ON RADIAL SLIT DISC MAPPINGS

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§ 1. Introduction.

1. In our previous paper [13] we discussed canonical conformal mappings of a plane domain and some properties of its image domain. In the present paper we shall deal with supplementary problems related to radial slit disc mappings.

Let Ω be a plane domain and C be a boundary component of it. As was remarked by Grötzsch [4], the general radial slit disc mapping function has no extremal property about the minimum modulus on C contrary to the case that C is an isolated continuum, even if the extremal radius of C is finite [10, 7]. We intend to define a functional of a normalized univalent function in Ω which is regarded as an essential minimum modulus on C with respect to the extremal distance and to show that the radial slit disc mapping function is the unique function maximizing the quantity among the family of the normalized schlicht functions. To the purpose we shall remark the following property of the incisions of the image domain of Ω under the radial slit disc mapping which is a direct result from Ohtsuka's theorem [6], pp. 132: The extremal distance of the incisions from a compact disc in the image domain is infinite.

Secondly we shall give a characterization of the radial slit domain with its extremal radius infinite.

Recently Oikawa [8] obtained another interesting characterization as follows: The radial slit disc mapping function is the unique function minimizing the maximum modulus on C among the univalent functions which map the boundary components $\partial\Omega - C$ onto a quasi-minimal set [13] of radial slits. A similar extremal problem for the annulus was treated by Strebel [11].

§ 2. Extremal distances and metrics.

2. Let Γ be a family of locally rectifiable curves in Ω which is called simply a curve family Γ and $P(\Gamma)$ be the class of (measurable) admissible metrics with respect to the L-normalization i.e.

$$(1) \qquad \int_\gamma \rho |dz| \geqq 1, \qquad \gamma \in \Gamma.$$

The *module* of the curve family Γ is defined by

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$$\operatorname{mod} \Gamma = \inf_{\rho \in P(\Gamma)} \|\rho\|_\Omega^2 = \inf_{\rho \in P(\Gamma)} \iint_\Omega \rho^2 dxdy,$$

the *extremal length* denoted by $\lambda(\Gamma)$ is defined by its reciprocal and $\lambda(\Gamma)$ is called the *extremal distance* of two sets if Γ is the family of curves joining them.

We state Hersch's theorem as a lemma which is essential in the sequel.

LEMMA 1. (Hersch [5]). *Let $\{\Gamma_n\}$ be at most a countable number of curve families. Then*

$$\operatorname{mod} (\bigcup_n \Gamma_n) \leqq \sum_n \operatorname{mod} \Gamma_n$$

3. Let $P^*(\Gamma)$ be the closure of the intersection of $P(\Gamma)$ with the l_2-space which is called a generalized admissible class. $P^*(\Gamma)$ is a closed convex set. Strebel [11] showed that there exists always a unique metric ρ_0 called generalized extremal metric which satisfies

$$\|\rho\|^2 = \min_{\rho \in P^*(\Gamma)} \|\rho\|^2 = \operatorname{mod} \Gamma$$

if $P^*(\Gamma)$ is not void.

The deviation of $\rho \in P^*(\Gamma)$ from ρ_0 is evaluated by the following inequality:

$$\|\rho - \rho_0\|^2 \leqq \|\rho\|^2 - \|\rho_0\|^2 \tag{2}$$

which was given with a numerical constant 1/2 in [13]. In fact we can take a competing function $(\rho_0 + \varepsilon\rho)/(1+\varepsilon)$ from the convexity of $P^*(\Gamma)$ for $\varepsilon \geqq 0$ and the standard argument shows it.

We call a metric $\rho \in P^*(\Gamma)$ *measurable* on Γ if the integral in (1) is defined for all $\gamma \in \Gamma$. From the method of Ohtsuka [6] mentioned in the introduction we have

THEOREM 1. *Let $\rho \in P^*(\Gamma)$ be measurable on Γ and Λ be a subfamily of Γ defined by*

$$\Lambda = \left\{ \gamma \,\middle|\, \int_\gamma \rho |dz| < 1,\ \gamma \in \Gamma \right\}.$$

Then the module of Λ is equal to zero.

