$$

outside a set of finite measure, where $a(z)$ is a meromorphic function satisfying $T(r, a)=o(m(r, e^{H}))$ outside a set of finite measure.

In the first place assume that $a_1(z)\equiv 3(f'(z)/f(z)+H'(z)-L'(z))\not\equiv 0$. From (5.4), (5.5) and (5.7) we can apply lemma 1 to the identity (5.6). Therefore lemma 1 gives

$$a_1(z)e^{3H(z)+3L(z)}+c_2a_2(z)e^{2H(z)+2L(z)}+c_3a_3(z)e^{H(z)+L(z)}+c_0a_0(z)=0,$$

where c_2, c_3, c_0 are constants and $a_2(z)\equiv\eta_1(3\varphi_1(z)f'(z)/f(z)+2\varphi_1(z)H'(z)+\varphi_1'(z)-2\varphi_1(z)L'(z))$, $a_3(z)\equiv\eta_2(3\varphi_2(z)f'(z)/f(z)+\varphi_2(z)H'(z)+\varphi_2'(z)-\varphi_2(z)L'(z))$, $a_0(z)\equiv\eta_3(3\varphi_3(z)\cdot f'(z)/f(z)+\varphi_3'(z))$. Since $T(r, a_j)=o(m(r, e^{H}))$ $(j=0, 1, 2, 3)$ outside a set of finite measure, we have

$$m(r, e^{H+L})=o(m(r, e^{H}))$$

outside a set of finite measure. Since $\eta_3\neq 0$, writing the identity (5.6) in the form

$$\begin{aligned}&3\eta_3(f'/f+H')e^{3H}+[\eta_2(3f'/f+3H'-L')e^{H+L}+\eta_3(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1')]e^{2H}\\ &+[\eta_1(3f'/f+3H'-2L')e^{2H+2L}+\eta_2(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1'-\varphi_1 L')e^{H+L}\\ &+\eta_3(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2')]e^{H}+[3(f'/f+H'-L')e^{3H+3L}+\eta_1(3\varphi_1 f'/f+2\varphi_1 H'\\ &+\varphi_1'-2\varphi_1 L')e^{2H+2L}+\eta_2(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-\varphi_2 L')e^{H+L}+\eta_3(3\varphi_3 f'/f+\varphi_3')]\\ &+[(3\varphi_1 f'/f+2\varphi_1 H'+\varphi_1'-3\varphi_1 L')e^{3H+3L}+\eta_1(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-2\varphi_2 L')e^{2H+2L}\\ &+\eta_2(3\varphi_3 f'/f+\varphi_3'-\varphi_3 L')e^{H+L}]e^{-H}+[(3\varphi_2 f'/f+\varphi_2 H'+\varphi_2'-3\varphi_2 L')e^{3H+3L}\\ &+\eta_1(3\varphi_3 f'/f+\varphi_3'-2\varphi_3 L')e^{2H+2L}]e^{-2H}+[(3\varphi_3 f'/f+\varphi_3'-3\varphi_3 L')e^{3H+3L}]e^{-3H}=0,\end{aligned}$$

lemma B gives

$$f'(z)/f(z)+H'(z)=0, \quad \text{that is,} \quad f(z)=de^{-H(z)},$$

where d is a non-zero constant. Then the identity (5.1) reduces to

$$(d^3-\eta_3)e^{3H}+(d^3\varphi_1-\eta_2 e^{H+L})e^{2H}+(d^3\varphi_2-\eta_1 e^{2H+2L})e^{H}+d^3\varphi_3-e^{3H+3L}=0.$$

Hence lemma B gives

$$d^3=\eta_3 \quad \text{and} \quad d^3\varphi_3(z)=e^{3H(z)+3L(z)}.$$

Since $\varphi_3(z)$ has at least one zero, this is impossible.

Next assume that $a_1(z)\equiv 0$. Then we get $f(z)=de^{L(z)-H(z)}$, where d is a non-zero constant. Here the identity (5.1) reduces to

$$(5.8) \qquad (1-d^3)e^{3H+3L}+\eta_1 e^{3H+2L}+\eta_2 e^{3H+L}+\eta_3 e^{3H}-d^3\varphi_1 e^{2H+3L}-d^3\varphi_2 e^{H+3L}-d^3\varphi_3 e^{3L}=0.$$

