f-STRUCTURES INDUCED ON SUBMANIFOLDS IN SPACES, ALMOST HERMITIAN OR KAEHLERIAN

By HISAO NAKAGAWA

Introduction.

It is known that, if the tangent space of a submanifold of an almost complex space is invariant by the almost complex structure, the submanifold admits an almost complex structure. As a generalization of an almost complex structure, Yano [14] has introduced the concept of an f-structure in a differentiable manifold.

The purpose of this paper is to show that a submanifold of an almost complex space admits an f-structure under certain conditions and to study the induced f-structure.

The author wishes to express here his hearty thanks to Prof. K. Yano, Prof. S. Ishihara and Prof. I. Mogi for their valuable suggestions and guidance.

1. f-structures [14], [15], [17].

Let M^n be an n-dimensional connected differentiable manifold of class C^∞ and $\{\eta^c\}$ local coordinates. If there exists a non-vanishing tensor field f of type (1, 1) and of class C^∞ satisfying

$$f^3+f=0, \tag{1.1}$$

and the rank of f is constant everywhere and is equal to s, then we call[1] such a structure an f-structure of rank s. We put

$$l=-f^2, \qquad m=f^2+1, \tag{1.2}$$

where 1 denotes the unit tensor, then we have

$$l+m=1, \qquad l^2=l, \qquad m^2=m, \qquad lm=ml=0. \tag{1.3}$$

These equations mean that the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators and there exist complementary distributions L and M corresponding to the operators l and m respectively. Then the distribution L is s-dimensional and M is $(n-s)$-dimensional. Further we get

$$\begin{aligned} fl=lf=f, &\qquad fm=mf=0, \\ f^2l=-l, &\qquad f^2m=0. \end{aligned} \tag{1.4}$$

Received November 1, 1965.

1) Yano [14].

It is known that a manifold admitting an f-structure of rank s induces a positive definite Riemannian metric tensor g satisfying

$$g_{ed}f_c{}^e f_b{}^d + m_{cb} = g_{cb}. \tag{1.5}$$

Then the structure is said to be an (f, g)-structure.

The Nijenhuis tensor $N(X, Y)$[2] of an f-structure f of rank s is given by

$$N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] - l[X, Y], \tag{1.6}$$

where X and Y are any two vector fields on M^n. Denoting by $N_{cb}{}^a$ the components of the Nijenhuis tensor $N(X, Y)$, we have in terms of components as follows:

$$N_{cb}{}^a = f_c{}^e \partial_e f_b{}^a - f_b{}^e \partial_e f_c{}^a - (\partial_c f_b{}^e - \partial_b f_c{}^e) f_e{}^a, \tag{1.6$'$}$$

where $\partial_c = \partial/\partial\eta^c$.

Taking account of the integrability conditions given by Yano and Ishihara [17], we see that the following results are valid;

THEOREM A. *It is necessary and sufficient for the distribution M to be integrable that*

$$N(mX, mY) = 0, \tag{1.7}$$

and for L to be integrable that

$$mN(lX, lY) = 0 \tag{1.8}$$

for any two vector fields X and Y.

If the distribution L is integrable and takes an arbitrary vector field v which is tangent to an integral manifold of L, then the vector field fv belongs to the same integral manifold. If we define an operator f' by

$$f'v = fv$$

in each tangent space of each integral manifold of L, then it is seen that f' is an almost complex structure in each integral manifold of L. When the distribution L is integrable and the almost complex structure f' induced from the f-structure on each integral manifold of L is also integrable, then the f-structure is said to be partially integrable. Concerning the partial integrability, we have proved in [17]

THEOREM B. *It is necessary and sufficient for the structure f to be partially integrable that*

$$N(lX, lY) = 0 \tag{1.9}$$

for any two vector fields X and Y.

We suppose now that there exists in each coordinate neighbourhood a coordinate system in which an f-structure f has numerical components

2) Nijenhuis [4] and Yano and Ishihara [17].

$$(f_b{}^a)=\begin{pmatrix} 0 & -1_t & 0 \\ 1_t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 1_t denotes the $t\times t$ unit matrix, $s=2t$ being the rank of f. In this case, the f-structure is said to be integrable. Yano and Ishihara have proved also in [17] the following

THEOREM C. *It is necessary and sufficient for the f-structure to be integrable that*

$$N(X, Y)=0$$

for any two vector fields X and Y.

Let U be a coordinate neighbourhood of a differentiable manifold admitting an f-structure of rank $n-r$. There exist $n-r$ vector fields $f_q{}^j$ spanning the distribution L and r vector fields $f_y{}^j$ spanning the distribution M.[3] If we denote by $\begin{pmatrix} f^p{}_i \\ f^x{}_i \end{pmatrix}$ the inverse image of $(f_q{}^j, f_y{}^j)$, we have

(1.10) $$m_i{}^j=f^y{}_i f_y{}^j,\text{[4]} \qquad f^x{}_i f_y{}^i=\delta_y^x,$$

from which we get

(1.11) $$\begin{aligned} &f_j{}^h f_i{}^j=-\delta_i^h+f_y{}^h f^y{}_i, \\ &f_j{}^h f_y{}^j=0, \qquad f^x{}_j f_i{}^j=0. \end{aligned}$$

The ordered set $\{f_y{}^j\}$ is called an r-frame in U and the ordered set $\{f^x{}_i\}$ is called an r-coframe in U. The set $(f_i{}^j, \{f_y{}^j\}, \{f^x{}_i\})$ of the structure f, an r-frame in U and an r-coframe in U is called a local f-structure with complementary frame. We simply denote it by (f, f_y, f^x).

We assume now that, in a differentiable manifold admitting an f-structure of rank r, there exist globally an $(n-r)$-frame and an $(n-r)$-coframe. The set (f, f_y, f^x) is called an f-structure with complementary frame, which satisfies (1.10) and (1.11).

We suppose that there exist given two r-frames $\{f_y{}^j\}$ and $\{\bar{f}_w{}^j\}$ in the given f-manifold M^n. If we have

(1.12) $$\bar{f}_w{}^j=c_w{}^y f_y{}^j$$

everywhere, with constant $c_w{}^y$, we say that the two frames $\{f_y{}^j\}$ and $\{\bar{f}_w{}^j\}$ are equivalent to each other and that the two f-manifold $(M^n, \{f_y{}^j\})$ and $(M^n, \{\bar{f}_w{}^j\})$ with complementary frame are equivalent to each other.

We take two f-structure (f, f_y, f^x) and $(f, \bar{f}_w, \bar{f}^z)$ with complementary frame so that r-frames $\{f_y{}^j\}$ and $\{\bar{f}_w{}^j\}$ are equivalent to each other. Then, the following equations about the covariant vector fields must be satisfied:

3) The indices $p, q, \cdots$ run over the range $1, 2, \cdots, n-r$ and the indices $x, y, \cdots$ run over the range $n-r+1, \cdots, n$.

4) We use the summation convention.

(1. 13) $$\bar{f}^{z}{}_{i}=d_{y}{}^{z}f^{y}{}_{i},$$

$d_y{}^z$ being constant. From the last equation of (1. 11), (1. 12) and (1. 13), it follows that

(1. 14) $$c_{y}{}^{x}d_{z}{}^{y}=\delta_{z}^{x}.$$

Since the projection operator m of the structure f depends only on f, the given two f-structures have the same operator m. Consequently we have

$$m_{i}{}^{j}=\bar{f}^{y}{}_{i}\bar{f}_{y}{}^{j}=f^{z}{}_{i}f_{z}{}^{j}.$$

Substituting (1. 12) and (1. 13) into the equation above and then transvecting with $f^{w}{}_{j}f_{u}{}^{i}$, we have

(1. 15) $$c_{u}{}^{z}d_{z}{}^{w}=\delta_{u}^{w}.$$

2. The Nijenhuis tensor of the product space $M^n\times E^r$.

Let M^n be an n-dimensional differentiable manifold admitting an f-structure with complementary frames. We consider the product space $M^n\times E^r$, where E^r is an r-dimensional Euclidean space. We take a sufficiently small open covering $\{U_\alpha\}$ of M^n by coordinate neighbourhoods. If we denote by $\{\eta^a\}$ a coordinate system of U_α in $\{U_\alpha\}$ and by $\{\eta^x\}$ a cartesian coordinate of E^r, then $\{\eta^a, \eta^x\}$ can be considered a set of coordinates of $U_\alpha\times E^r$ and $\{U_\alpha\times E^r\}$ constitutes an open covering of $M^n\times E^r$ by coordinate neighbourhoods. We take local coordinate systems $\{U_\alpha, \eta^a\}$ and $\{U_{\alpha'}, \eta^{a'}\}$, where U_α and $U_{\alpha'}$ belong to $\{U_\alpha\}$ and $U_\alpha\cap U_{\alpha'}\neq\phi$. Let

(2. 1) $$\eta^{a'}=\eta^{a'}(\eta^1, \cdots, \eta^n)$$

be the coordinate transformation in $U_\alpha\cap U_{\alpha'}$. We define the coordinate transformation in $(U_\alpha\times E^r)\cap(U_{\alpha'}\times E^r)$ by

(2. 2) $$\begin{cases} \eta^{a'}=\eta^{a'}(\eta^1, \cdots, \eta^n), \\ \eta^{x'}=\eta^{x} \qquad \text{for any } x. \end{cases}$$

For the f-structure with complementary frames (f, f_y, f^x), we put

$$\tilde{F}=\begin{pmatrix} f & -f_y \\ f^x & 0 \end{pmatrix}$$

in every coordinate neighbourhood $\{U_\alpha\times E^r\}$. As the Jacobian matrix of the coordinate transformation (2. 2) is given by

$$\begin{pmatrix} \partial\eta^{a'}/\partial\eta^{b} & 0 \\ 0 & 1_r \end{pmatrix},$$

it is easily seen that $\tilde{F}$ defines a tensor field on the product manifold $M^n\times E^r$ and is denoted by $\tilde{F}_B{}^A$ in terms of components $(A, B, \cdots=1, 2, \cdots, n+r)$.

