THE CARTAN-BRAUER-HUA THEOREM FOR ALGEBRAS

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The Cartan-Brauer-Hua theorem is saying: If H is a skew field contained in the skew field K, and if every inner automorphism of K maps H into itself, then H is either K, or H belongs to the center of K.

This theorem has been generalized in various forms by Amitsur [1], Faith [3], Kasch [6] and others. In the present note we shall give a generalization of the theorem for algebras as follows. In the following, we assume that Z is a field containing an infinite number of elements.

Theorem 1. *Let A be an algebra over Z with a unit element and of finite rank, and let H be a skew field contained in A possessing an infinite number of elements in Z. If every inner antomorphism of A maps H into itself, then H is either A, or H belongs to the center of A.*

We first prove the following lemma:

Lemma 1. *Let A be an algebra over Z with a unit element and of finite rank, and let b be an arbitrary element in A. Then, in the set of elements $\{b+c_1, b+c_2, \cdots\}$ where c_i's are elements of Z, there exist an infinite number of regular elements.*

Proof. In a regular representation of A in Z, these elements $b+c_1, b+c_2, \cdots$ are represented as follows:

$$(b+c_i)[u_1, u_2, \cdots u_n]=[u_1, u_2, \cdots u_n](B+c_iE)$$

where b corresponds to B, and $u_1, u_2, \cdots u_n$ are a basis of A over Z. If $B+c_iE$ is nonsingular, then $b+c_i$ is a regular element. Since the number of roots of the equation $|B+xE|=0$ in Z is at most $[A: Z]=n$, there exist an infinite number of regular elements in them.

Proof of Theorem 1. If H is neither A, nor H belongs to the center of A, then there exists an element d in H not in the center of A. As additive groups, we obtain the next relations of indices:

$$[A^+: H^+]=\infty, \qquad [A^+: V(d)^+]=\infty,$$

where $V(d)$ is the commutator of d in A. Then, by Lemma 5 in Okuzumi [8], there exists an element b in A not in $H\smile V(d)$. So, by Lemma 1, we have two regular elements $b+c_1, b+c_2$ such that

$$(b+c_1)d=h_1(b+c_1), \qquad (b+c_2)d=h_2(b+c_2),$$

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where $h_i \in H$, and $c_i \in H \frown Z$. Then we have:

$$(c_1 - c_2)d = (h_1 - h_2)b + h_1 c_1 - h_2 c_2.$$

Consequently, if $h_1 = h_2$, it contradicts with $bd \neq db$, and if $h_1 \neq h_2$, it contradicts with $b \notin H$.

Next, we modify Lemma 1 in Nagahara [7] for algebras as follows, and then prove Faith's form of Theorem 1.

LEMMA 2. *Let A be an algebra with a unit element over Z, and let H be a proper skew subfield of A containing an infinite number of elements of Z. If a and b are two elements of A such that $ba \neq ab$ and $b \notin H$. Then in the set of regular elements $b+c_1, b+c_2, \cdots, c_i \in Z \frown H$, there exist at most two $(b+c_i)$'s which transform a into H. If a is in H, then there exists at most one.*

THEOREM 2. *Let A be an algebra with a unit element over Z, and let H be a proper skew subfield containing an infinite number of elements in Z and not contained in the center of A. Then, A contains infinitely many subfields conjugate to H.*

Proof. First, we take an element a in H not contained in the center of A. If the number of conjugate subfields is finite, by Lemma 5 in Okuzumi [8], there exists an element b such that $ab \neq ba$ and not contained in these conjugate subfields. Then, in the set of elements $b+c_1, b+c_2, \cdots$, we have an infinite number of regular elements by Lemma 1. Consequently, by Lemma 2, there exists another subfield conjugate to H. This contradicts with the assumption of finiteness.

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