Since $\eta_3\neq 0$, lemma 1 gives

$$(5.9) \qquad \eta_3 e^{3H(z)}-c_2 d^3\varphi_3(z)e^{3L(z)}+c_3(1-d^3)e^{3H(z)+3L(z)}=0,$$

where c_2 and c_3 are constants. If $c_2c_3(1-d^3)\neq 0$, then writing the identity (5.9) in the form

$$\eta_3 e^{3H-3L}+c_3(1-d^3)e^{3H}=c_2 d^3\varphi_3,$$

and using lemma A, we have

$$c'\eta_3 e^{3H-3L}+c_3'c_3(1-d^3)e^{3H}=0, \quad \text{that is,} \quad c_1'\eta_3 e^{-3L}+c_3'c_3(1-d^3)=0,$$

where c_1' and c_3' are constants which are not all zero. This contradicts $L(z)\not\equiv$const.. If $c_2=c_3=0$, then the identity (5.9) is clearly impossible because of $\eta_3\neq 0$. If $c_2=0$ and $c_3(1-d^3)\neq 0$, then the identity (5.9) reduces to $\eta_3+c_3(1-d^3)e^{3L}=0$, which is impossible. If $c_2\neq 0$ and $c_3(1-d^3)=0$, then we have

$$m(r, e^{H-L})=o(m(r, e^H))$$

outside a set of finite measure. The identity (5.8) reduces to

$$e^{3L-3H}(1-d^3)e^{6H}+e^{2L-2H}(\eta_1-d^3\varphi_1 e^{L-H})e^{5H}+e^{L-H}(\eta_2-d^3\varphi_2 e^{2L-2H})e^{4H}$$
$$+(\eta_3-d^3\varphi_3 e^{3L-3H})e^{3H}=0.$$

Hence lemma B gives

$$d^3=1 \quad \text{and} \quad \varphi_3(z)=\eta_3 e^{3H(z)-3L(z)}.$$

Since $\varphi_3(z)$ has at least one zero, this is impossible.

Thus we have proved the impossibility of the identity (5.1), that is, the validity of theorem 2.

§6. Let R and S be two regularly branched three-sheeted covering Riemann surfaces defined by two equations $y^3=G(z)$ and $u^3=g(w)$, respectively, where $G(z)$ and $g(w)$ are two entire functions having no zero other than an infinite number of simple or double zeros. Then Mutō [3] has established the following perfect condition for the existence of analytic mappings from R into S:

THEOREM A. *If there exists an analytic mapping φ from R into S, then there exists an entire function $h(z)$ satisfying $f_2(z)^3G(z)=g\circ h(z)$ or $f_3(z)^3G(z)^2=g\circ h(z)$, where $f_2(z)$ is an entire function and $f_3(z)$ is a single-valued regular function* *excepting at most all the double zeros of* $G(z)$ *at which* $f_3(z)$ *has simple poles. The converse holds also good.*

Suppose that $P(R)=P(S)=6$. Then by a characterization, which has been given by Hiromi and the author [1], of R with $P(R)=6$, we can put

$$F(z)^3G(z)=(e^{H(z)}-\alpha)(e^{H(z)}-\beta)^2, \qquad H(z)\not\equiv\text{const.}, \tag{6. 1}$$
$$H(0)=0, \qquad \alpha\beta(\alpha-\beta)\neq 0,$$

with two entire functions $F(z)$ and $H(z)$ and two constants α and β, and

$$f(w)^3g(w)=(e^{L(w)}-\gamma)(e^{L(w)}-\delta)^2, \qquad L(w)\not\equiv\text{const.}, \tag{6. 2}$$
$$L(0)=0, \qquad \gamma\delta(\gamma-\delta)\neq 0,$$

with two entire functions $f(w)$ and $L(w)$ and two constants γ and δ.
Now we shall prove the following theorem and its corollary:

THEOREM 3. *Let R and S be two regularly branched three-sheeted covering Riemann surfaces with* $P(R)=P(S)=6$. *Then there exists an analytic mapping* φ *from R into S if and only if there exists an entire function* $h(z)$ *satisfying one of the conditions*

$$\text{(a)} \qquad H(z)=L\circ h(z)-L\circ h(0), \qquad \gamma=e^{L\circ h(0)}\alpha, \quad \delta=e^{L\circ h(0)}\beta,$$
$$\text{(a}'\text{)} \qquad H(z)=L\circ h(z)-L\circ h(0), \qquad \gamma=e^{L\circ h(0)}\beta, \quad \delta=e^{L\circ h(0)}\alpha,$$
$$\text{(b)} \qquad H(z)=-L\circ h(z)+L\circ h(0), \qquad \alpha\gamma=e^{L\circ h(0)}, \quad \beta\delta=e^{L\circ h(0)},$$
$$\text{(b}'\text{)} \qquad H(z)=-L\circ h(z)+L\circ h(0), \qquad \alpha\delta=e^{L\circ h(0)}, \quad \beta\gamma=e^{L\circ h(0)},$$

where R and S are defined by $y^3=G(z)$ *and* $u^3=g(w)$ *with* $G(z)$ *and* $g(w)$ *satisfying* (6.1) *and* (6.2), *respectively.*