Making use of (1. 12), we have

$$\tilde{F}_{E}{}^{A}\tilde{F}_{B}{}^{E}=-\delta_{B}^{A},$$

that is, the tensor $\tilde{F}$ is an almost complex structure on the product manifold $M^n \times E^r$. The Nijenhuis tensor $N_{CB}{}^A$ of the almost complex structure $\tilde{F}$ is given by

$$N_{CB}{}^A = \tilde{F}_C{}^E \partial_E \tilde{F}_B{}^A - \tilde{F}_B{}^E \partial_E \tilde{F}_C{}^A - (\partial_C \tilde{F}_B{}^E - \partial_B \tilde{F}_C{}^E)\tilde{F}_E{}^A. \tag{2. 3}$$

If we calculate the components of this tensor by grouping the indices in two groups $(a, b, \cdots)$ and $(x, y, \cdots)$, denoting by $f_y{}^a$ and $f^x{}_b$ the components of the vector fields f_y and f^x respectively, we get

$$\left\{\begin{aligned}
S_{cb}{}^a &= N_{cb}{}^a + (\partial_c f^w{}_b - \partial_b f^w{}_c) f_w{}^a, \\
S_{cb}{}^x &= f_c{}^e \partial_e f^x{}_b - f_b{}^e \partial_e f^x{}_c - (\partial_c f_b{}^e - \partial_b f_c{}^e) f^x{}_e, \\
S_{cy}{}^a &= -f_c{}^e \partial_e f_y{}^a + f_y{}^e \partial_e f_c{}^a + \partial_c f_y{}^e \cdot f_e{}^a, \\
S_{cy}{}^x &= f_y{}^e \partial_e f^x{}_c + \partial_c f_y{}^e \cdot f^x{}_e, \\
S_{zy}{}^a &= f_z{}^e \partial_e f_y{}^a - f_y{}^e \partial_e f_z{}^a.
\end{aligned}\right. \tag{2. 4}$$

We assume now that an affine connection $\Gamma_c{}^a{}_b$ is given on the manifold M^n. We denote by $T_{cb}{}^a = (\Gamma_c{}^a{}_b - \Gamma_b{}^a{}_c)/2$ the torsion tensor of the affine connection and by ∇ the covariant derivation with respect to the affine connection, then the five sets of the components $S_{cb}{}^a$, $S_{cb}{}^x$, $S_{cy}{}^a$, $S_{cy}{}^x$ and $S_{zy}{}^a$ can be rewritten as follows:

$$\left\{\begin{aligned}
S_{cb}{}^a &= f_c{}^e \nabla_e f_b{}^a - f_b{}^e \nabla_e f_c{}^a - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^a \\
&\quad + (\nabla_c f^z{}_b - \nabla_b f^z{}_c) f_z{}^a + 2(T_{cb}{}^a - T_{ed}{}^a f_c{}^e f_b{}^d + f_d{}^a T_{eb}{}^d f_c{}^e + f_d{}^a T_{ce}{}^d f_b{}^e), \\
S_{cb}{}^x &= f_c{}^e \nabla_e f^x{}_b - f_b{}^e \nabla_e f^x{}_c - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f^x{}_e \\
&\quad + 2(f^x{}_d T_{eb}{}^d f_c{}^e + f^x{}_d T_{ce}{}^d f_b{}^e), \\
S_{cy}{}^a &= -f_c{}^e \nabla_e f_y{}^a + f_y{}^e \nabla_e f_c{}^a + \nabla_c f_y{}^e \cdot f_e{}^a \\
&\quad + 2(T_{ed}{}^a f_c{}^e f_y{}^d - f_e{}^a T_{cd}{}^e f_y{}^d), \\
S_{cy}{}^x &= f_y{}^e \nabla_e f^x{}_c + \nabla_c f_y{}^e \cdot f^x{}_e - 2 f^x{}_d T_{ce}{}^d f_y{}^e, \\
S_{zy}{}^a &= f_z{}^e \nabla_e f_y{}^a - f_y{}^e \nabla_e f_z{}^a - 2 T_{ed}{}^a f_z{}^e f_y{}^d,
\end{aligned}\right. \tag{2. 5}$$

where the covariant derivative of the vector fields f_y and f^x along the submanifold M^n with respect to the affine connection are define by

$$\nabla_c f_y{}^a = \partial_c f_y{}^a + \Gamma_c{}^a{}_e f_y{}^e, \qquad \nabla_c f^x{}_b = \partial_c f^x{}_b - \Gamma_c{}^e{}_b f^x{}_e.$$

From (2. 5) it is seen that S's are tensors defined on the submanifold M^n.

In particular, when the connection $\Gamma_c{}^a{}_b$ is symmetric, we get (2. 5) with vanishing torsion tensor T.

In the rest of this section, we shall investigate the properties of the tensors S. First we take the Lie derivative of f^x, f_z and f with respect to the vector f_y. Then, taking account of the third, the fourth and the last equations of (2. 4) we get

$$\mathcal{L}(f_y) f^x{}_c = S_{cy}{}^x, \qquad \mathcal{L}(f_y) f_z{}^a = -S_{zy}{}^a, \tag{2. 6}$$

and

(2. 7) $$\mathfrak{L}(f_y)f_c{}^a = S_{cy}{}^a,$$

where $\mathfrak{L}(f_y)$ denotes the Lie derivative with respect to the vector field f_y. Thus we have

THEOREM 2. 1. *It is necessary and sufficient for the tensor $S_{cy}{}^x$, $S_{zy}{}^a$ and $S_{cy}{}^a$ to vanish identically that $\mathfrak{L}(f_y)f^x{}_c=0$, $\mathfrak{L}(f_y)f_z{}^a=0$ and $\mathfrak{L}(f_y)f_c{}^a=0$, respectively.*

As is well known, the Nijenhuis tensor $N_{CB}{}^A$ of the induced almost complex structure $\tilde{F}$ is hybrid with respect to the indices A and C, and pure with respect to B and C. Since hybridness of $N_{CB}{}^A$ is expressed by

$$N_{EB}{}^A\tilde{F}_C{}^E = -N_{CB}{}^E\tilde{F}_E{}^A,$$

we find

(2. 8) $$\left\{\begin{array}{l} S_{eb}{}^a f_c{}^e + S_{wb}{}^a f^w{}_c + f_e{}^a S_{cb}{}^e - f_w{}^a S_{cb}{}^w = 0, \\ -S_{eb}{}^a f_z{}^e + f_e{}^a S_{zb}{}^e - f_w{}^a S_{zb}{}^w = 0, \\ S_{ey}{}^a f_c{}^e + S_{wy}{}^a f^w{}_c + f_e{}^a S_{cy}{}^e - f_w{}^a S_{cy}{}^w = 0, \\ S_{eb}{}^x f_c{}^e + S_{wb}{}^x f^w{}_c + f^x{}_e S_{cb}{}^e = 0, \\ -S_{eb}{}^x f_z{}^e + f^x{}_e S_{zb}{}^e = 0, \\ S_{ey}{}^x f_c{}^e + f^x{}_e S_{cy}{}^e = 0, \\ -S_{ey}{}^a f_z{}^e + f_e{}^a S_{zy}{}^e = 0, \\ -S_{ey}{}^x f_z{}^e + f^x{}_e S_{zy}{}^e = 0. \end{array}\right.$$

Since purity of $N_{CB}{}^A$ is expressed by

$$N_{EB}{}^A\tilde{F}_C{}^E = N_{CE}{}^A\tilde{F}_B{}^E,$$

we find

(2. 9) $$\left\{\begin{array}{l} S_{eb}{}^a f_c{}^e + S_{wb}{}^a f^w{}_c - S_{ce}{}^a f_b{}^e - S_{cw}{}^a f^w{}_b = 0, \\ -S_{eb}{}^a f_z{}^e - S_{ze}{}^a f_b{}^e - S_{zw}{}^a f^w{}_b = 0, \\ S_{ey}{}^a f_c{}^e + S_{wy}{}^a f^w{}_c + S_{ce}{}^a f_y{}^e = 0, \\ S_{eb}{}^x f_c{}^e + S_{wb}{}^x f^w{}_c - S_{ce}{}^x f_b{}^e - S_{cw}{}^x f^w{}_b = 0, \\ -S_{eb}{}^x f_z{}^e - S_{ze}{}^x f_b{}^e = 0, \\ S_{ey}{}^x f_c{}^e + S_{ce}{}^x f_y{}^e = 0, \\ -S_{ey}{}^a f_z{}^e + S_{ze}{}^a f_y{}^e = 0, \\ -S_{ey}{}^x f_z{}^e + S_{ze}{}^x f_y{}^e = 0. \end{array}\right.$$