Proof. There exists a sequence $\{\rho_\nu\}$ contained in $P(\Gamma)$ converging to ρ strongly. Let Λ_n be the subfamily of Λ defined by

$$\Lambda_n = \left\{ \gamma \,\middle|\, \int_\gamma \rho |dz| < 1 - \frac{1}{n},\ \gamma \in \Lambda \right\}$$

Then a metric $\mu_\nu = n|\rho_\nu - \rho|$ is admissible for Λ_n and hence $\operatorname{mod} \Lambda_n = 0$, since $\|\mu_\nu\|^2 \to 0$. We get $\operatorname{mod} \Lambda = 0$ from Lemma 1 because $\Lambda = \bigcup \Lambda_n$.

A similar class defined by the property in Theorem 1 was considered by Fuglede [3] in the more general situations. It is easily verified that his class essentially coincides with out class $P^*(\Gamma)$.

4. Let ρ_1 and ρ_2 belong to $P^*(\Gamma)$. We put $\rho_1 \vee \rho_2 = \max(\rho_1, \rho_2)$ which is easily verified to belong to $P^*(\Gamma)$. Being concerned with the extremal distrance between two sets, we take the family of curves joining them. Let A and B be two sets on the complex sphere. We mean by a curve γ: $z(t)$, $0<t<1$, starting from the set A such a curve that any neighbourhood of A taken in the sphere contains its suitable subarc $z=z(t)$, $0<t<t_0$. Such terminologies as joining, ending etc. can be defined by their neighbourhoods.

As to an open Riemann surface R, we take a compactification of R which makes each boundary component an ideal point, that is so called Stoilow's compactification of R and written by $\hat{R}$ ([2], Chap. I). For two disjoint closed sets A and B on $\hat{R}$, the above terminologies are defined similarly. Especially, let C be a closed set of ideal point. Then we can construct a sequence of open sets $\{\Delta_n\}$ on R such that i) $\Delta_n \cup C \supset \bar{\Delta}_{n+1}$, ii) Δ_n consists of a finite number of subdomains of R whose relative boundaries are a finite number of closed Jordan analytic curves, iii) $R-\bar{\Delta}_n$ is connected and iv) $\cap \bar{\Delta}_n = C$, we call $\{\Delta_n\}$ a defining sequence of C. An exhaustion of R, given by $R_n = R-\bar{\Delta}_n$ is called an exhaustion of R in the direction to C.

We define the distance of A and B by the quantity

$$d_\rho(A, B) = \inf_{\gamma\in\Gamma} \int_\gamma \rho|dz|,$$

if ρ is measurable on the curve family Γ joining A and B, and define a new metric $\hat{\rho}$ by

$$\hat{\rho} = \hat{\rho}_A = \begin{cases} 0 \text{ on } G(A) = \{z | d_\rho(A, z) > 1\}, \\ \rho \text{ elsewhere}, \end{cases}$$

we call $\hat{\rho}$ a modified metric with respect to A.

LEMMA 2. *Let Γ be the family of curves joining A and B within Ω. If $\rho \in P^*(\Gamma)$ and if ρ is measurable on Γ then so does $\hat{\rho}$ constructed above.*

Proof. We first remark that the function $d_\rho(A, z)$ is lower semicontinuous in Ω and the set $G(A)$ defined above is open. In fact, let $\{z_\nu\}$ be a sequence converging to z_0 satisfying $d_\rho(A, z_\nu) \to d_0$. Then we have $d_\rho(A, z_0) \leqq d_0$ which is shown by the similar method due to Strebel [12, pp. 10–11]. Next we take such a compact set K within $G(A)$ that $\|\rho\|_{G(A)} < \|\rho\|_K + \varepsilon/2$. From Theorem 1 there exists a metric satisfying $\|\mu\|_\Omega < \varepsilon/2$, where Λ is the curve family defined in Theorem 1, if $\Lambda \neq \phi$. If $\Lambda = \phi$, we take $\mu \equiv 0$. We put

$$\rho_\varepsilon = \begin{cases} 0 \text{ on } K, \\ \rho \vee \mu \text{ elsewhere}. \end{cases}$$

It is easily verified that ρ_ε belongs to $P(\Gamma)$ and $\|\hat{\rho} - \rho_\varepsilon\| < \varepsilon$, which implies the assertion.