COROLLARY. *Let R be a regularly branched three-sheeted covering Riemann surface with* $P(R)=6$ *defined by*

$$y^3=f(z)^3(e^{H(z)}-\gamma)(e^{H(z)}-\delta)^2, \qquad \gamma\delta(\gamma-\delta)\neq 0, \qquad H(0)=0,$$

with a non-constant entire function $H(z)$ *and a meromorphic function* $f(z)$. *Let* φ *be an analytic mapping from R into itself. Then* φ *is a univalent conformal mapping from R onto itself and the corresponding entire function* $h(z)$ *is a linear function of the form*

$$e^{2\pi ip/q}z+b$$

with a suitable rational number p/q.

Proof of Theorem 3. First suppose that there exists an analytic mapping φ from R into S. Then from theorem A there exists an entire function $h(z)$ satisfying either $f_2(z)^3G(z)=g\circ h(z)$ or $f_3(z)^3G(z)^2=g\circ h(z)$, where $f_2(z)$ and $f_3(z)$ are two functions having the properties described in theorem A, respectively.

Case I. $f_2(z)^3G(z)=g\circ h(z)$. In the case from (6.1) and (6.2) we get an equation

$$f^*(z)^3(e^{H(z)}-\alpha)(e^{H(z)}-\beta)^2=(e^{L\circ h(z)}-\gamma)(e^{L\circ h(z)}-\delta)^2, \tag{6.3}$$

where $f^*(z)=f_2(z)f\circ h(z)/F(z)$ is a meromorphic function having zeros and poles possibly at the zeros of order at least 3 of $(e^{L\circ h}-\gamma)(e^{L\circ h}-\delta)^2$ and $(e^H-\alpha)(e^H-\beta)^2$, respectively. Evaluating similarly as in § 5, we have

$$m(r, e^H)\sim m(r, e^{L\circ h}) \tag{6.4}$$

and

$$T(r, f^{*\prime}/f^*)=o(m(r, e^H)+m(r, e^{L\circ h})) \tag{6.5}$$

outside a set of finite measure. Hence this case is similarly treated as in the process of proof of theorem 2. Therefore from the reasoning of § 5 it is sufficient to consider the following two cases:

(I. I) $m(r, e^{H+L\circ h})=o(m(r, e^H))$ outside a set of finite measure, and $f^*(z)=de^{-H(z)}$, where d is a non-zero constant. Then the identity (6.3) reduces to

$$(d^3-\eta_3)e^{3H}+(d^3\zeta_1-\eta_2e^{H+L\circ h})e^{2H}+(d^3\zeta_2-\eta_1e^{2H+2L\circ h})e^H+d^3\zeta_3-e^{3H+3L\circ h}=0,$$

where $\zeta_1=-(\alpha+2\beta)$, $\zeta_2=2\alpha\beta+\beta^2$, $\zeta_3=-\alpha\beta^2$, $\eta_1=-(\gamma+2\delta)$, $\eta_2=2\gamma\delta+\delta^2$ and $\eta_3=-\gamma\delta^2$. Hence lemma B gives

$$d^3=\eta_3,\ d^3\zeta_1=\eta_2e^{H(z)+L\circ h(z)},\ d^3\zeta_2=\eta_1e^{2H(z)+2L\circ h(z)}\ \text{ and }\ d^3\zeta_3=e^{3H(z)+3L\circ h(z)}.$$

Therefore the function $H(z)+L\circ h(z)$ must be the constant $L\circ h(0)$. Then we have

$$\gamma\delta^2(\alpha+2\beta)=(2\gamma\delta+\delta^2)e^{L\circ h(0)},\ \gamma\delta^2(2\alpha\beta+\beta^2)=(\gamma+2\delta)e^{2L\circ h(0)}\ \text{ and }\ \gamma\delta^2\alpha\beta^2=e^{3L\circ h(0)}.$$

These relations yield $\alpha\gamma=e^{L\circ h(0)}$ and $\beta\delta=e^{L\circ h(0)}$. Thus we attain to the case (b) in our theorem.