Transvecting $f_z{}^c$ and $f^x{}_a$ to the first equation of (2. 8), we get

$$S_{zb}{}^a=-f_e{}^a S_{cb}{}^e f_z{}^c+f_w{}^a S_{cb}{}^w f_z{}^c, \tag{2. 10}$$

and

$$S_{cb}{}^x=f^x{}_a S_{eb}{}^a f_c{}^e+f^x{}_a S_{wb}{}^a f^w{}_c, \tag{2. 11}$$

respectively. On the other hand, transvecting $f_y{}^b$ and $f^x{}_a$ to the second equations of (2. 9) and (2. 8), we have

$$S_{zy}{}^a=-S_{eb}{}^a f_z{}^e f_y{}^b, \qquad S_{zb}{}^x=-f^x{}_a S_{eb}{}^a f_z{}^e, \tag{2. 12}$$

respectively. It follows from the fifth equations of (2. 8) and (2. 9) that

$$f^x{}_e S_{zb}{}^e=-S_{ze}{}^x f_b{}^e. \tag{2. 13}$$

Similarly it follows from the third equations of (2. 8) and (2. 9) that

$$f_e{}^a S_{cy}{}^e-f_w{}^a S_{cy}{}^w=S_{ce}{}^a f_y{}^e. \tag{2. 14}$$

Making use of the equations from (2. 11) to (2. 14), we have

THEOREM 2. 2. *If the tensor $S_{cb}{}^a$ vanishes identically, then so do the other S's.*

By virtue of the sixth, the seventh and the last equations of (2. 8), we have

THEOREM 2. 3. *If the tensors $S_{cy}{}^a$ and $S_{cy}{}^x$ vanish identically, then so does $S_{zy}{}^a$. If the tensors $S_{cy}{}^a$ and $S_{zy}{}^a$ vanish identically, then so does $S_{cy}{}^x$.*

Making use of the Nijenhuis tensor $N_{CB}{}^A$ of the almost complex structure $\tilde{F}$ on the product space $M^n\times E^r$ and taking account of Theorem 2. 2, we can prove the following

THEOREM 2. 4. *In a differentiable manifold M^n admitting an f-structure with complementary frame, it is necessary and sufficient for the almost complex structure $\tilde{F}$ defined on the product space $M^n\times E^r$ to be complex that the tensor $S_{cb}{}^a$ vanishes identically.*

We suppose now that a manifold M^n admits two equivalent f-structures (f, f_y, f^x) and $(f, \bar{f}_w, \bar{f}^z)$ with complementary frames. Let $N_{CB}{}^A$ and $\bar{N}_{CB}{}^A$ be the Nijenhuis tensors of almost complex structures induced on $M^n\times E^r$ by given f-structures, respectively. Making use of (1. 13) and (1. 14), and taking account of (2. 4), we get

$$\left\{\begin{array}{ll} \bar{S}_{cb}{}^a=S_{cb}{}^a, & \bar{S}_{cb}{}^{\bar{x}}=d_x{}^{\bar{x}} S_{cb}{}^x, \\ \bar{S}_{c\bar{y}}{}^a=c_{\bar{y}}{}^y S_{cy}{}^a, & \bar{S}_{c\bar{y}}{}^{\bar{x}}=c_{\bar{y}}{}^y d_x{}^{\bar{x}} S_{cy}{}^x, \\ \bar{S}_{\bar{z}\bar{y}}{}^a=c_{\bar{z}}{}^z c_{\bar{y}}{}^y S_{zy}{}^a, & \end{array}\right. \tag{2. 15}$$

$\bar{S}'$ being components of the Nijenhuis tensor $\bar{N}_{CB}{}^A$. Thus we get

THEOREM 2. 5. *In a differentiable manifold M^n admitting equivalent f-structures (f, f_y, f^x) and $(f, \bar{f}_w, \bar{f}^z)$ with complementary frames, $S_{cb}{}^a=0$ and $\bar{S}_{cb}{}^a=0$ are equivalent to each other.*

3. f-submanifolds of almost complex and almost Hermitian spaces.

Let M^m be an m-dimensional almost complex space with local coordinate system $\{\xi^h\}$ and F be an almost complex structure defined on M^m. Let M^n be an n-dimensional submanifold of M^m represented by $\xi^h=\xi^h(\eta^a)$ for a local coordinate $\{\eta^a\}$ in M^n. We put

$$B_b{}^h=\partial\xi^h/\partial\eta^b, \tag{3.1}$$

which span the tangent space of M^n at each point.

We assume that the tangent space $T(M^n)_p$ of M^n at each point p satisfies

$$\dim\,(T(M^n)_p\cap F(T(M^n)_p))=n-r>0 \tag{3.2}$$

everywhere on M^n. Then we can choose locally linearly independent unit vector fields $u_{n+1},\cdots,u_{n+r}$ normal to the submanifold M^n such that $Fu_{n+1},\cdots,Fu_{n+r}$[5] are tangent to M^n and $m-n-r$ mutually orthogonal vector fields $C_{n+r+1},\cdots,C_m$ belonging the complement of the tangent subspace $T(M^n)_p+F(T(M^n)_p)$ at each point in M^n. If this is the case, we say that M^n is locally framed in M^m. Let v_y be an inverse image of u_y under the transformation F. Then the vector field v_y is given by $Fv_y=u_y$, from which we see that v_y is a tangent vector of M^n. Let $\tilde{v}^x$ be a covariant vector field such that $\tilde{v}^x(v_y)=\delta_y^x$ and $\tilde{u}^x$ be an inverse image of $\tilde{v}^x$ under the transformation F, that is, $\tilde{u}^xF=\tilde{v}^x$. Consequently, we have

$$\left\{\begin{array}{ll} Fv_y=u_y, & Fu_y=-v_y, \\ \tilde{u}^xF=\tilde{v}^x, & \tilde{v}^xF=-\tilde{u}^x. \end{array}\right. \tag{3.3}$$

Summing up, we have

LEMMA 3.1. *If a submanifold M^n is locally framed in an almost complex space M^m, then the tangent subspace $T(M^n)_p\cap F(T(M^n)_p)$ and $T(M^n)_p+F(T(M^n)_p)$ for any point p on the submanifold M^n are invariant under the transformation F.*

We denote by B, u and C the matrices $(B_b{}^h)$, $(u_y{}^h)$ and $(C_\beta{}^h)$, respectively. A matrix (B,u,C) is of maximal rank, and hence there exists an inverse matrix, which is denoted by

$$\begin{pmatrix} B^* \\ \tilde{u} \\ C^* \end{pmatrix}=\begin{pmatrix} B^a{}_i \\ \tilde{u}^x{}_i \\ C^\alpha{}_i \end{pmatrix}.$$

5) Throughout this paper, indices run over the ranges as follows:

$$\begin{array}{lllll} i,j, & \cdots: & 1,2, & \cdots, & m, \\ a,b, & \cdots: & 1,2, & \cdots, & n, \\ x,y, & \cdots: & n+1, & \cdots, & n+r, \\ A,B, & \cdots: & 1,2, & \cdots, & n+r, \\ \alpha,\beta, & \cdots: & n+r+1, & \cdots, & m, \\ \lambda,\mu, & \cdots: & n+1, & \cdots, & m, \end{array}$$

unless otherwise stated.

The vector fields $B^a{}_i$ span the cotangent space of M^n. We put $A=(B, u)$ and $A^*=\begin{pmatrix} B^* \\ \tilde{u} \end{pmatrix}$. It follows from the construction that we get the equations

$$(3.4) \qquad \begin{cases} B^a{}_i B_b{}^i = \delta^a_b, & \tilde{u}^x{}_i u_y{}^i = \delta^x_y, \\ B^a{}_i u_y{}^i = 0, & \tilde{u}^x{}_i B_y{}^i = 0. \end{cases}$$

If, in a submanifold M^n framed in an almost complex space M^m, a tangent field T, say $T_{ji}{}^h$, has components of the form

$$(3.5) \qquad T_{ji}{}^h = C^\alpha{}_j P_{\alpha i}{}^h + C^\alpha{}_i Q_{j\alpha}{}^h + C_\beta{}^h R_{ji}{}^\beta$$

with certain tensor fields P, Q and R, then the relation (3.5) is expressed in a simplified form as

$$(3.6) \qquad T_{ji}{}^h \equiv 0,$$

and $U_{ji}{}^h - V_{ji}{}^h \equiv 0$, as $U_{ji}{}^h \equiv V_{ji}{}^h$. Using this notations, we get

$$(3.7) \qquad B_a{}^j B^a{}_i \equiv \delta^j_i - u_y{}^j \tilde{u}^y{}_i,$$

because a matrix $\begin{pmatrix} B^* \\ \tilde{u} \\ C^* \end{pmatrix}$ is an inverse of (B, u, C).