§ 3. Minimal slit discs.

5. We introduce a quantity called the extremal radius due to Strebel [12]. Let a closed disc $|z-a|\leqq q$ be contained in Ω and Γ_q be a curve family joining the disc and C. Then a quantity $2\pi(\lambda(\Gamma_q)+\log q)$ increases with ressect to decreasing q and its limit value is denoted by $\log R(a, C)$ $(\leqq\infty)$ [7, 12]. $R(a, C)$ is called the extremal radius of C with respect to a.

We now state its relation to the radial slit disc mapping. The general radial slit disc mapping was dealt by Strebel [12], Reich [9] and recently by the author [13]. Let $\{\Omega_n\}$ be an exhaustion of Ω in the direction to C and C_n be its boundary curve enclosing C. Then there exists a unique function g_{aC_n} with the normalization $g_{aC_n}(a)=1-g'_{aC_n}(a)=0$ and mapping Ω_n onto a minimal radial slit disc with its radius $R(a, C_n)$. In the general case we give the following Strebel-Reich's

THEOREM 2. *If $R(a, C)$ is finite, then there exists a unique function $g_{aC}(z)$ with the following property*:

i) $\|g'_{aC}/g_{aC}-g'_{aC_n}/g_{aC_n}\|_{\Omega_n}\to 0$ *as* $n\to\infty$, *where* g_{aC_n} *is defined above.*

ii) *The images of the boundary components other than C under g_{aC} are a quasi-minimal set of radial slits.*

iii) *The image of C is a circle with radius $Q=R(a, C)$ having possible incisions of angular measure zero emanating from it.*

iv) *The module of the curve family joining a small circle $|z|=q$ and $g_{aC}(C)$ is equal to* $2\pi/\log(Q/q)$ [9, 12, 13].

We mean by a quasi-minimal set a set of slit whose compact subset is minimal [13].

We now show a property of the incisions of $g_{aC}(\Omega)$.

THEOREM 3. *Let a closed disc $|w|\leqq q$ be contained in the image domain $g_{aC}(\Omega)$ and let Λ_q be the family of curves joining the circle $|w|=q$ and the points of the outer boundary component contained in the disc $|w|<R(a, C)$. Then the module of Λ_q is equal to zero. In other words the extremal distance of the set of all incisions is infinite from the circle.*

We have already shown an altenative form of Ohtsuka's theorem [6] for metrics in no. 3 and this can be proved word for word as in his proof, but we give a simple proof adapting for the present situation.

Proof. Let $\{\Omega_\nu\}$ be an exhaustion of Ω in the direction to C and g_{aC_ν} be its radial slit disc mapping function, where C_ν is the distinguished boundary curve enclosing C. Let C_q be the inverse image of the circle $|w|=q$ under g_{aC}. We denote by Λ_q^n the curve family joining C_q and C:

$$\Lambda_q^n=\left\{\gamma\,\middle|\,\overline{\lim_{z\in\gamma}}\log|g_{aC}(z)|<\log R(a, C)-\frac{1}{n}\right\}.$$

We define the value of g_{aC_ν} by the constant $R(a, C_\nu)$ in $\Omega-\Omega_\nu$ and put $u_\nu=\log|g_{aC_\nu}/g_{aC}|$. Then because of the uniform convergence of g_{aC_ν} on C_q and of the convergence of $R(a, C_\nu)$ to $R(a, C)$ the oscillation of u_n on $\gamma\in\Lambda_q^n$ exceeds $1/2n$ for sufficiently large ν and hence the metric $\rho_\nu=2n|\operatorname{grad} u_\nu|$ is admissible for Λ_q^n. The property i) in Theorem 2 shows that $\operatorname{mod}\Lambda_q^n=0$ and we get $\operatorname{mod}\Lambda=0$ by Lemma 1, since $\Lambda=\cup\Lambda_q^n$.

By the above theorem we have easily a simple proof of the Strebel's result about the image domain $g_{aC}(\Omega)$ [11], which seemed to contain a gap in the original paper and was proved alternatively in our previous paper [13].