(I. II) $f^*(z)=de^{L\circ h(z)-H(z)}$, where d is a non-zero constant, and $m(r, e^{H-L\circ h})=o(m(r, e^H))$ outside a set of finite measure. Then the identity (6.3) reduces to

$$e^{3L\circ h-3H}(1-d^3)e^{6H}+e^{2L\circ h-2H}(\eta_1-d^3\zeta_1e^{L\circ h-H})e^{5H}$$
$$+e^{L\circ h-H}(\eta_2-d^3\zeta_2e^{2L\circ h-2H})e^{4H}+(\eta_3-d^3\zeta_3e^{3L\circ h-3H})e^{3H}=0.$$

We deduce from lemma B that the function $L\circ h(z)-H(z)$ is the constant $L\circ h(0)$ and the following relations hold:

$$\gamma+2\delta=(\alpha+2\beta)e^{L\circ h(0)},\ 2\gamma\delta+\delta^2=(2\alpha\beta+\beta^2)e^{2L\circ h(0)}\ \text{ and }\ \gamma\delta^2=\alpha\beta^2e^{3L\circ h(0)}.$$

These relations yield $\gamma=e^{L\circ h(0)}\alpha$ and $\delta=e^{L\circ h(0)}\beta$. Thus we attain to the case (a) in our theorem.

Case II. $f_3(z)^3G(z)^2=g\circ h(z)$. In the case from (6.1) and (6.2) we get an equation

$$f^*(z)^3(e^{H(z)}-\alpha)^2(e^{H(z)}-\beta)=(e^{L\circ h(z)}-\gamma)(e^{L\circ h(z)}-\delta)^2, \tag{6.6}$$

where $f^*(z)=f_3(z)(e^{H(z)}-\beta)f\circ h(z)/F(z)^2$. Here $f_3(z)$ has simple poles at most at the double zeros of $G(z)$, that is, at the simple zeros of $e^{H(z)}-\beta$ or at the double zeros of $e^{H(z)}-\alpha$. However from the equation (6.6) and lemma C we see that $f_3(z)$ has simple poles at almost all simple zeros of $e^{H(z)}-\beta$. Hence $f^*(z)$ satisfies the condition (6.5). And in this case the relation (6.4) holds also true. Therefore by virtue of the case I, we attain to the cases (a′) and (b′) in our theorem.

Conversely, suppose that there exists an entire function $h(z)$ satisfying (a) or (b) or (a′) or (b′). Then we have

$$\left(\frac{e^{L\circ h(0)}F(z)}{f\circ h(z)}\right)^3 G(z)=g\circ h(z)$$

if (a) is the case, or

$$\left(\frac{-e^{L\circ h(z)}F(z)}{\sqrt[3]{\alpha\beta^2}\, e^{H(z)} f\circ h(z)}\right)^3 G(z)=g\circ h(z)$$

if (b) is the case, or

$$\left(\frac{e^{L\circ h(0)}F(z)^2}{(e^{H(z)}-\beta)f\circ h(z)}\right)^3 G(z)^2=g\circ h(z)$$

if (a′) is the case, or

$$\left(\frac{-e^{L\circ h(0)}F(z)^2}{\sqrt[3]{\alpha^2\beta}\, e^{H(z)}(e^{H(z)}-\beta)f\circ h(z)}\right)^3 G(z)^2=g\circ h(z)$$

if (b′) is the case. Since zeros of $G(z)$ are all simple or double, $e^{L\circ h(0)}F(z)/f\circ h(z)$ and $-e^{L\circ h(0)}F(z)/(\sqrt[3]{\alpha\beta^2}\, e^{H(z)} f\circ h(z))$ are two entire functions and $e^{L\circ h(0)}F(z)^2/((e^{H(z)}-\beta)$ $f\circ h(z))$ and $-e^{L\circ h(0)}F(z)^2/(\sqrt[3]{\alpha^2\beta}\, e^{H(z)}(e^{H(z)}-\beta)f\circ h(z))$ are two single-valued functions having the properties of $f_3(z)$ in theorem A. Therefore from theorem A there exists an analytic mapping φ from R into S. Thus we have just proved theorem 3.

Proof of Corollary. By virtue of theorem 3 there exists an entire function $h(z)$ satisfying either $H(z)=H\circ h(z)-H\circ h(0)$ or $H(z)=-H\circ h(z)+H\circ h(0)$. Then we have $h(z)=az+b$ by using lemma D if $H(z)$ is a transcendental entire function or directly if $H(z)$ is a polynomial. This implies the first part of corollary, that is, φ is a univalent conformal mapping from R onto itself.

By considering its iteration $\varphi_n=\varphi\circ\varphi_{n-1}$ as in the proof of theorem 2 in Ozawa [7], we can say that

$$h(z)=e^{2\pi i p/q}z+b$$

with a suitable rational number p/q. Q.E.D.

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