Now we put

$$(3.8) \qquad f_b{}^a = B^a{}_j F_i{}^j B_b{}^i, \qquad f_y{}^a = B^a{}_j v_y{}^j, \qquad f^x{}_b = \tilde{v}^x{}_i B_b{}^i,$$

which define a tensor field of type (1, 1), a contravariant vector field and a covariant vector field on M^n. Making use of (3.8) and the properties of the tangent vector v_y, we obtain

$$(3.9) \qquad f_y{}^a f^x{}_a = \delta^x_y.$$

We put

$$(3.10) \qquad \tilde{F} = A^* F A,$$

then we get

$$(3.11) \qquad (\tilde{F}_B{}^A) = \begin{pmatrix} f_b{}^a & -f_y{}^a \\ f^x{}_b & 0 \end{pmatrix}.$$

From the definition (3.10) of $\tilde{F}$ it follows that

$$(3.12) \qquad \tilde{F}^2 = -1.$$

We express the equation above in terms of components, then we have

$$(3.13) \qquad \begin{cases} f_e{}^a f_b{}^e - f_y{}^a f^y{}_b = -\delta^a_b, \\ -f_e{}^a f_y{}^e = 0, \qquad f^x{}_e f_c{}^e = 0, \qquad f_y{}^e f^x{}_e = \delta^x_y. \end{cases}$$

This implies that the following equation is valid:

$$f_e{}^a f_b{}^e f_c{}^b + f_c{}^a = 0,$$

that is, the tensor field f on the submanifold M^n satisfies $f^3+f=0$ and $f^2=-1$

$+f^y \otimes f_y$. This means that the submanifold M^n admits an f-structure f with complementary frames. Thus we find

THEOREM 3.2. *If the submanifold M^n is locally framed in an almost complex space M^m, then M^n admits an f-structure of rank $n-r$ with complementary frames.*

We call the f-structure the induced f-structure with complementary frames and the submanifold M^n admitting an f-structure the f-submanifold.

Next, let M^m be an almost Hermitian space and (F, G) be an almost Hermitian structure. We put

$$\tilde{G}=A^tGA. \tag{3.14}$$

We denote by G_{ji} the components of the Riemannian metric tensor G of the almost Hermitian structure. If we write down the components of both sides of (3.14), we get

$$\tilde{G}_{CB}=G_{kl}A_C{}^kA_B{}^l. \tag{3.14$'$}$$

By grouping their indices in two groups $(a, b, \cdots)$ and $(x, y, \cdots)$, we get

$$\tilde{G}_{CB}=\begin{pmatrix} g_{cb} & 0 \\ 0 & g'_{xy} \end{pmatrix}: \quad \tilde{G}=\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}, \tag{3.15}$$

where $g_{cb}=G_{ji}B_c{}^jB_b{}^i$ and $g'_{xy}=G_{ji}u_x{}^ju_y{}^i$ define a tensor field of type (0, 2).

It follows from (3.14) and (3.15) that we have

$$G\equiv(B^*)^tgB^*+u\cdot g'\cdot\tilde{u}. \tag{3.16}$$

We express both sides of the equation above in terms of the components, then we get

$$G^{ji}\equiv g^{cb}B_c{}^jB_b{}^i+g'^{xy}u_y{}^ju_x{}^i. \tag{3.16$'$}$$

Making use of (3.8) and (3.14), we get

$$\tilde{G}=\tilde{F}^t\tilde{G}\tilde{F},$$

which means that we get

$$g_{cb}=g_{ed}f_c{}^ef_b{}^d+g'_{yx}f^y{}_cf^x{}_b. \tag{3.17}$$

This shows that the tensor fields f and g defined on the submanifold M^n satisfy (1.5). Thus we find

THEOREM 3.3. *If the f-submanifold M^n is locally framed in an almost Hermitian space M^m, then M^n admits an (f, g)-structure of rank $n-r$ with complementary frames.*

If, in particular, $r=0$, then the submanifold M^n is invariant.[6] Tashiro [11] has treated of the case that $r=1$.

6) Yano and Schouten [20].

4. f-submanifolds of almost Kaehlerian and Kaehlerian spaces.

In this section, let M^m be an almost Kaehlerian space and (F, G) be an almost Kaehlerian structure. Let M^n be an f-submanifold of M^m. We can choose locally $m-n$ unit vector fields $C_{n+1}, \cdots, C_m$ along M^n, which are perpendicular to M^n and linearly independent at each point of M^n. The Riemannian connection $\{ {}^{\,a}_{cb} \}$ induced on the submanifold from the given Riemannian connection $\{ {}^{\,h}_{ji} \}$ is by definition given by

$$\{ {}^{\,a}_{cb} \}=(\partial_c B_b{}^h+B_c{}^j B_b{}^i\{ {}^{\,h}_{ji} \})B^a{}_h.$$

We define the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ along the submanifold by

$$\nabla_c B_b{}^h=\partial_c B_b{}^h+B_c{}^j B_b{}^i\{ {}^{\,h}_{ji} \}-B_a{}^h\{ {}^{\,a}_{cb} \}.$$

We put

$$\nabla_c B_b{}^j=h_{cb}{}^\lambda C_\lambda{}^j, \tag{4. 1}$$

where $h_{cb}{}^\lambda$ are second fundamental tensors of M^n. Concerning the second fundamental tensor, it is known that the following result is valid:

$$h_{cb}{}^\lambda=h_{bc}{}^\lambda \qquad \text{for any } \lambda. \tag{4. 2}$$

We put

$$h_c{}^{a\lambda}=g^{ae}h_{ce}{}^\lambda, \tag{4. 3}$$

then we have

$$\nabla_c f_b{}^a=-h_{cb}{}^\lambda p_\lambda{}^a+h_c{}^{a\lambda}p_{\lambda b}+B^a{}_j\nabla_k F_i{}^j\cdot B_c{}^k B_b{}^j, \tag{4. 4}$$

where $p_\lambda{}^a=-B^a{}_j F_i{}^j C_\lambda{}^i$ and $p_{\lambda b}=g_{ab}p_\lambda{}^a$. From which we get

$$f_{cba}\overset{def}{=}\nabla_c f_{ba}+\nabla_b f_{ac}+\nabla_a f_{cb}=0. \tag{4. 5}$$

Thus we find

LEMMA 4. 1. *In an f-submanifold M^n of an almost Kaehlerian space M^m the exterior differential form $f_{cb}dx^c\wedge dx^b$ is closed.*

Making use of Lemma 4. 1, we can prove

THEOREM 4. 2. *In an f-submanifold M^n of an almost Kaehlerian space M^m the distribution M corresponding to the projection operator m given by the f-structure is integrable.*

Hereafter we call such distributions L and M the induced distributions.

Proof. Since it follows from (1. 6)$'$ that the components of the Nijenhuis tensor of the structure f satisfy

$$N_{cb}{}^a=f_c{}^e\nabla_e f_b{}^a-f_b{}^e\nabla_e f_c{}^a-(\nabla_c f_b{}^e-\nabla_b f_c{}^e)f_e{}^a, \tag{4. 6}$$

we get

$$N_{cb}{}^a m_e{}^c m_d{}^b = -(\nabla_c f_{bf} + \nabla_b f_{fc}) f^{fa} m_e{}^c m_d{}^b,$$

from which, by virtue of Lemma 4.1, we obtain $N_{cb}{}^a m_e{}^c m_d{}^b = 0$. The equation means that $N(mX, mY)$ vanishes identically. Because of Theorem A, this completes the proof.

Next we consider the case that M^m is a Kaehlerian space. Let M^n be an f-submanifold of M^m. Substituting (4.4) into the components (4.6) of the Nijenhuis tensor of the f-structure, we have

$$\text{(4.7)} \qquad \left\{ \begin{aligned} N_{cb}{}^a = & f_c{}^e(-h_{eb}{}^\lambda p_\lambda{}^a + h_e{}^{a\lambda} p_{\lambda b}) - f_b{}^e(-h_{ec}{}^\lambda p_\lambda{}^a + h_e{}^{a\lambda} p_{\lambda c}) \\ & -(h_c{}^{e\lambda} p_{\lambda b} - h_b{}^{e\lambda} p_{\lambda c}) f_e{}^a. \end{aligned} \right.$$

We assume that the vector fields p_λ belong to the distribution M. Then, transvecting $m_a{}^f l_e{}^c l_d{}^b$ and $l_a{}^f l_e{}^c l_d{}^b$ to this, we get

$$\text{(4.8)} \qquad m_a{}^f N_{cb}{}^a l_e{}^c l_d{}^b = -h_{cb}{}^\lambda p_\lambda{}^f f_e{}^c l_d{}^b + h_{cb}{}^\lambda p_\lambda{}^f l_e{}^c f_d{}^b,$$

and

$$\text{(4.9)} \qquad l_a{}^f N_{cb}{}^a l_e{}^c l_d{}^b = 0,$$

respectively. Since we see that the left hand side of the equation (4.8) is the components of the tensor $mN(lX, lY)$, it follows from Theorem A that we have

LEMMA 4.3. *If in an f-submanifold M^n of a Kaehlerian space M^m the vector fields p_λ belong to the distribution M, then it is necessary and sufficient for the induced distribution L to be integrable that*

$$f_b{}^e f_a{}^d (f_e{}^c h_{cd}{}^\lambda - f_d{}^c h_{ce}{}^\lambda) = 0.$$

The equation (4.9) means that $lN(lX, lY) = 0$ for any two vector fields X and Y, and therefore, making use of this result and Theorem 4.2 we can obtain the following

LEMMA 4.4. *If in an f-submanifold M^n of a Kaehlerian space M^m the vector fields p_λ belong to the distribution M, then we have*

$$N(X, Y) = mN(lX, lY) + N(lX, mY) + N(mX, lY)$$

for any two vector fields X and Y.