Corollary 1. *Let a closed disc $|w|\leqq q$ be contained in the image domain of Ω under a radial slit disc mapping function g_{aC} with a finite radius $R(a, C)$. Let Γ_q' be a curve family joining two circles with radii q and $R(a, C)$ in it. Then the module of Γ_q' is equal to $2\pi/\log(R(a, C)/q)$.*

Proof. It is obvious from the following inequality due to Lemma 1:

$$\operatorname{mod}\Gamma_q'\leqq\operatorname{mod}\Gamma_q\leqq\operatorname{mod}\Gamma_q'+\operatorname{mod}\Lambda_q,$$

where Γ_q and Λ_q are the curve families joining $|w|=q$ with the outer boundary and the possible incisions respectively. Hence $\operatorname{mod}\Gamma_q'=\operatorname{mod}\Gamma_q$ since $\operatorname{mod}\Lambda_q=0$.

We called the image domain under g_{aC} a quasi-minimal radial slit disc and gave a geometrical characterization in [13]. Summing up the characterizations given by Strebel [11, 12] and by us, we have from Theorems 2 and 3

Corollary 2. *Let Ω be a bounded domain and Q be the least upper bound of $|z|$ in Ω. Then Ω is a quasi-minimal radial slit disc, if any two of the following three conditions are complied:*

i) *$\partial\Omega-C$ is a quasi-minimal set of radial slits.*

ii) *$\operatorname{mod}\Gamma_q=2\pi/\log(Q/q)$, where Γ_q is defined in Theorem 2.*

iii) *The module of its subfamily Λ_q of curves joining the circle $|z|=q$ and the boundary points contained in the disc $|z|<Q$ is equal to zero.*

Conversely all the three conditions are valid if Ω is a quasi-minimal slit disc with radius Q.

Proof. We only show the characterizations, since the necessity is contained in Theorems 2 and 3. The characterization by i) and ii) was given by Strebel [12, 13] and that by ii) and iii) coincides with the condition of Theorem 15 in [13] if we use the arguments in Corollary 1. Thus we assume the conditions i) and iii). Then we have $\operatorname{mod}\Lambda_q=2\pi/\log(Q/q)$. Indeed the metric $\rho=(|z|\log(Q/q))^{-1}$ belongs to $P^*(\Gamma_q)$ from iii), since $0\in P^*(\Gamma_q)$ and $\rho^\vee 0=\rho$, and hence $\operatorname{mod}\Gamma_q\leqq 2\pi/\log(Q/q)$. Next from Strebel's inequality [11, 13]

$$\operatorname{mod}\Gamma_q\geqq\int_0^{2\pi}\frac{d\theta}{l(\theta)}, \tag{3}$$

where $l(\theta)$ is the logarithmic length of the segment from the circle to the outer boundary component of Ω lying on the ray $z=te^{i\theta}$, $t\geqq 0$. Since $l(\theta)\leqq\log(Q/q)$, we get $\mathrm{mod}\,\Gamma_q\geqq 2\pi/\log(Q/q)$, which implies the condition ii).

We shall give another characterization by means of an extremal property in § 4.

6. These results are easily generalized to the potentials on Riemann surfaces. In fact let R be a Riemann surface and C be a closed set of $\hat{R}-R$. Let a be a point in R with its value of a local parameter zero. Then a potential function $P(p, a, C)$ can be constructed as follows: Let $\{R_n\}$ be an exhaustion of R in the direction to C. Denoting by C_n the relative boundary of R_n separating C, we first make a potential function $P(p, a, C_n)$ such that it is harmonic in R_n-a, $P(p, a, C_n)+\log|z|$ is harmonic and is zero at a, $P(p, a, C_n)$ is constant on C_n and its normal derivative vanishes on the ideal boundary ∂R_n-C_n. Such a function can be constructed by means of Sario's linear operator method [2]. The constant value of $P(p, a, C_n)$ on C_n coincides with $\log R(a, C_n)$ and $R(a, C_n)$ is increasing with n. The limit value $R(a, C)$ $(\leqq\infty)$ is also called the *extremal radius* of C with respect to a. If $R(a, C)$ is finite, then $P(p, a, C_n)$ converges to a function $P(p, a, C)$ in such a way that $\|dP(p, a, C_n)-dP(p, a, C)\|_{R_n}\to 0$, as $n\to\infty$. We remark that such repeated procedures are indispensable. Indeed Oikawa [8] showed that there exists a normal exhaustion $\{R_n\}$ such that $dP(p, a, C_n)$ does not converge in the above sense. The translation of our result on Riemann surface R is as follows:

Let Λ be the family of curves γ joining a compact neighbourhood of a and boundary components C and satisfying that

$$\overline{\lim_{p\to C}}\, P(p, a, C)<\log R(a, C)$$

along γ. Then the module of Λ is equal to zero.

§ 4. An extremal problem.

7. We now define an essential minimum of a univalent function on C. Let $\mathfrak{F}_{aC}$ be a family of univalent functions $f(z)$ with the normalizations: $f(a)=1-f'(a)=0$ for $a\in\Omega$ which maps C onto the outer boundary component of its image domain. We put for a curve γ joining a compact simply connected neighbourhood $\bar{U}$ of a and C

$$M_\gamma(f)=\overline{\lim_{z\to C}}\,|f(z)|$$

along γ and define by $m^*(f)$ the least upper bound of m such that $M_\gamma(f)\geqq m$ except a curve family of its module zero. Let Λ_m be the curve family satisfying $M_\gamma(f)<m$. Then from Lemma 1 we obtain $\mathrm{mod}\,\Lambda_{m^*(f)}=0$. In the image domain Δ the extremal distance of the boundary points contained in the disc $|w|<m^*(f)$ is infinite from a compact neighbourhood of the origin.

We remark that the family of curves of module zero in the above definition is independent of the choice of the neighbourhood U. Such a fact is not so trivial as it seems and we discussed similar problems related to the criterion of minimal sets in [13]. In fact, if the neighbourhood U contains a neighbourhood V, the wellknown monotonity of the module shows the assertion. If not, we take a large simply connected neighbourhood V_1 with an analytic boundary and containing both $\bar{U}$ and V. We now construct a quasiconformal mapping which maps $\Omega-\bar{U}$ onto $\Omega-V_1$ and fixes the boundary of Ω: We first map by a function $\varphi(z)$ the ring domain $V_1-\bar{U}$ onto an annulus $s<|w|<S<1$ in such a way that the mapping $\varphi(z)$ is extended to a larger neighbourhood V_2 such that $\bar{V}_2-U$ is mapped onto $s<|w|\leqq 1$. The quasi-conformal mapping

$$\Phi(w)=|w|^{\log S/\log s}e^{i\arg w}$$

maps the annulus $s<|w|\leqq 1$ onto $S<|w|\leqq 1$ fixing the circle $|w|=1$. The function

$$\Psi(z)=\begin{cases}\varphi^{-1}\circ\Phi\circ\varphi & \text{in } V_2-\bar{U},\\ z & \text{in } \Omega-V_2\end{cases}$$

is a desired function because of the continuation theorem of quasiconformal mapping [1]. The invariance of vanishing of the module under a quasiconformal mapping implies the conclusions.

8. We can state the extremal property of the radial slit disc mapping of finite extremal radius.

THEOREM 4. *The radial slit disc mapping g_{aC} is the unique function maximizing the quantity $m^*(f)$ among the family $\mathfrak{F}_{aC}$ and its maximum value is equal to $R(a,C)$.*

Proof. Since $m^*(g_{aC})=R(a,C)$ from Theorem 3, we only show the extremal property. Let $f(z)$ be a function in $\mathfrak{F}_{aC}$ and Δ be the image domain of Ω under $f(z)$. We take a closed disc $|w|\leqq q$ which is contained in Δ and denote by m_q the minimum modulus of $g_{aC}\circ f^{-1}(w)$ on it. The metric $\rho_0=|g'_{aC}/g_{aC}\log(R(a,C)/m_q)|$ is an extremal metric for the module problem of the family of curves joining the closed curve $|g_{aC}(z)|=m_q$ and the boundary component C in Ω. The metric $\rho=|f'/f\log(m^*(f)/q)|$ in the intersection Ω with the domain $q<|f(z)|$ and $=0$ elsewhere is admissible for it in the sense of no. 3. Taking the modified metric $\hat{\rho}$ with respect to the curve $|f(z)|=q$ in Lemma 2, we have by the fundamental inequality (2)