Lemma 4.4 implies that we get $N(lX, lY) = mN(lX, lY)$ and this means that the integrability of the distribution L is equivalent to the partial integrability of the structure f. Thus we find

THEOREM 4.5. *If in an f-submanifold M^n of a Kaehlerian space M^m, the vector fields p_λ belong to the distribution M, then it is necessary and sufficient for the structure f to be partially integrable that the distribution L is integrable.*

Now, suppose that the f-submanifold M^n is totally umbilical. From the hypothesis $h_{cb}{}^\lambda = A^\lambda g_{cb}$ for any λ and (4.8), we get

$$m_a{}^f N_{cb}{}^a l_e{}^c l_d{}^b = 2 f_{ed} A^\lambda p_\lambda{}^f.$$

Therefore, if the induced distribution L is integrable, then $A^\lambda=0$ for any λ. Thus we find

COROLLARY 4. 6. *If in a totally umbilical f-submanifold M^n of a Kaehlerian space M^m the induced distribution L is integrable and the vector fields p_λ belong to the distribution M, then the submanifold is totally geodesic.*

In particular, we assume that the f-submanifold M^n of a Kaehlerian space M^m is totally geodesic. It follows from (4. 5) that $\nabla_c f_b{}^a=0$, and this implies that the Nijenhuis tensor $N_{cb}{}^a$ vanishes identically. Consequently the (f, g)-structure f is integrable.

5. *f*-submanifolds framed in a Kaehlerian space.

Let M^m be a Kaehlerian space and (F, G) a Kaehlerian structure. Let M^n be a submanifold of M^m. We denote by $N(M^n)$ an orthogonal complementary subbundle of $T(M^n)$ in $T(M^n)+F(T(M^n))$.

In this section, we assume that the normal bundle to M^n is the product of $N(M^n)$ and its complement $\bar{N}(M^n)$. Then, there exist r globally vector fields u_y of M^m defined on M^n, which belong to the subbundle $N(M^n)$ and are lineary independent, and moreover $m-n-r$ mutually orthogonal vector fields C_β belonging the complement of $N(M^n)$ in the normal bundle. We call such a submanifold M^n the submanifold framed in M^m.

Taking account of the vector fields u_y and C_β introduced above in the submanifold M^n in a Kaehlerian space, as discussed in the section 3, the submanifold M^n admits an (f, g)-structure $\{f, f_y\}$ of rank $n-r$ with complementary frames. Hence M^n is an f-submanifold and the induced f-structure $\{f, f_y\}$ satisfies (1. 5).

In an f-submanifold M^n framed in a Kaehlerian space, we put

$$\nabla_c B_b{}^j = h_{cb}{}^x u_x{}^j + h_{cb}{}^\alpha C_\alpha{}^j, \tag{5. 1}$$

where $h_{cb}{}^x$ and $h_{cb}{}^\alpha$ are second fundamental tensors of M^n, then we get

$$\nabla_c f_b{}^a = -h_{cb}{}^x f_x{}^a + h_c{}^{ax} f_{xb}, \tag{5. 2}$$

where $f_y{}^a = B^a{}_j F_i{}^j u_y{}^i$ and $f_{xb}=f_x{}^e g_{eb}$, from which we have

$$\left\{\begin{aligned} N_{cb}{}^a = & f_c{}^e(-h_{eb}{}^x f_x{}^a + h_e{}^{ax} f_{xb}) - f_b{}^e(-h_{ec}{}^x f_x{}^a + h_e{}^{ax} f_{xc}) \\ & -(h_c{}^{ex} f_{xb} - h_b{}^{ex} f_{xc}) f_e{}^a. \end{aligned}\right. \tag{5. 3}$$

By the similar method as the section 3, we obtain the following properties:

LEMMA 5. 1. *In an f-submanifold M^n framed in a Kaehlerian space M^m it is necessary and sufficient for the induced distribution L to be integrable that*

$$f_b{}^e f_a{}^d (f_e{}^c h_{cd}{}^x - f_d{}^c h_{ce}{}^x) = 0.$$

LEMMA 5. 2. *In an f-submanifold M^n framed in a Kaehlerian space M^m we have*

$$N(X,\ Y) = mN(lX,\ lY) + N(lX,\ mY) + N(mX,\ lY)$$

for any two vector fields X and Y.

THEOREM 5. 3. *In an f-submanifold M^n framed in a Kaehlerian space M^m, it is necessary and sufficient for the structure f to be partially integrable that the distribution L is integrable.*

Since an f-submanifold M^n framed in a Kaehlerian space M^m admits an f-structure with complementary frames $\{f, f_y\}$, it follows that the product space $M^n \times E^r$ admits an almost complex structure $\tilde{F}$ induced from the f-structure with complementary frames. We denote the five sets of the components of the Nijenhuis tensor of the almost complex structure $\tilde{F}$ by $S_{cb}{}^a$, $S_{cb}{}^x$, $S_{cy}{}^a$, $S_{cy}{}^x$ and $S_{zy}{}^a$, and then we have (2. 5) with vanishing T.

Suppose that $S_{cy}{}^a=0$. By making use of Theorem 2. 1, this implies that the structure f is invariant under the transformations generated by the infinitesimal transformation f_y. On the other hand, when we write down the components of $l(\mathcal{L}(mY)f)l$ for any vector field Y, we get $l_a{}^d(\mathcal{L}(m_f{}^e Y^f)f_b{}^a)l_c{}^b$. Consequently, we show, under the assumption that $S_{cy}{}^a=0$, that $l(\mathcal{L}(mY)f)l=0$. Since it is known[7] that the last equation is valid if and only if $N(lX, mY)=0$, it follows from Lemma 5. 2 that

$$N(X, Y)=mN(lX, lY) \tag{5. 4}$$

for any two vector fields X and Y. Thus we have

LEMMA 5. 4. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensor $S_{cy}{}^a$ vanishes identically, then we get* (5. 4).

Taking account of the property of the induced distribution L and the condition for the f-structure to be integrable and making use of Lemma 5. 4, we get

THEOREM 5. 5. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensor $S_{cy}{}^a$ vanishes identically, then it is necessary and sufficient for the induced f-structure to be integrable that so does the induced distribution L.*

Next, in an f-submanifold M^n framed in a Kaehlerian space M^m, we shall investigate the relation between the tensors $S_{cb}{}^a$ and $S_{cy}{}^a$. We first assume that $S_{cy}{}^a=0$ and $S_{cy}{}^x=0$ ($S_{cy}{}^a=0$ and $S_{zy}{}^a=0$). Then it follows from Theorem 2. 3 that $S_{zy}{}^a=0$ (and $S_{cy}{}^x=0$), respectively. Making use of (4. 8) and (5. 4) and the definition of the tensor $S_{cb}{}^a$, we have

$$S_{cb}{}^a=-(f_c{}^e l_b{}^d-f_b{}^e l_c{}^d)h_{ed}{}^x f_x{}^a+(\nabla_c f^x{}_b-\nabla_b f^x{}_c)f_x{}^a. \tag{5. 5}$$

On the other hand, we get from the fourth equation of (2. 8)

$$S_{eb}{}^x f_c{}^e+f^x{}_e S_{cb}{}^e=0.$$

Substituting the second equation of (2. 5) and (5. 5) into the equation above and then transvecting $f_d{}^c f_a{}^b$, we have

$$-(\nabla_d f^x{}_a-\nabla_a f^x{}_d)-(f_a{}^b l_d{}^c-f_d{}^b l_a{}^c)h_{bc}{}^x=0,$$

7) Yano and Ishihara [17].

from which it implies that

$$\nabla_c f^x{}_b - \nabla_b f^x{}_c = (f_c{}^e l_b{}^d - f_b{}^e l_c{}^d) h_{ed}{}^x.$$

By substituting this equation into (5. 5), it is seen that $S_{cb}{}^a = 0$. Thus we find

THEOREM 5. 6. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensors $S_{cy}{}^a$ and $S_{cy}{}^x$ ($S_{cy}{}^a$ and $S_{zy}{}^a$) vanish identically, then so does the tensor $S_{cb}{}^a$.*

Next, we prove the following

THEOREM 5. 7. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the induced structure f is integrable and the tensor $S_{cy}{}^x$ vanishes identically, then the tensor $S_{cb}{}^a$ vanishes identically and each of exterior differential forms $f^x{}_c dx^c$ is closed.*

Proof. Since the induced structure f is integrable, we get $N(lX, mY)=0$. This means $l(\mathcal{L}(mY)f)l=0$, that is,

$$l_a{}^d(\mathcal{L}(f_z) f_b{}^a) l_c{}^b = 0. \tag{5. 6}$$

By virtue of the fifth equation of (2. 5), (2. 6) and Lemma 4. 1, we get

$$(\mathcal{L}(f_z) f_b{}^a) f_y{}^b = -f_b{}^a S_{zy}{}^b = -g^{ac} f_z{}^e f_y{}^b \nabla_c f_{eb} = 0,$$

and

$$f^x{}_a(\mathcal{L}(f_z) f_b{}^a) = S_{za}{}^x f_b{}^a = 0.$$

Substituting these equations into (5 6), we have $\mathcal{L}(f_z) f_b{}^a = 0$, which implies from Theorem 2. 1 that $S_{zb}{}^a = 0$. By virtue of Theorem 5. 6, it is seen that $S_{cb}{}^a = 0$.

From $S_{cb}{}^a = 0$ and the first equation of (2. 5), it is easily seen that $\nabla_c f^x{}_b - \nabla_b f^x{}_c = 0$ for any x.

Hence we prove Theorem 5. 7.

In particular, we consider the case $r=1$, then the induced (f, g)-structure is almost contact metric.[8] If we assume that the induced almost contact metric structure is integrable and $S_b = 0$, then, taking account of Theorem 5. 7, we get

$$\nabla_c f_b - \nabla_b f_c = 0.$$

On the other hand, when, for an almost contact metric structure, both 1-form $\theta_1 = f_c dx^c$ and 2-form $\theta_2 = (1/2) f_{cb} dx^c \wedge dx^b$ are closed, the structure is said[9] to be of K_1-type. When the tensor $f_b{}^a$ and the vector f^a defining the almost contact metric structure are both covariant constant, the structure is said[10] to be of K_2-type. Since it is seen[11] that a normal almost contact metric structure of K_1-type is of K_2-type, we get

COROLLARY 5. 8. *If in an f-submanifold ($r=1$) framed in a Kaehlerian space M^m the induced almost contact metric structure is integrable and the tensor S_b vanishes identically, then the structure is normal and moreover of K_2-type.*

8) Concerning the almost contact metric structure, see Sasaki and Hatakeyama [8] and [9].

9), 10), 11) Nakagawa [3].

6. Weingarten's formula.

Let M^m be a Kaehlerian space and (F, G) be a Kaehlerian structure. Let M^n be the f-submanifold of M^m. Then n vector fields B_b are tangent to the submanifold M^n and there exist $m-n$ local vector fields C_λ in M^m defined on M^n such that the vector fields are orthogonal to M^n. Therefore we get $G_{ji}B_b{}^jC_\lambda{}^i=0$. If we put

$$g_{cb}=G_{ji}B_c{}^jB_b{}^i, \qquad g_{\lambda\mu}=G_{ji}C_\lambda{}^jC_\mu{}^i,$$

then g_{cb} is a Riemannian metric in M^n and $g_{\lambda\mu}$ is a metric tensor. Differentiating this covariantly along the submanifold M^n and using the second fundamental tensor, we get

$$g_{\lambda\mu}h_{cb}{}^\mu+G_{ji}B_b{}^j\nabla_c C_\lambda{}^i=0,$$

where

$$\nabla_c C_\lambda{}^i=\partial_c C_\lambda{}^i+B_c{}^j\{{}^{\,i}_{jk}\}C_\lambda{}^k-\Gamma_c{}^\mu{}_\lambda C_\mu{}^i, \qquad \Gamma_c{}^\mu{}_\lambda=C^\mu{}_h(\partial_c C_\lambda{}^h+\{{}^{\,h}_{ji}\}C_\lambda{}^jB_c{}^i).$$

For the induced (f, g)-structure in the f-submanifold M^n, transvecting $B_a{}^hg^{ab}$ to the equation above, we have

$$g_{\lambda\mu}B_a{}^hh^a{}_c{}^\mu+\nabla_c C_\lambda{}^h-G^{hk}C^\mu{}_kG_{ji}\nabla_c C_\lambda{}^i\cdot C_\mu{}^j=0.$$

We put $G_{ji}\nabla_c C_\lambda{}^j\cdot C_\mu{}^i=l_{c\lambda\mu}$, which are the third fundamental tensors of the submanifold M^n. Thus we get

$$\nabla_c C_\lambda{}^h=-g_{\lambda\mu}B_e{}^hh^e{}_c{}^\mu+l_{c\lambda\mu}C^\mu{}_jG^{jh}, \tag{6. 1}$$

which is a generalization of well known Weingarten's formula in the theory of surfaces.

It is easily seen that the third fundamental tensor $l_{c\lambda\mu}$ is skew-symmetric in λ and μ, that is,

$$l_{c\lambda\mu}+l_{c\mu\lambda}=0. \tag{6. 2}$$

In this section, we assume that the f-submanifold M^n is framed in M^m. Accordingly there exist linearly independent vector fields B_b, u_y and C_β, and taking account of (6. 1) we get

$$\nabla_c u_y{}^h=-g_{yx}B_e{}^hh^e{}_c{}^x+l_{cyx}u^x{}_jG^{jh}+l_{cy\alpha}C^\alpha{}_jG^{jh}, \tag{6. 1$'$}$$

where $g_{zy}=G_{ji}u_z{}^ju_y{}^i$ and $g_{\gamma\beta}=G_{ji}C_\gamma{}^jC_\beta{}^i$, which are metric tensors in the vector bundles $N(M^n)$ and $\bar{N}(M^n)$. Making use of Gauss and Weingarten equations, we see that the following identities are valid:

$$\nabla_c f^x{}_b=-f_b{}^eh_{ce}{}^x+g^{xy}l_{cyz}f^z{}_b \tag{6. 3}$$

and

$$\nabla_c f_y{}^a=g_{xy}f_e{}^ah_c{}^{ex}+f^{za}l_{cyz}, \tag{6. 4}$$

where g^{xy} is defined by $g^{xz}g_{zy}=\delta^x_y$. Substituting (4. 5), (6. 3) and (6. 4) into (2. 5) without T, we have

$$(6.5)\quad \begin{cases} S_{cb}{}^{a}=(-f_{e}{}^{a}h_{c}{}^{ew}+f_{c}{}^{e}h_{e}{}^{aw})f_{wb}+(-f_{b}{}^{e}h_{e}{}^{aw}+f_{e}{}^{a}h_{b}{}^{ew})f_{wc} \\ \qquad +(l_{czw}f^{w}{}_{b}-l_{bzw}f^{w}{}_{c})f^{za}, \\ S_{cb}{}^{x}=-(h_{c}{}^{ew}f_{wb}-h_{b}{}^{ew}f_{wc})f^{x}{}_{e}+g^{xz}(f_{c}{}^{e}l_{czw}f^{w}{}_{b}-f_{b}{}^{e}l_{ezw}f^{w}{}_{c}), \\ S_{cy}{}^{a}=f_{b}{}^{e}h_{e}{}^{aw}f_{wc}-g_{zy}(h_{c}{}^{az}+f_{c}{}^{e}f_{d}{}^{a}h_{e}{}^{dz})-f_{c}{}^{e}f^{wa}l_{cyw} \\ \qquad -f_{y}{}^{e}h_{ec}{}^{w}f_{w}{}^{a}+g_{zy}f_{w}{}^{a}f^{w}{}_{e}h_{c}{}^{ez}, \\ S_{cy}{}^{x}=-f_{y}{}^{e}f_{c}{}^{d}h_{ed}{}^{x}+g^{xz}(f_{y}{}^{e}l_{ezw}f^{w}{}_{c}+l_{cyz}), \\ S_{zy}{}^{a}=f_{d}{}^{a}(g_{wy}f_{z}{}^{e}h_{e}{}^{dw}-g_{wz}f_{y}{}^{e}h_{e}{}^{dw})+f^{wa}(f_{z}{}^{e}l_{eyw}-f_{y}{}^{e}l_{ezw}). \end{cases}$$

Suppose that the tensors $S_{cy}{}^{x}$ and $S_{zy}{}^{a}$ vanish identically. Then, transvecting f_{xa} to the last equation of (6. 5), we get

$$(6.6)\qquad f_{z}{}^{e}l_{eyx}=f_{y}{}^{e}l_{ezx}.$$

It follows from (6. 2) that we have $f_{z}{}^{e}l_{eyx}=-f_{y}{}^{e}l_{exz}$. Making use of (6. 6), we see that the right hand side is symmetric in y and x and furthermore the left hand side is skew-symmetric in y and x. Accordingly we get

$$(6.7)\qquad f_{z}{}^{e}l_{eyx}=0,$$

from which we have

$$(6.8)\qquad f_{a}{}^{d}h_{ed}{}^{x}(g_{xy}f_{z}{}^{e}-g_{xz}f_{y}{}^{e})=0.$$

Thus it follows from the fourth equation of (6. 5), (6. 7) and (6. 8) that

$$-g_{zx}f_{y}{}^{e}f_{c}{}^{d}h_{ed}{}^{z}+l_{cyx}=0,$$

and the first term of this equation is symmetric in x and y. Consequently the third fundamental tensor l_{cyx} must vanish. Thus we find