$$\|\rho_0-\hat{\rho}\|^2\leqq\frac{2\pi}{\log\dfrac{m^*(f)}{q}}-\frac{2\pi}{\log\dfrac{R(a,C)}{m_q}}$$

since the support of $\hat{\rho}$ is contained in the annulus $q<|w|<m^*(f)$, which implies $\|\hat{\rho}\|^2\leqq 2\pi/\log(m^*(f)/q)$. Multiplying by it the denominators on the right hand side and letting q tend to zero, we have

$$\left\|\left|\frac{f'}{f}\right|-\left|\frac{g'_{aC}}{g_{aC}}\right|\right\|^2_{\bar{U}}\leqq\log\frac{R(a,C)}{m^*(f)}$$

where $\bar{U}$ is a compact neighbourhood of the origin. Therefore $R(a,C)\geqq m^*(f)$ and the equality implies $f\equiv g_{aC}$ from the normalization.

The above quantity $m^*(g_{aC})=R(a, C)$ may be different from the true minimum modulus of g_{aC} on C, but Oikawa [7] showed that $R(a, C)$ is the least upper bound of the geometrical minimum modulus of $f\in\mathfrak{F}_{aC}$.

§ 5. A characterization of a radial slit disc with infinite extremal radius.

9. We discuss the case in which $R(a, C)=\infty$. In this case the sequence of the radial slit disc mapping functions g_{aC_n} of the exhausting domains Ω_n of Ω in the direction to C contains a convergent subsequence and its limit function, say $g_0(z)$, maps Ω onto a radial slit domain in such a way that the image of C in the point at infinity with possible radial slit incisions emanating from it and the image of the boundary components other than C is a quasi-minimal set of radial slits [12, 13]. Let Ω be a finite domain with the above mentioned property of the image damain under g_0. Then we can construct such an exhaustion $\{\Omega_n\}$ in the direction to C that their radial slit functions $g_{0C_n}(z)$ converge to the function $w=z$.

THEOREM 5. *Let Ω be a finite domain containing the origin and let C be its boundary component containing the point at infinity. If $R(0, C)=\infty$, and if the union of boundary components other than C is a quasi-minimal set of radial slits, then there exists such an exhaustion $\{\Omega_n\}$ in the direction to C that its radial slit disc mapping function g_{aC_n} tends to the function $w=z$ uniformly on any compact set in Ω.*

It should be noted that the conditions in Theorem 5 are degenerated conditions in Corollary 2 in which $Q=\infty$, and ii) coincides with iii).

Proof. We first remark that the condition that $R(0, C)=\infty$ implies the following: The module of the curves joining C and a compact disc $|z|\leqq q$ vanishes and the boundary component C is the point at infinity having possible radial incisions of angular measure zero emanating from it. In fact taking an arbitrary exhaustion of Ω in the direction to C, we can conclude that the extremal distance of the boundary component C from a compact neighbourhood of the origin is infinite, since $R(0, C_n)$ tends to infinity. Then Strebel's inequality (3) shows that the boundary component C is the point at infinity with possible radial slit incisions of angular measure zero.