THEOREM 6. 1. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensor $S_{cy}{}^{x}$ and $S_{zy}{}^{a}$ vanish identically, then so does the third fundamental tensor l_{cyx}.*

Under the assumption of Theorem 6. 1, (6. 5) can be simplified as follows:

$$(6.9)\quad \begin{cases} S_{cb}{}^{a}=(-f_{e}{}^{a}h_{c}{}^{ew}+f_{c}{}^{e}h_{e}{}^{aw})f_{wb}+(-f_{b}{}^{e}h_{e}{}^{aw}+f_{e}{}^{a}h_{b}{}^{ew})f_{wc}, \\ S_{cb}{}^{x}=-(h_{c}{}^{ew}f_{wb}-h_{b}{}^{ew}f_{wc})f^{x}{}_{e}, \\ S_{cy}{}^{a}=f_{y}{}^{e}h_{e}{}^{aw}f_{wc}-g_{xy}(h_{c}{}^{ax}+f_{c}{}^{e}f_{d}{}^{a}h_{e}{}^{dx})-f_{y}{}^{e}h_{ec}{}^{w}f_{w}{}^{a}+g_{xy}f_{w}{}^{a}f^{w}{}_{e}h_{c}{}^{ex}. \end{cases}$$

Next, we prove the following

THEOREM 6. 2. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensor $S_{cb}{}^{a}$ vanishes identically, then any vector field f_y is Killing.*

Proof. By virtue of Theorem 2. 3, we see that the other tensors S vanish identically. Therefore, transvecting f_{ba} to the third equation of (6. 9) and taking account of $f_{y}{}^{e}f_{c}{}^{d}h_{edx}=0$, we have

$$f_b{}^e h_{ce}{}^x + f_c{}^e h_{be}{}^x = 0,$$

from which we have

$$\nabla_c f^x{}_b + \nabla_b f^x{}_c = 0 \qquad \text{for any } x.$$

This completes the proof.

COROLLARY 6. 3. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the induced f-structure is integrable and if the tensor $S_{cy}{}^x$ vanishes identically, then the f-structure $\{f, f_y\}$ with complementary frames is covariant constant.*

Proof. Making use of Theorem 5. 7 and Theorem 6. 2, we see that the vector f_y is covariant constant. This implies from (6. 3) that $f_b{}^e h_{ce}{}^x = 0$. Since the tensor $S_{cb}{}^x$ vanishes identically, we have $h_c{}^{ew} f_{wb} - h_b{}^{ew} f_{wc} = 0$. This means from (4. 4) that the tensor $f_b{}^a$ is covariant constant. Thus the f-structure $\{f, f_y\}$ is covariant constant.

7. Kaehlerian spaces of constant holomorphic curvature.

A Kaehlerian space M^m is said[12)] to be of constant holomorphic curvature if the holomorphic sectional curvature is always constant with respect to any plane section at each point on the manifold M^m and its curvature tensor is given by

$$K_{kjih} = k[G_{kh}G_{ji} - G_{ki}G_{jh} + F_{kh}F_{ji} - F_{ki}F_{jh} - 2F_{kj}F_{ih}], \tag{7. 1}$$

k being constant. In this section, we consider the f-submanifold M^n of a Kaehlerian space M^m of constant holomorphic curvature.

Substituting (7. 1) into the Gauss and Codazzi equations

$$K_{dcba} = K_{kjih} B^{kjih}_{dcba} + (h_{da}{}^\lambda h_{cb\lambda} - h_{db}{}^\lambda h_{ca\lambda}), \tag{7. 2}$$

where $B^{kjih}_{dcba} = B^k_d B^j_c B^i_b B^h_a$, and

$$K_{kjih} B^{kji}_{dcb} C_\mu{}^h = \nabla_d h_{cb\mu} - \nabla_c h_{db\mu} + h_{db}{}^\lambda l_{c\mu\lambda} - h_{cb}{}^\lambda l_{d\mu\lambda}, \tag{7. 3}$$

which are obtained by using Weingarten equation (6. 1), we have

$$K_{dcba} = k[g_{da}g_{cb} - g_{db}g_{ca} + f_{da}f_{cb} - f_{db}f_{ca} - 2f_{dc}f_{ba}] + (h_{da}{}^\lambda h_{cb\lambda} - h_{db}{}^\lambda h_{ca\lambda}), \tag{7. 4}$$

and

$$k[p_{\mu d}f_{cb} - p_{\mu c}f_{db} - 2f_{dc}p_{\mu b}] = \nabla_d h_{cb\mu} - \nabla_c h_{db\mu} + h_{db}{}^\lambda l_{c\mu\lambda} - h_{cb}{}^\lambda l_{d\mu\lambda}, \tag{7. 5}$$

where $p_\mu{}^a = -B^a{}_j F_i{}^j C_\mu{}^i$ and $p_{\mu b} = g_{be} p_\mu{}^e$.

We now assume that the second fundamental tensor $h_{cb}{}^\lambda$ satisfies

$$h_{cb}{}^\lambda = A^\lambda g_{cb} + B^\lambda m_{cb} \text{[13)]} \qquad \text{for any } \lambda, \tag{7. 6}$$

where A^λ and B^λ are scalar functions. Substituting (7. 6) into (7. 4) and (7. 5), we get

12) Yano and Mogi [19].

13) As is well known [11], if in the pseudo-invariant submanifold of a Kaehlerian manifold the induced almost contact metric structure is Sasakian, then the second fundamental tensor has the form (7. 6).

$$
(7.7) \qquad \begin{cases} K_{dcba}=(k+A_\lambda A^\lambda)(g_{da}g_{cb}-g_{db}g_{ca})+k(f_{da}f_{cb}-f_{db}f_{ca} \\ \qquad -2f_{dc}f_{ba})+A^\lambda B_\lambda(g_{da}m_{cb}-g_{db}m_{ca}+g_{cb}m_{da}-g_{ca}m_{db}) \\ \qquad +B^\lambda B_\lambda(m_{da}m_{cb}-m_{db}m_{ca}), \end{cases}
$$

and

$$
\begin{aligned} k(p_{\mu d}f_{cb}-p_{\mu c}f_{db}-2f_{dc}p_{\mu b}) &= \nabla_d A_\mu\cdot g_{cb}-\nabla_c A_\mu\cdot g_{db}+\nabla_d B_\mu\cdot m_{cb}-\nabla_c B_\mu\cdot m_{db} \\ &\quad +B_\mu(\nabla_d m_{cb}-\nabla_c m_{db})+h_{db}{}^\lambda l_{c\mu\lambda}-h_{cb}{}^\lambda l_{d\mu\lambda}. \end{aligned}
$$

We assume that the vector fields p_μ belong to the distribution M. Then, transvecting $f^{dc}p_\lambda{}^b$ to the last equation, we get

$$
(7.8) \qquad (n-r)kp_{\mu b}p_\lambda{}^b=B_\mu\nabla_d m_b{}^c\cdot f_e{}^d p_\lambda{}^b.
$$

Because, provided that we substitute (7.6) into $h_{db}{}^\lambda l_{c\mu\lambda}\cdot f_y{}^b f^{dc}$, this term vanishes identically.

From now on, we assume that an f-submanifold M^n is framed in a Kaehlerian space of constant holomorphic curvature.

Then (7.8) is simplified as follows:

$$
(7.8)' \qquad (n-r)kg_{\lambda y}=B_\lambda\nabla_d f_y{}^e\cdot f_e{}^d.
$$

Substituting (6.4) into $\nabla_d f_y{}^e\cdot f_e{}^a$, we get

$$
(7.9) \qquad \nabla_d f_y{}^e\cdot f_e{}^d=-(n-r)A_y,
$$

and hence, from (7.8)′ and (7.9), we have

$$
(7.10) \qquad kg_{\lambda y}=-A_y\cdot B_\lambda.
$$

On the other hand, making use of the property of the induced f-structure, we show that

$$
(7.11) \qquad A_x+B_x=0 \qquad \text{for any } x,
$$

where $r\geqq 2$. In fact, it follows from (5.2) that

$$
\nabla_d f_y{}^e\cdot f_e{}^d=-(n-1)A_y-(r-1)B_y.
$$

Taking account of this equation and (7.9), we see that (7.11) is valid.

By virtue of (7.10) and (7.11), we state the following

THEOREM 7.1. *If, in an f-submanifold M^n framed in a Kaehlerian space M^m of constant holomorphic curvature, all second fundamental tensors $h_{cb}{}^\lambda$ satisfy*

$$
h_{cb}{}^\lambda=A^\lambda g_{cb}+B^\lambda m_{cb} \qquad \text{for any } \lambda,
$$

then the holomorphic sectional curvature k is non-negative constant, where $r\geqq 2$.

Especially, if the space M^m is Euclidean, then $h_{cb}{}^x=0$ for any x, and if k is negative then there exists at least an index x such that $h_{cb}{}^x\neq A^x g_{cb}+B^x m_{cb}$.