We now construct an exhaustion of Ω stated in the theorem. To this end, for given ε and Q we construct such an auxiliary domain Ω_Q^ε that all the points of its outer boundary except on the incisions of Ω_Q^ε lie outside the circle $|z|=Q$, its inner boundary consists of the slits of Ω and its reduced logarithmic area is less than $2\pi\log Q+\varepsilon/2$. Such a domain may be constructed as follows. Taking an annulus $Q<|z|<Q'$ such that $2\pi\log(Q'/Q)<\varepsilon/4$, we denote by S the set of all slits intersecting both the circles $|z|=Q$ and $|z|=Q'$. The area of S is zero. Considering a suitable member of the defining sequence of each slit of S and using Lindelöf's covering theorem, we can find at most a countable number of Jordan domains with analytic boundries $\{J_\nu\}$ such that $\partial J_\nu \subset \Omega$, $S \subset \cup J_\nu$ and the logarithmic area of $\cup J_\nu$ is less than $\varepsilon/4$. Then $\{|z|<Q'\} \cup J_1$ is possibly a domain with finite connectivity n. Connecting its inner boundary components with its outer boundary by analytic Jordan arcs within $\Omega \cap J_1$ we have a simply connected domain Ω_Q^1. Next we apply the same procedure to $\Omega_Q^1 \cup J_2$, and so on. Then we have an increasing sequence of simple connected domains $\{\Omega_Q^\nu\}$. $\Omega_Q^0 = \cup_\nu \Omega_Q^\nu$ is a simply connected domain and $\Omega_Q^0 \cap \Omega$ is desired domain Ω_Q^ε, since $\Omega_Q^\nu \cap \Omega$ is connected.

We denote by Ω_{qQ}^ε the intersection $\Omega_Q^\varepsilon \cap \{|z|>q\}$. Then we have

$$\iint_{\Omega_{qQ}^\varepsilon} \frac{1}{|z|^2}\, dxdy \leqq 2\pi \log \frac{Q}{q} + \frac{\varepsilon}{2}. \tag{4}$$

Denoting by Γ_q the family of curves joining the circle $|z|=q$ and the outer boundary of Ω_{qQ}^ε, say C_Q, we have from Strebel's inequality (3)

$$\operatorname{mod} \Gamma_q \geqq \frac{4\pi^2}{\displaystyle\int_0^{2\pi} l(\theta)d\theta} = \frac{4\pi^2}{\text{logarithmic area of } \Omega_{qQ}^\varepsilon}$$

using Schwarz's inequality and the inequality (4) and the definition of the extremal radius in no. 5 we get

$$\log R(0, C_Q) < \log Q + \frac{\varepsilon}{4\pi}. \tag{5}$$

Let $g_{0C_Q}(z)$ be the radial slit disc mapping function of Ω_Q^ε and m_q be the minimum value of g_{0C_Q} on the circle $|z|=q$. We put

$$\rho = \begin{cases} \left(|z| \log \dfrac{Q}{q}\right)^{-1} & \text{in the annulus } q<|z|<Q, \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\rho_0 = |g'_{0C_Q}| / (|g_{0C_Q}| \log R(0, C_Q)/m_q).$$

Then ρ_0 is the generalized extremal metric for the curve family joining the inverse image of the circle $|g_{0C_Q}|=m_q$ and the boundary component C_Q, and ρ is admissible for it, since the module of the family of the curves ending at the boundary points in $|z|<Q$ vanishes. From the fundamental inequality (2) we have

$$\|\rho - \rho_0\|^2 \leqq \frac{2\pi}{\log \dfrac{Q}{q}} - \frac{2\pi}{\log \dfrac{R(0, C_Q)}{m_q}},$$

and multiplying it by $\log(Q/q) \log(R(0, C_Q)/m_q)$ and tending q to zero we get from (5)

$$\left\| \frac{1}{|z|} - \left| \frac{g'_{0C_Q}}{g_{0C_Q}} \right| \right\|^2 \leqq \pi \log \frac{R(0, C_Q)}{Q} < \frac{\varepsilon}{2}.$$

Next we take an exhaustion of Ω'_Q in the direction C_Q and for a sufficiently near domain Ω_1 with its outer boundary C_1, and we have for a neighbourhood of the origin

$$\left\| \left| \frac{1}{|z|} \right| - \left| \frac{g'_{0C_1}}{g_{0C_1}} \right| \right\|_U < \varepsilon.$$

Let $\{Q_n\}$ be an increasing sequence tending to infinity and $\{\varepsilon_n\}$ be a decreasing sequence tending to zero. Then we can construct an exhaustion $\{\Omega_n\}$ of Ω in the direction to C satisfying that $\| \, |g'_{0C_n}/g_{0C_n}| - 1/|z| \, \|^2 < \varepsilon_n$, which implies the uniform convergence of g_{0C_n} on any compact set on Ω.

After we have completed this article we find that Marden and Rodin discuss related problems in more general situations in Acta Math. **115** (1966), 237–269.

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