In particular, it follows from (7.10) that the following result is valid:

THEOREM 7. 2. *In a non-Euclidean Kaehlerian space M^m of constant holomorphic curvature, there exists no totally umbilical f-submanifold framed in M^m.*

Suppose that $k=0$ and $r=m-n\geqq 2$. Then, making use of Theorem 7. 1, we get $A^\lambda=0$ and $B^\lambda=0$ for any λ. Thus we find

COROLLARY 7. 3. *Under the assumption of Theorem 7. 1 if a space M^m is Euclidean with a natural Kaehlerian structure and $r=m-n\geqq 2$, then the f-submanifold is locally Euclidean also.*

Finally, we assume that $r=1$. Then, we get $m_{da}m_{cb}-m_{db}m_{ca}=0$, that is, the last term of the right hand side in (7. 7) vanishes identically. On the other hand, as mentioned in the previous section, the condition $r=1$ means that the submanifold admits an almost contact metric structure. An almost contact metric space which has the curvature tensor of the form (7. 7) without the last term is called[14] a locally C-Fubinian space. Thus we have

COROLLARY 7. 4. *Under the assumption of Theorem 7. 1, if $r=1$, then the submanifold is locally C-Fubinian.*

8. Flat distributions.

In this section, let M^m be a Kaehlerian space and M^n be the f-submanifold framed in M^m. If a distribution of M^n is parallel when we displace in any direction contained in M, the distribution is said to be parallel along M. When we translate a vector contained in the distribution M parallelly along itself, if the translated vector is always belonged in M, the distribution M is said[15] to be flat. It is known that the condition for the distribution M to be flat is

$$m_c{}^e \nabla_e m_b{}^a=0. \tag{8. 1}$$

Now, taking account of the definition (1. 2) of the projection operator m, we get

$$\nabla_c m_b{}^a=\nabla_c f_b{}^e\cdot f_e{}^a+f_b{}^e\nabla_c f_e{}^a.$$

Making use of (5. 2), we easily get

$$\nabla_c m_b{}^a=-h_{ce}{}^z\,(f_z{}^a f_b{}^e+f_{zb}f^{ae}). \tag{8. 2}$$

Suppose that $S_{cy}{}^x$ and $S_{zy}{}^a$ vanish identically. Under this assumption, Theorem 6. 1 and the fourth equation of (6. 5) mean that $f_y{}^e f_c{}^d h_{ed}{}^x=0$. The last equation shows that the right hand side of (8. 2) vanishes identically. This implies that (8. 1) is valid. Thus we find

THEOREM 8. 1. *If in an f-submanifold M^n framed in a Kaehlerian space M^m the tensors $S_{cy}{}^x$ and $S_{zy}{}^a$ vanish identically, then the induced distribution M is flat.*

14) Okumura [7].

15) Concerning flat distributions, see Kotō [2], Walker [12] and Yano [16].

9. Hypersurfaces of a Kaehlerian space.

As is well known, an odd dimensional almost contact metric manifold is similar to an almost Hermitian space in formal aspect. Giving attention to this fact, the present author has defined several structures in an almost contact metric manifold which correspond to restricted almost Hermitian structures in the previous paper [3] and investigated the mutual relations among these structures. In this section, we shall afford the examples of the new structures defined on the almost contact metric manifold by studying the structure of the almost contact metric hypersurface of a Kaehlerian space, whose hypersurface is one of the important examples about the almost contact metric structure it is possible for us to give.

Let M^{m-1} be an almost contact metric manifold and $(\phi_c{}^b, \xi^b, \eta_c, g_{cb})$ be an almost contact metric structure. If in the almost contact metric manifold M^{m-1} a tensor field ϕ satisfies $\nabla_c\phi_b{}^c=0$ and the divergence of a vector field ξ vanishes identically, then the structure is said[16] to be of A-type. As mentioned already in the previous sections, if the 1-form θ_1 and 2-form θ_2 are both closed, then the structure is said[17] to be of K_1-type. In the last, the almost contact metric structure is said to be of K_2-type, if the given tensor field ϕ and the vector field η are both covariantly constant.

Now, let M^m be a Kaehlerian space. Let M^{m-1} be the almost contact metric hypersurface of M^m and $(f_b{}^a, f^a, f_b, g_{cb})$[18] be the induced almost contact metric structure. Making use of Gauss and Weingarten equations of the hypersurface, we see[19] that the following equations

$$\nabla_c f_b = -h_{ce} f_b{}^e \tag{9. 1}$$

and

$$\nabla_c f_b{}^a = -h_{cb} f^a + h_c{}^a f_b \tag{9. 2}$$

are always valid, where h_{cb} is the second fundamental tensor of the hypersurface. In an almost contact metric hypersurface M^{m-1} of a Kaehlerian manifold M^m, the four tensor $S_{cb}{}^a$, $S_b{}^a$, S_{cb} and S_b introduced by Sasaki and Hatakeyama [9] are useful.

From now on, we assume that the tensor S_b vanish identically, that is, the vector field f^a defines a principal direction of the second fundamental tensor h_{cb} of the hypersurface. Let α be a characteristic root of the second fundamental tensor h_{cb} with respect to the principal direction f^a. Contracting with respect to a and c in (9. 2), in the almost contact metric hypersurface under the assumption we get

$$\nabla_c f_b{}^c = (h_c{}^c - \alpha) f_b, \tag{9. 3}$$

from which we have

THEOREM 9. 1. *If in the almost contact metric hypersurface M^{m-1} of a Kaehl-*

16) Concerning the structure of each type, see Nakagawa [3].
17) The structure of K_1-type is studied by Okumura [6] and Takizawa [10].
18) In this section, indices $a, b, \cdots$ run over the range $1, 2, \cdots, m-1$.
19) Okumura [5].

erian space M^m the tensor S_b vanishes identically, then it is necessary and sufficient for the induced almost contact metric structure to be of A-type that the first mean curvature of the hypersurface is equal to $\alpha/(m-1)$.

Next we require the condition under which the almost contact metric hypersurface of a Kaehlerian space is of K_1-type. Suppose that $\nabla_c\alpha=\beta f_c$, where β is a scalar field. Differentiating $\nabla_c\alpha=\beta f_c$ along the hypersurface, we get

$$\nabla_b\nabla_c\alpha=\nabla_b\beta\cdot f_c+\beta\nabla_b f_c.$$

Since the left hand side of the equation above is symmetric in b and c, we have $\beta(\nabla_c f_b-\nabla_b f_c)=f_c\nabla_b\beta-f_b\nabla_c\beta$. Transvecting f^b to this, we have $\nabla_c\beta=(f^b\nabla_b\beta)f_c$, from which we get

$$\beta(\nabla_c f_b-\nabla_b f_c)=0.$$

Thus we find

LEMMA 9. 2. *If in the analytic almost contact metric hypersurface M^{m-1} of a Kaehlerian space M^m the tensor S_b vanishes identically and moreover if $\nabla_c\alpha=\beta f_c$, then the scalar α is constant or the induced almost contact metric structure is of K_1-type.*

Taking account of this lemma, we can prove the following

THEOREM 9. 3. *If in the analytic almost contact metric hypersurface M^{m-1} of a Kaehlerian space M^m of constant holomorphic curvature the tensor S_b vanishes identically and moreover α is non-constant, then the induced almost contact metric structure is of K_1-type.*

Proof. Under the assumption of this theorem, it is seen that we have

$$(\nabla_b\alpha\cdot f_c-\nabla_c\alpha\cdot f_b)+\alpha(\nabla_b f_c-\nabla_c f_b)=-2kf_{bc}-2h_{ce}h_{bd}f^{ed}. \tag{9. 4}$$

Transvecting f^b to this,we show that the equation $\nabla_c\alpha=\beta f_c$ is valid. Consequently (9. 4) implies that

$$\alpha(\nabla_b f_c-\nabla_c f_b)=-2kf_{bc}-2h_{ce}h_{bd}f^{ed}. \tag{9. 5}$$

As α is non-constant, Lemma 9. 2 follows that $\nabla_c f_b-\nabla_b f_c=0$. Taking account of Lemma 4. 1, we show that 2-form $\theta_2=(1/2)f_{cb}dx^c\wedge dx^b$ is closed. Thus the induced almost contact metric structure is K_1-type. This completes the proof.

Finally, we consider the case that the second fundamental tensor h_{cb} is of rank one, that is,

$$h_{cb}=\alpha f_c f_b.$$

About this property, it is known[20] that the following theorem is valid:

THEOREM 9. 4. *If in the almost contact metric hypersurface M^{m-1} of a Kaehlerian space M^m the tensor S_b vanishes identically, then it is necessary and sufficient for the induced almost contact metric structure to be of K_2-type that the*

20) Yano and Ishihara [18].

second fundamental tensor is of rank one.

Making use of Theorem 9. 3 and Theorem 9. 4 and taking account of (9. 5), we obtain the following

COROLLARY 9. 5. *If in the almost contact metric hypersurface M^{m-1} of a Kaehlerian space of constant holomorphic curvature the tensor S_b vanishes identically and α is non-constant, then the hypersurface admits the almost contact metric structure of K_1-type and not of K_2-type.*

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TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY.