THE f-STRUCTURE INDUCED ON SUBMANIFOLDS OF COMPLEX AND ALMOST COMPLEX SPACES

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Let there be given, in a differentiable manifold V, a tensor field f of type (1, 1), a vector field U and a 1-form ω such that

$$f^2X+X=\omega(X)U,$$

$$fU=0, \qquad \omega(fX)=0, \qquad \omega(U)=1$$

for an arbitrary vector field X. The structure defined by f, U and ω is called an almost contact structure (Cf. [4], [6], [7], [8], [9], [10], [11], [12], [21]).

It is easily seen that when the manifold V admits an almost contact structure, the product space $V\times R$ admits an almost complex structure, R being the real line. When this almost complex structure is integrable, the original almost contact structure is said to be normal. The notion of the normality introduced by Sasaki and Hatakeyama [9] plays an important role in the study of differentiable manifolds with almost contact structure.

For example, a hypersurface in an almost complex space admits an almost contact structure and hypersurfaces in an even-dimensional Euclidean space are found to form a very interesting and important class of hypersurfaces (Cf. [4], [6], [7], [10], [11], [12], [21]).

When f, U and ω define an almost contact structure, we can easily obtain $f^3+f=0$ from the first and the second equations above. Conversely, if a tensor field f of type (1, 1) and of rank $n-1$ everywhere in an n-dimensional orientable manifold satisfies $f^3+f=0$, then it defines an almost contact structure in the manifold.

A tensor field of type (1, 1) satisfying $f^3+f=0$ and of rank r everywhere is called an f-structure of rank r. The normality of f-structure has been defined and studied by one of the present authors [2].

The main purpose of the present paper is to show first of all that a general submanifold in an almost complex space admits what we call an f-structure under certain conditions and then to study the properties of f-structures on submanifolds in complex and almost complex spaces, that is, those of f-structures on submanifolds in a locally flat complex space, in an almost Hermitian space, in a Kählerian space and in a Fubini space.

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CONTENTS

§ 1. *f*-structure and its integrability conditions.

Let there be given, in an n-dimensional differentiable manifold V of class C^∞, a non-null tensor field f of type (1, 1) and of class C^∞ satisfying the equation

$$f^3+f=0. \tag{1.1}$$

We call such a structure f an *f-structure* of rank r, when the rank of f is constant everywhere in the manifold and is equal to r, r being necessarily even (Cf. [17], [18]).

If we put

$$l=-f^2, \qquad m=f^2+1,$$

then we find immediately

$$l^2=l, \qquad m^2=m,$$

$$l+m=1, \qquad lm=ml=0,$$

where 1 denotes the Kronecker's unit tensor. These equations show that the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators. Thus there exist in the manifold complementary distributions L and M corresponding to the projection operators l and m respectively. When the rank of f is r, the distribution L is r-dimensional and M is $(n-r)$-dimensional.

The Nijenhuis tensor $N_{cb}{}^a$ of the f-structure f is by definition[1)]

$$N_{cb}{}^a=(f_c{}^e\nabla_e f_b{}^a-f_b{}^e\nabla_e f_c{}^a)-(\nabla_c f_b{}^e-\nabla_b f_c{}^e)f_e{}^a, \tag{1.2}$$

$f_b{}^a$ being the components of f and ∇_b denoting covariant differentiation with respect to a *symmetric* linear connection. It is easily seen that $N_{cb}{}^a$ does not depend on the symmetric linear connection ∇_b involved. Denote by $l_b{}^a$ and $m_b{}^a$ the components of l and m respectively. We have proved in [3] the following theorems.

1) The indices a, b, c, d, e, f run over the range $\{1, 2, \cdots, n\}$.

THEOREM A. *A necessary and sufficient condition for the distribution M to be integrable is that*

$$m_c{}^e m_b{}^d N_{ed}{}^a = 0,$$

or equivalently

$$m_c{}^e m_b{}^d N_{ed}{}^f f_f{}^a = 0.$$

THEOREM B. *A necessary and sufficient condition for the distribution L to be integrable is that*

$$N_{cb}{}^e m_e{}^a = 0,$$

or equivalently

$$l_c{}^e l_b{}^d N_{ed}{}^f m_f{}^a = 0.$$

THEOREM C. *A necessary and sufficient condition for both of two distributions L and M to be integrable is that $N_{cb}{}^a$ has the form*

$$N_{cb}{}^a = l_c{}^e l_b{}^d N_{ed}{}^f l_f{}^a + l_c{}^e m_b{}^d N_{ed}{}^a + m_c{}^e l_b{}^d N_{ed}{}^a.$$

Suppose that the distribution L is integrable. Then, since $fl = lf$ and $f^2 l = -l$, f acts as an almost complex structure on each integrable manifold of L. If L is integrable and this almost complex structure is also integrable in each integral manifold of L, we say that the f-structure is *partially integrable.*

THEOREM D. *A necessary and sufficient condition for an f-structure f to be partially integrable is that*

$$l_c{}^e l_b{}^d N_{ed}{}^a = 0.$$

Suppose that for any point of the manifold there exists a coordinate neighborhood of the manifold with respect to which f has the numerical components

$$(f_b{}^a) = \begin{pmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

E being an $m \times m$ unit matrix, where $r = 2m$ is the rank of f. In this case, we say that the f-structure f is *integrable.*

THEOREM E. *A necessary and sufficient condition for an f-structure f to be integrable is that*

$$N_{cb}{}^a = 0.$$

Let there be given a positive definite Riemannian metric a_{cb} in a differentiable manifold admitting an f-structure $f_b{}^a$. Putting

$$\bar{a}_{cb}=\frac{1}{2}[a_{cb}+(l_c{}^e-m_c{}^e)(l_b{}^d-m_b{}^d)a_{ed}],$$

we easily see that

$$g_{cb}=\frac{1}{2}[\bar{a}_{cb}+f_c{}^e f_b{}^d \bar{a}_{ed}+m_c{}^e m_b{}^d \bar{a}_{ed}] \tag{1.3}$$

is also a positive definite Riemannian metric. It is easily verified that

$$\left\{\begin{array}{c} f_c{}^e f_b{}^d g_{ed}+m_{cb}=g_{cb}, \\ f_{cb}=-f_{bc}, \qquad l_{cb}=l_{bc}, \qquad m_{cb}=m_{bc}, \\ l_c{}^e m_b{}^d g_{ed}=0, \end{array}\right. \tag{1.4}$$

where, by definition,

$$f_{cb}=f_c{}^e g_{eb}, \qquad l_{cb}=l_c{}^e g_{eb}, \qquad m_{cb}=m_c{}^e g_{eb}.$$

We call a pair of such an f-structure $f_b{}^a$ and a Riemannian metric g_{cb} an (f, g)-*structure*. Thus we have

THEOREM 1. 1. *There always exists in a differentiable manifold admitting an f-structure $f_b{}^a$ a positive definite Riemannian metric g_{cb} such that $f_b{}^a$ and g_{cb} form an (f, g)-structure.*

§ 2. Normal f-structure.

Let U be a coordinate neighborhood of an n-dimensional differentiable manifold V admitting an f-structure $f_b{}^a$ of rank r and (η^a) local coordinates defined in U. There exist, in U, r (contravariant) vector fields $f_q{}^a$ spanning the distribution L and $n-r$ vector fields $f_y{}^a$ spanning the distribution M.[2)]

If we denote by $(f^p{}_b, f^x{}_b)$ the inverse of the matrix $(f_q{}^a, f_y{}^a)$, we have

$$l_b{}^a=f^p{}_b f_p{}^a, \qquad m_b{}^a=f^x{}_b f_x{}^a,$$

from which

$$f_b{}^c f_c{}^a=-\delta_b^a+f^x{}_b f_x{}^a$$

by virtue of $m=f^2+1$. We also have

$$f^p{}_c f_y{}^c=0, \qquad f^x{}_a f_q{}^a=0, \qquad f^x{}_a f_y{}^a=\delta_y^x,$$

2) The indices p, q run over the range $\{1, 2, \cdots, r\}$ and the indices u, v, w, x, y, z the range $\{r+1, r+2, \cdots, n\}$.

from which

$$f_c{}^a f_y{}^c=0, \qquad f^x{}_a f_c{}^a=0$$

by virtue of $l_b{}^a=f^p{}_b f_p{}^a$ and $fl=f$.

Summing up, we have

$$\left\{\begin{aligned} &f_b{}^c f_c{}^a=-\delta_b^a+f^y{}_b f_y{}^a, \qquad f_y{}^c f_c{}^a=0, \\ &f_b{}^c f^x{}_c=0, \qquad f_y{}^c f^x{}_c=\delta_y^x. \end{aligned}\right. \tag{2. 1}$$

The ordered set $\{f_x{}^a\}$ is called an $(n-r)$*-frame* in U and the ordered set $\{f^y{}_b\}$ the $(n-r)$*-coframe* dual to $\{f_x{}^a\}$.

Let v^a be a vector field belonging to the distribution M at each point. Then, v^a is expressed in U uniquely as

$$v^a=v^x f_x{}^a,$$

which is a linear combination of $f_x{}^a$, and v^x is called the *components* of v^a with respect to the $(n-r)$-frame $\{f_x{}^a\}$. Consider a covector field ϕ_b such that

$$\phi_e f_b{}^e=0$$

or equivalently

$$\phi_e l_b{}^e=0.$$

We say that ϕ_b is *transversal to* the distribution L. If we transvect the first equation of (2. 1) with ϕ_b, we see that ϕ_b is expressed uniquely as

$$\phi_b=\phi_y f^y{}_b,$$

where

$$\phi_y=\phi_b f_y{}^b.$$

We call ϕ_y the *components* of ϕ_b with respect to the $(n-r)$-frame $\{f_x{}^a\}$.

Denoting by $M(V)$ the vector bundle consisting of all vectors which belong to the distribution M, we see that $M(V)$ is a subbundle of the tangent bundle $T(V)$ of the manifold V. Let $M^*(V)$ be the vector bundle over V which is dual to $M(V)$. Then, it is easily seen that $M^*(V)$ can be identified, in a natural way, with the covector bundle consisting of all covectors which are transversal to the distribution L. In this sense, $M^*(V)$ is regarded as a subbundle of the cotangent bundle $T^*(V)$ of V.

Let there be given a tensor field $T_{cb}{}^a$ such that it is expressed in each coordinate neighborhood by

$$T_{cb}{}^a=T_{cy}{}^x f^y{}_b f_x{}^a.$$

Then, transvecting $T_{cb}{}^a$ with an arbitrary vector field v^c, we have

$$v^c T_{cb}{}^a = (v^c T_{cy}{}^x) f^y{}_b f_x{}^a,$$

which is a cross-section of the tensor bundle $M^*(V)\otimes M(V)$. We call such a tensor field $T_{cb}{}^a$ an *$M^*(V)\otimes M(V)$-valued tensor field of type* (0, 1) and $T_{cy}{}^x$ its *components* with respect to $(\eta^a, f_x{}^a)$. An $M^*(V)\otimes M(V)$-valued tensor field of type (1, 1) has components of the form $T_d{}^c{}_y{}^x$ and is given by

$$T_d{}^c{}_b{}^a = T_d{}^c{}_y{}^x f^y{}_b f_x{}^a.$$

Similarly, we can define tensor fields of any such mixed type (Cf. Ishihara [2]).

Let there be given a connection ω^* in the vector bundle $M(V)$ and denote by $\Gamma_c{}^x{}_y$ the components of ω^* with respect to $(\eta^a, f_x{}^a)$ in each coordinate neighborhood of the manifold V. Let v^a be a vector field belonging to the distribution M. Then, v^a is regarded as a cross-section of the vector bundle $M(V)$ and is expressed as

$$v^a = v^x f_x{}^a$$

in each coordinate neighborhood. If we put

$$\nabla_c v^x = \partial_c v^x + \Gamma_c{}^x{}_y v^y, \tag{2.2}$$

it is easily verified that

$$(\nabla_c v^x) f_x{}^a$$

is an $M(V)$-valued tensor field of type (0, 1), which is globally defined in V (Cf. Ishihara [2]). We call the tensor field $(\nabla_c v^x) f_x{}^a$ or, simply, $\nabla_c v^x$ the *covariant derivative* of v^x with respect to the connection ω^*.

Let there be given a linear connection ω in the manifold V and $\Gamma_c{}^a{}_b$ its components in local coordinates (η^a). If, taking an $M^*(V)\otimes M(V)$-valued tensor field $T^{ax}{}_y$ of type (1, 0), we put

$$\nabla_c T^{ax}{}_y = \partial_c T^{ax}{}_y + \Gamma_c{}^a{}_e T^{ex}{}_y + \Gamma_c{}^x{}_z T^{az}{}_y - \Gamma_c{}^z{}_y T^{ax}{}_z, \tag{2.3}$$

then we can easily verify that $(\nabla_c T^{ax}{}_y) f^y{}_b f_x{}^d$ or, simply, $\nabla_c T^{ax}{}_y$ is an $M^*(V)\otimes M(V)$-valued tensor field of type (1, 1), which is globally defined in V (Cf. Ishihara [2]). We call the tensor field $\nabla_c T^{ax}{}_y$ the *convariant derivative* of $T^{ax}{}_y$ with respect to both of the connections ω and ω^*.

We assume hereafter that the connection $\Gamma_c{}^a{}_b$ is *symmetnic*. On putting

$$L_{cb}{}^x = f_c{}^e(\nabla_e f^x{}_b - \nabla_b f^x{}_e), \tag{2.4}$$

we call $L_{cb}{}^x$ *Levi tensor*, which is an $M(V)$-valued tensor field of type (0, 2). We have introduced in [2] the following five tensor fields:

$$
(2.5) \qquad \begin{cases} S_{cb}{}^{a}=N_{cb}{}^{a}+(\nabla_{c}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{c})f_{x}{}^{a}, \\ S_{cb}{}^{x}=L_{cb}{}^{x}-L_{bc}{}^{x}, \\ S_{cy}{}^{a}=f_{y}{}^{e}\nabla_{e}f_{c}{}^{a}-f_{c}{}^{e}\nabla_{e}f_{y}{}^{a}+f_{e}{}^{a}\nabla_{c}f_{y}{}^{e}, \\ S_{cy}{}^{x}=f_{y}{}^{e}(\nabla_{e}f^{x}{}_{c}-\nabla_{c}f^{x}{}_{e}), \\ S_{zy}{}^{a}=f_{z}{}^{e}\nabla_{e}f_{y}{}^{a}-f_{y}{}^{e}\nabla_{e}f_{z}{}^{a}, \end{cases}
$$

where $N_{cb}{}^{a}$ is the Nijenhuis tensor of the given f-structure $f_{b}{}^{a}$. These five tensors S's do not depend on the symmetric connection $\Gamma_{c}{}^{a}{}_{b}$ in V, but they are determined by giving the connection $\Gamma_{c}{}^{x}{}_{y}$ in the vector bundle $M(V)$.

We shall now find out identities involving these tensors L and S's. Taking account of (2. 4) and (2. 5), we find

$$
(2.6) \qquad \begin{cases} f_{y}{}^{c}L_{cb}{}^{x}=0, \\ L_{cb}{}^{x}f_{y}{}^{b}=f_{c}{}^{e}(\nabla_{e}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{e})f_{y}{}^{b} \\ \qquad =-f_{c}{}^{e}S_{ey}{}^{x}, \\ f_{c}{}^{e}L_{eb}{}^{x}=-(\nabla_{c}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{c})+f^{y}{}_{c}S_{by}{}^{x}, \\ L_{ce}{}^{x}f_{b}{}^{e}=(\nabla_{e}f^{x}{}_{d}-\nabla_{d}f^{x}{}_{e})f_{c}{}^{e}f_{b}{}^{d}, \\ L_{ed}{}^{x}f_{c}{}^{e}f_{b}{}^{d}=L_{bc}{}^{x}+S_{ey}{}^{x}f^{y}{}_{c}f_{b}{}^{e}. \end{cases}
$$

From the first equation of (2. 5), we obtain

$$
(2.7) \qquad S_{cb}{}^{a}f^{x}{}_{a}=-(\nabla_{e}f^{x}{}_{d}-\nabla_{d}f^{x}{}_{e})f_{c}{}^{e}f_{b}{}^{d}+(\nabla_{c}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{c}),
$$

$$
(2.8) \qquad S_{cb}{}^{a}f_{y}{}^{b}=S_{cy}{}^{e}f_{e}{}^{a}-S_{cy}{}^{x}f_{x}{}^{a},
$$

from which we have, transvecting (2. 8) with $f^{z}{}_{a}$,

$$
(2.9) \qquad S_{cy}{}^{x}=-S_{cb}{}^{a}f^{x}{}_{a}f_{y}{}^{b}.
$$

Substituting into (2. 7) equations

$$
\nabla_{c}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{c}=-f_{c}{}^{e}L_{eb}{}^{x}+f^{y}{}_{c}S_{by}{}^{x},
$$

$$
(\nabla_{e}f^{x}{}_{d}-\nabla_{d}f^{x}{}_{e})f_{c}{}^{e}f_{b}{}^{d}=L_{ce}{}^{x}f_{b}{}^{e}
$$

obtained respectively from the third and the fourth equations of (2. 6), we find

$$
(2.10) \qquad S_{cb}{}^{a}f^{x}{}_{a}=-f_{c}{}^{e}L_{eb}{}^{x}-L_{ce}{}^{x}f_{b}{}^{e}+f_{c}{}^{y}S_{by}{}^{x}.
$$

If we transvect (2. 10) with $f_{d}{}^{b}$ and make use of the second and the fifth equations of (2. 6), we find

$$
(2.11) \qquad \begin{aligned} S_{cb}{}^{x}&=L_{cb}{}^{x}-L_{bc}{}^{x} \\ &=S_{ce}{}^{a}f_{b}{}^{e}f^{x}{}_{a}-S_{ey}{}^{x}f_{c}{}^{e}f^{y}{}_{b}, \end{aligned}
$$

from which we have

$$S_{cy}{}^{x}=S_{ed}{}^{x}f_{c}{}^{e}f_{y}{}^{d}+S_{zy}{}^{e}f^{z}{}_{c}f^{x}{}_{e} \tag{2.12}$$

as a consequence of the fourth and the fifth equations of (2.5).

Transvecting the first equation of (2.5) with $f_{z}{}^{c}f_{y}{}^{b}$, we get

$$S_{zy}{}^{a}=-S_{cb}{}^{a}f_{z}{}^{c}f_{y}{}^{b}. \tag{2.13}$$

By means of the third, the fourth and the fifth equations of (2.5), we obtain

$$S_{cy}{}^{e}f_{e}{}^{a}+f_{c}{}^{d}S_{dy}{}^{a}=S_{cy}{}^{x}f_{x}{}^{a}+f^{z}{}_{c}S_{yz}{}^{a}$$

and, substituting (2.8) in this equation,

$$f_{c}{}^{d}S_{dy}{}^{a}=-S_{ce}{}^{a}f_{y}{}^{e}+f^{z}{}_{c}S_{yz}{}^{a}. \tag{2.14}$$

Transvecting (2.14) with $f_{b}{}^{c}$, we obtain

$$S_{by}{}^{a}=S_{ed}{}^{a}f_{b}{}^{e}f_{y}{}^{d}-S_{zy}{}^{e}f^{z}{}_{b}f_{e}{}^{a} \tag{2.15}$$

by virtue of

$$f_{z}{}^{e}S_{ey}{}^{a}=S_{zy}{}^{e}f_{e}{}^{a}$$

obtained from the third and the fifth equations of (2.5).

Furthermore, we find

$$S_{cy}{}^{x}=S_{ey}{}^{d}f_{c}{}^{e}f^{x}{}_{d}+S_{zy}{}^{d}f^{z}{}_{c}f^{x}{}_{d} \tag{2.16}$$

by making use of the third, the fourth and the fifth equations of (2.5).

From (2.9), (2.11), (2.13) and (2.15), we have

PROPOSITION 2.1. *If $S_{cb}{}^{a}$ vanishes, then all the other S's, i.e. $S_{cb}{}^{x}$, $S_{cy}{}^{a}$, $S_{cy}{}^{x}$, $S_{zy}{}^{a}$ vanish* (Cf. Ishihara [2]).

From (2.16), we have

PROPOSITION 2.2. *If $S_{cy}{}^{a}$ and $S_{zy}{}^{a}$ vanish, then $S_{cy}{}^{x}$ vanishes* (Cf. Ishihara [2]).

From (2.12), we have

PROPOSITION 2.3. *If $S_{cb}{}^{x}$ and $S_{zy}{}^{a}$ vanish, then $S_{cy}{}^{x}$ vanishes* (Cf. Ishihara [2]).

When there is given a connection $\Gamma_{c}{}^{x}{}_{y}$ in the vector bundle $M(V)$, there exists an almost complex structure F in the bundle space of $M(V)$ (Cf. Ishihara [2]). If the almost complex structure F is complex analytic, then we say that the given f-structure is *normal* with respect to the connection $\Gamma_{c}{}^{x}{}_{y}$ in $M(V)$. We have proved in [2].

THEOREM F. *A necessary and sufficient condition for an f-structure f to be* *normal with respect to a connection $\Gamma_c{}^x{}_y$ in $M(V)$ is that*

$$S_{cb}{}^a=0$$

be satisfied and the given connection $\Gamma_c{}^x{}_y$ be of zero curvature.

In a manifold with an f-structure $f_b{}^a$, a tensor field, say, $T_{cb}{}^a$ (or $T_{cb}{}^x$) is said to *be congruent to zero with respect to* $f^x{}_c$, if it has components of the form

$$T_{cb}{}^a=f^y{}_cP_{yb}{}^a+f^y{}_bQ_{yc}{}^a \quad (\text{or } T_{cb}{}^x=f^y{}_cP_{yb}{}^x+f^y{}_bQ_{yc}{}^x), \tag{2.17}$$

$P_{yb}{}^a$ and $Q_{yc}{}^a$ (or $P_{yb}{}^x$ and $Q_{yc}{}^x$) being certain local tensor fields. In such a case, the relation (2. 17) will be expressed in a simplified form as

$$T_{cb}{}^a\equiv 0 \qquad (\text{or } T_{cb}{}^x\equiv 0). \tag{2.18}$$

The relation $U_{cb}{}^a-V_{cb}{}^a\equiv 0$ (or $U_{cb}{}^x-V_{cb}{}^x\equiv 0$) is expressed as

$$U_{cb}{}^a\equiv V_{cb}{}^a \qquad (\text{or } U_{cb}{}^x\equiv V_{cb}{}^x).$$

It is easily seen that (2. 18) is valid if and only if we have

$$T_{cb}{}^aw^cv^b=0 \qquad (\text{or } T_{cb}{}^xw^cv^b=0)$$

for any vector fields v^a and w^a such that $f^x{}_av^a=0$, $f^x{}_aw^a=0$, i.e. for any vector fields v^a and w^a belonging to the distribution L. We have from (2. 11)

PROPOSITION 2. 4. *If $S_{cb}{}^a\equiv 0$, then $S_{cb}{}^x\equiv 0$, that is*

$$L_{cb}{}^x\equiv L_{bc}{}^x.$$

It is easily seen from (2. 5) that each of the three conditions

$$S_{cb}{}^a\equiv 0, \qquad S_{cb}{}^x\equiv 0 \qquad \text{and} \qquad L_{cb}{}^x\equiv L_{bc}{}^x$$

does not depend on both of the connections $\Gamma_c{}^a{}_b$ and $\Gamma_c{}^x{}_y$ involved.

The distribution L is integrable, if and only if

$$\partial_cf^x{}_b-\partial_bf^x{}_c\equiv 0.$$

Consequently, taking account of (2. 6), we have from the definition (2. 4) of $L_{cb}{}^x$ and the definition (2. 5) of $S_{cb}{}^a$

PROPOSITION 2. 5. *A necessary and sufficient condition for the distribution L to be integrable is that one of the following three conditions is satisfied:*

$$S_{cb}{}^a-N_{cb}{}^a\equiv 0, \qquad (S_{cb}{}^a-N_{cb}{}^a)f^x{}_a\equiv 0,$$
$$f_c{}^eL_{eb}\equiv 0.$$

It is easily seen from (2. 5) that the three equivalent conditions stated in Proposition 2. 5 are independent of the connections $\Gamma_c{}^a{}_b$ and $\Gamma_c{}^x{}_y$ involved.

If we take account of the definition (2. 5) of $S_{cb}{}^a$, we see from Theorem A given in § 1 that the distribution M is integrable if and only if

$$S_{cb}{}^e f_z{}^c f_y{}^b f_e{}^a = 0$$

is satisfied. Thus we have

PROPOSITION 2. 6. *If an f-structure is normal, or if* $S_{cb}{}^a=0$, *then the distribution M is integrable.*

Let there be given an (f, g)-structure $(f_b{}^a, g_{cb})$ in a differentiable manifold. Let $\{f_x{}^a\}$ be an $(n-r)$-frame and $\{f^y{}_b\}$ the $(n-r)$-coframe dual to $\{f_x{}^a\}$. On putting

$$g_{ba} f_x{}^a = f_{xb}, \qquad g^{ab} f^y{}_b = f^{ya},$$
$$g_{yx} = f_y{}^c f_x{}^b g_{cb},$$

we have

$$f_{xb} = f^y{}_b g_{yx}, \qquad f^{ya} = f_x{}^a g^{xy}$$

by means of (1. 4), where g^{ba} and g^{yx} are defined by

$$(g^{ba}) = (g_{ba})^{-1} \qquad \text{and} \qquad (g^{yx}) = (g_{yx})^{-1}$$

respectively.

Let there be given a connection $\Gamma_c{}^x{}_y$ in the vector bundle $M(V)$ and suppose that

$$\nabla_c g_{yx} = \partial_c g_{yx} - \Gamma_c{}^z{}_y g_{zx} - \Gamma_c{}^z{}_x g_{yz} = 0$$

is satisfied. We call such a connection $\Gamma_c{}^x{}_y$ a *metric connection* in the bundle $M(V)$. Denoting by $\{{}_c{}^a{}_b\}$ the Riemannian connection determined by g_{cb}, we see that

$$\Gamma_c{}^x{}_y = [\partial_c f_y{}^a + \{{}_c{}^a{}_b\} f_y{}^b] f^x{}_a$$

define a metric connection in the vector bundle $M(V)$.

If we assume that

$$\nabla_c f^x{}_b - \nabla_b f^x{}_c = 2A^x f_{cb} \tag{2. 19}$$

is valid with a certain vector field A^x, $f_{cb}=-f_{bc}$ being defined by

$$f_{cb} = f_c{}^a g_{ab},$$

then we find

$$L_{cb}{}^x = 2A^x(-g_{cb} + m_{cb}). \tag{2. 20}$$

In such a case, we have

$$\nabla_c v_b - \nabla_b v_c = 2(A^x v_x) f_{cb},$$

i.e.

$$d(v_b d\eta^b)=af_{cb}d\eta^c\wedge d\eta^b, \qquad a=\Lambda^x v_x$$

for any vector field $v_b=f_b{}^x v_x$ belonging to the distribution M and satisfying $\nabla_b v_x=0$.

§ 3. Surfaces in an almost complex space.

Let W be an N-dimensional differentiable manifold of class C^∞ with an almost complex structure $F=(F_i{}^h)$[3] of class C^∞, i.e.

$$F_i{}^k F_k{}^h=-\delta_i^h, \tag{3.1}$$

N being necessarily even.

Let there be given an n-dimensional submanifold V differentiably immersed in W, and denote by $T_P(V)$ the tangent space of V at a point P belonging to V. We suppose that

$$H_P(V)=T_P(V)\cap F(T_P(V))\neq\{0\}$$

and that $r=\dim H_P(V)$ is constant everywhere in V, r being necessarily even, where we have put

$$F(T_P(V))=\{FX|X\in T_P(V)\}.$$

If this is the case, we call the submanifold V an *f-submanifold* in the almost complex space W. The vector space $H_P(V)$ is called the *holomorphic tangent space* of V at P. On putting

$$T_P^H(V)=T_P(V)+F(T_P(V)), \tag{3.2}$$

we call this subspace $T_P^H(V)$ the *holomorphic extension of tangent space* $T_P(V)$. It is easily seen that, if $\dim H_P(V)=r$, we have $\dim T_P^H(V)=2n-r$.

If $2n>N$, we have $r>0$ because of $N\geqq\dim T_P^H(V)$. Thus we know that, in an almost complex space W of N dimensions, a submanifold of n dimensions is always an f-submanifold if $2n>N$ and $\dim H_P(V)$ is constant.

Let there be given an f-submanifold V in an almost complex space W. Then, there exists a subspace N_P of the holomorphic extension $T_P^H(V)$ of the tangent space $T_P(V)$ at each point P belonging to V such that

$$F(N_P)\subset T_P(V), \qquad T_P^H(V)=T_P(V)+N_P \qquad \text{(direct sum)}, \tag{3.3}$$

where $F(N_P)=\{FX|X\in N_P\}$. If $T_P(W)\neq H_P(V)$, in the tangent space $T_P(W)$ of the enveloping space W at P, there exists a subspace $\bar{N}_P$ such that

$$F(\bar{N}_P)=\bar{N}_P, \qquad T_P(W)=T_P^H(V)+\bar{N}_P \qquad \text{(direct sum)}, \tag{3.4}$$

3) The indices h, i, j, k, s, t run over the range $\{1, 2, \cdots, N\}$.

where $F(\bar{N}_P)=\{FX|X\in\bar{N}_P\}$. The subspaces N_P and $\bar{N}_P$ are respectively $(n-r)$-dimensional and $(N-2n+r)$-dimensional. Therefore, there exist along V two differentiable fields of such subspaces N_P and $\bar{N}_P$. If we put

$$N(V)=\bigcup_{P\in V} N_P, \qquad \bar{N}(V)=\bigcup_{P\in V} \bar{N}_P,$$

then $N(V)$ and $\bar{N}(V)$ are vector bundles over V. Letting $N(V)$ and $\bar{N}(V)$ be fixed, we call the set $\{V, N(V), \bar{N}(V)\}$ an *f-surface* and V its *base submanifold.* For the sake of simplicity, we denote sometimes an f-surface $\{V, N(V), \bar{N}(V)\}$ simply by V.

Let there be given an f-surface $\{V, N(V), \bar{N}(V)\}$ in an almost complex space W and its base submanifold V be expressed by equations

$$\xi^h=\xi^h(\eta^a)$$

in local coordinate (ξ^h) in W, where (η^a) is a system of local coordinates in V. If we put

$$B_a{}^h=\partial_a\xi^h, \qquad \partial_a=\partial/\partial\eta^a, \tag{3.5}$$

then $B_a{}^h$ are n local tangent vector fields in V and span the tangent space $T_P(V)$ of V at each point P of V. There exist locally along V $n-r$ vector fields $C_y{}^h$ and $N-2n+r$ vector fields $C_\beta{}^h$ which span respectively N_P and $\bar{N}_P$ at each point P of V[4]. Denoting by $(B^a{}_i, C^x{}_i, C^\alpha{}_i)$ the inverse of the matrix $\begin{pmatrix} B_b{}^h \\ C_y{}^h \\ C_\beta{}^h \end{pmatrix}$, we have

$$\left\{\begin{array}{lll} B^a{}_hB_b{}^h=\delta_b^a, & B^a{}_hC_y{}^h=0, & B^a{}_hC_\beta{}^h=0, \\ C^x{}_hB_b{}^h=0, & C^x{}_hC_y{}^h=\delta_y^x, & C^x{}_hC_\beta{}^h=0, \\ C^\alpha{}_hB_b{}^h=0, & C^\alpha{}_hC_y{}^h=0, & C^\alpha{}_hC_\beta{}^h=\delta_\beta^\alpha \end{array}\right. \tag{3.6}$$

and

$$B_a{}^hB^a{}_i+C_x{}^hC^x{}_i+C_\alpha{}^hC^\alpha{}_i=\delta_i^h. \tag{3.7}$$

Taking account of (3. 3) and (3. 4), we can put

$$\left\{\begin{array}{l} F_i{}^hB_b{}^i=f_b{}^aB_a{}^h+f^x{}_bC_x{}^h, \\ F_i{}^hC_y{}^i=-f_y{}^aB_a{}^h, \\ F_i{}^hC_\beta{}^i=f_\beta{}^\alpha C_\alpha{}^h, \end{array}\right. \tag{3.8}$$

and, taking account of (3. 1), we find

$$\left\{\begin{array}{ll} f_b{}^cf_c{}^a=-\delta_b^a+f^x{}_bf_x{}^a, & f_b{}^cf^x{}_c=0, \\ f_y{}^cf_c{}^a=0, & f_y{}^cf^x{}_c=\delta_y^x, \\ \multicolumn{2}{c}{f_\beta{}^\gamma f_\gamma{}^\alpha=-\delta_\beta^\alpha,} \end{array}\right. \tag{3.9}$$

4) The indices u, v, x, y, z hereafter run over the range $\{n+1, n+2, \cdots, 2n-r\}$ and the indices $\alpha, \beta, \gamma, \delta, \varepsilon$ run over the range $\{2n-r+1, 2n-r+2, \cdots, N\}$.

which implies

$$f^3+f=0, \tag{3.10}$$

f being the tensor field of type (1, 1) defined in V by components $f_b{}^a$. Thus, taking account of (2. 1), we see from (3. 9) and (3. 10) that $f_b{}^a$ is an f-structure, $\{f_y{}^a\}$ is an $(n-r)$-frame and $\{f^x{}_b\}$ the $(n-r)$-coframe dual to $\{f_y{}^a\}$, all in the sense of § 2. The tensor field $f_b{}^a$ is called the *induced f-structure* of the given f-surface V.

By means of the second equation of (3. 8), we find that $F(N(V))$ coincides with the vector bundle $M(V)$ consisting of all vectors which belong to the distribution M determined by the projection operator $m_b{}^a=f_b{}^cf_c{}^a+\delta_b^a$. Thus, if we define a bundle isomorphism F^*: $N(V)\rightarrow M(V)$ by

$$F^*(X)=-F(X), \tag{3.11}$$

X being an arbitrary vector belonging to $N(V)$, then for any vector X having components $v^xC_x{}^h$ the transformed vector $F^*(X)$ has components $(v^xf_x{}^a)B_a{}^h$ as a consequence of the second equation of (3. 8).

Let there be given a *symmetric* linear connection $\Gamma_j{}^h{}_i$ in the enveloping space W. If we put

$$\Gamma_c{}^a{}_b=(\partial_cB_b{}^h+B_c{}^jB_b{}^i\Gamma_j{}^h{}_i)B^a{}_h, \tag{3.12}$$

then $\Gamma_c{}^a{}_b$ define a symmetric linear connection in the base submanifold V, which is called the *induced connection* of V.

If we put

$$\Gamma_c{}^x{}_y=(\partial_cC_y{}^h+B_c{}^jC_y{}^i\Gamma_j{}^h{}_i)C^x{}_h, \tag{3.13}$$

then $\Gamma_c{}^x{}_y$ define a connection in the vector bundle $N(V)$ which is called the *induced connection* in the vector bundle $N(V)$. Since, as is seen from (3. 11), there exists a bundle isomorphism F^*: $N(V)\rightarrow M(V)$, there exists in $M(V)$ a connection ω^* induced from $\Gamma_c{}^x{}_y$ by F^* and the connection ω^* induced in $M(V)$ is expressed by the same components $\Gamma_c{}^x{}_y$ with respect to the $(n-r)$-frame $\{f_y{}^a\}$. We call briefly the connection ω^* the *induced connection* of the vector bundle $M(V)$.

If we put

$$\Gamma_c{}^\alpha{}_\beta=(\partial_cC_\beta{}^h+B_c{}^jC_\beta{}^i\Gamma_j{}^h{}_i)C^\alpha{}_h, \tag{3.14}$$

the $\Gamma_c{}^\alpha{}_\beta$ define a connection in the vector bundle $N(V)$. We call the connection $\Gamma_c{}^\alpha{}_\beta$ the *induced connection* in $N(V)$.

We define the van der Wearden-Bortolotti covariant derivatives of $B_b{}^h$, $C_y{}^h$ and $C_\beta{}^h$ along V by

$$\begin{cases} \nabla_cB_b{}^h=\partial_cB_b{}^h+B_c{}^jB_b{}^i\Gamma_j{}^h{}_i-B_a{}^h\Gamma_c{}^a{}_b, \\ \nabla_cC_y{}^h=\partial_cC_y{}^h+B_c{}^jC_y{}^i\Gamma_j{}^h{}_i-C_x{}^h\Gamma_c{}^x{}_y, \\ \nabla_cC_\beta{}^h=\partial_cC_\beta{}^h+B_c{}^jC_\beta{}^i\Gamma_j{}^h{}_i-C_\alpha{}^h\Gamma_c{}^\alpha{}_\beta \end{cases} \tag{3.15}$$

respectively. Then $\nabla_c B_b{}^h$, $\nabla_c C_y{}^h$ and $\nabla_c C_\beta{}^h$ belong respectively to $N_P+\bar{N}_P$, $T_P(V)+\bar{N}_P$ and $T_P(V)+N_P$ at each point P of V. Thus we can put

$$(3.16)\qquad \begin{cases} \nabla_c B_b{}^h = & h_{cb}{}^x C_x{}^h + h_{cb}{}^\alpha C_\alpha{}^h, \\ \nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h & + h_{cy}{}^\alpha C_\alpha{}^h, \\ \nabla_c C_\beta{}^h = -h_c{}^a{}_\beta B_a{}^h - h_c{}^x{}_\beta C_x{}^h, \end{cases}$$

where h's are the so-called *second fundamental tensors* of the given f-surface V. It is easily seen that

$$h_{cb}{}^x = h_{bc}{}^x, \qquad h_{cb}{}^\alpha = h_{bc}{}^\alpha.$$

If we differentiate covariantly each member of (3.8) along V and take account of (3.8) and (3.16), we obtain

$$(3.17)\qquad \begin{aligned} (\nabla_j F_i{}^h)B_c{}^j B_b{}^i &= (\nabla_c f_b{}^a + h_{cb}{}^y f_y{}^a - h_c{}^a{}_y f^y{}_b)B_a{}^h \\ &\quad + (\nabla_c f^x{}_b + h_{ce}{}^x f_b{}^e)C_x{}^h \\ &\quad + (h_{cb}{}^\alpha - h_{cb}{}^\gamma f_\gamma{}^\alpha + h_c{}^a{}_y f^y{}_b)C_\alpha{}^h, \\ (\nabla_j F_i{}^h)B_c{}^j C_y{}^i &= -(\nabla_c f_y{}^a - h_c{}^e{}_y f_e{}^a)B_a{}^h \\ &\quad + (h_c{}^e{}_y f^x{}_e - h_{ce}{}^x f_y{}^e)C_x{}^h \\ &\quad - (h_{ce}{}^\alpha f_y{}^e + h_c{}^\beta{}_y f_\beta{}^\alpha)C_\alpha{}^h, \\ (\nabla_j F_i{}^h)B_c{}^j C_\beta{}^i &= (h_c{}^e{}_\beta f_e{}^a - h_c{}^a{}_\gamma f_\beta{}^\gamma - h_c{}^y{}_\beta f_y{}^a)B_a{}^h \\ &\quad + (h_c{}^e{}_\beta f^x{}_e - h_c{}^x{}_\gamma f_\beta{}^\gamma)C_x{}^h \\ &\quad + (\nabla_c f_\beta{}^\alpha)C_\alpha{}^h, \end{aligned}$$

where we have put

$$(3.18)\qquad \nabla_c f_\beta{}^\alpha = \partial_c f_\beta{}^\alpha + \Gamma_c{}^\alpha{}_\gamma f_\beta{}^\gamma - \Gamma_c{}^\gamma{}_\beta f_\gamma{}^\alpha.$$

The Nijenhuis tensor $N_{ji}{}^h$ of the almost complex structure $F_i{}^h$ is by definition

$$(3.19)\qquad N_{ji}{}^h = F_j{}^t \nabla_t F_i{}^h - F_i{}^t \nabla_t F_j{}^h - (\nabla_j F_i{}^s - \nabla_i F_j{}^s)F_s{}^h.$$

Taking account of (3.8), (3.17) and (3.19), we find

$$(3.20)\qquad \begin{aligned} & N_{ji}{}^h B_c{}^j B_b{}^i \\ &= [S_{cb}{}^a - f^x{}_c(h_b{}^e{}_x f_e{}^a - f_b{}^e h_e{}^a{}_x) + f^x{}_b(h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x)]B_a{}^h \\ &\quad + [S_{cb}{}^x - (f^y{}_c h_b{}^e{}_y - f^y{}_b h_c{}^e{}_y)f^x{}_e]C_x{}^h \\ &\quad + [(f_c{}^e f_b{}^y - f_b{}^e f_c{}^y)h_{ey}{}^\alpha - (h_{cy}{}^\beta f^y{}_b - h_{by}{}^\beta f^y{}_c)f_\beta{}^\alpha]C_\alpha{}^h \\ &\quad + f^x{}_c(\nabla_j F_i{}^h)C_x{}^j B_b{}^i - f^x{}_b(\nabla_j F_i{}^h)C_x{}^j B_c{}^i, \end{aligned}$$

$$
\begin{aligned}
& N_{ji}{}^{h}B_{c}{}^{j}C_{y}{}^{i} \\
&= [S_{cy}{}^{a} + (h_{c}{}^{a}{}_{y} + h_{e}{}^{d}{}_{y} f_{c}{}^{e} f_{d}{}^{a} - f^{z}{}_{c} h_{c}{}^{a}{}_{z} f_{y}{}^{e}] B_{a}{}^{h} \\
&\quad + [S_{cy}{}^{x} + f_{c}{}^{e} h_{e}{}^{d}{}_{y} f^{x}{}_{d}] C_{x}{}^{h} \\
&\quad + [-h_{cy}{}^{\alpha} - f_{c}{}^{e} h_{ey}{}^{\beta} f_{\beta}{}^{\alpha} + f^{z}{}_{c} h_{ez}{}^{\alpha} f_{y}{}^{e}] C_{\alpha}{}^{h} \\
&\quad + f^{z}{}_{c}(\nabla_{j} F_{i}{}^{h}) C_{z}{}^{j} C_{y}{}^{i} + (\nabla_{j} F_{i}{}^{t}) C_{y}{}^{j} B_{c}{}^{i} F_{t}{}^{h},
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
& N_{ji}{}^{h}B_{c}{}^{j}C_{\beta}{}^{i} \\
&= [h_{c}{}^{a}{}_{\beta} + h_{c}{}^{e}{}_{\gamma} f_{e}{}^{a} f_{\beta}{}^{\gamma} + h_{e}{}^{d}{}_{\beta} f_{c}{}^{e} f_{d}{}^{a} - h_{c}{}^{a}{}_{\gamma} f_{c}{}^{e} f_{\beta}{}^{\gamma} - h_{e}{}^{y}{}_{\beta} f_{c}{}^{e} f_{y}{}^{a}] B_{a}{}^{h} \\
&\quad + [(h_{c}{}^{d}{}_{\gamma} f_{\beta}{}^{\gamma} + f_{c}{}^{e} h_{e}{}^{d}{}_{\beta}) f^{x}{}_{d} + h_{c}{}^{x}{}_{\beta} - h_{e}{}^{x}{}_{\gamma} f_{c}{}^{e} f_{\beta}{}^{\gamma}] C_{x}{}^{h} \\
&\quad + [f_{c}{}^{e}(\nabla_{e} f_{\beta}{}^{\alpha}) - (\nabla_{c} f_{\beta}{}^{\gamma}) f_{\gamma}{}^{\alpha}] C_{\alpha}{}^{h} \\
&\quad + (\nabla_{j} F_{i}{}^{s}) C_{\beta}{}^{j} B_{c}{}^{i} F_{s}{}^{h} + (\nabla_{j} F_{i}{}^{h}) f^{z}{}_{c} C_{z}{}^{j} C_{\beta}{}^{i} - f_{\beta}{}^{\gamma} C_{\gamma}{}^{j} (\nabla_{j} F_{i}{}^{h}) B_{c}{}^{i},
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
& N_{ji}{}^{h}C_{z}{}^{j}C_{y}{}^{i} \\
&= S_{zy}{}^{a} B_{a}{}^{h} + [-(f_{z}{}^{e} h_{c}{}^{d}{}_{y} - f_{y}{}^{e} h_{c}{}^{d}{}_{z}) f_{d}{}^{x}] C_{x}{}^{h} \\
&\quad + [(f_{z}{}^{e} h_{ey}{}^{\beta} - f_{y}{}^{e} h_{ez}{}^{\beta}) f_{\beta}{}^{\alpha}] C_{\alpha}{}^{h} \\
&\quad + (\nabla_{j} F_{i}{}^{t} - \nabla_{i} F_{j}{}^{t}) C_{z}{}^{j} C_{y}{}^{i} F_{t}{}^{h},
\end{aligned} \tag{3.23}
$$

where S's are tensor fields given in (2. 5), *connections involved in S's being the induced connections* $\Gamma_{c}{}^{a}{}_{b}$ and $\Gamma_{c}{}^{x}{}_{y}$. We have immediately from (3. 20)

THEOREM 3. 1. *For an f-surface V in an almost complex space W, the vector field $(N_{ji}{}^{h}B_{c}{}^{j}B_{b}{}^{i})w^{c}v^{b}$ belongs to the holomorphic extension $T^{H}_{\mathrm{P}}(V)$ of tangent space $T_{\mathrm{P}}(V)$ at each point* P *of V, v^{a} and w^{a} being arbitrary vector fields in V satisfying the conditions $f^{x}{}_{a}v^{a}=0$, $f^{x}{}_{a}w^{a}=0$.*

THEOREM 3. 2. *For an f-surface V in an almost complex space W, the vector field $(N_{ji}{}^{h}B_{c}{}^{j}B_{b}{}^{i})w^{c}v^{b}$ is tangent to V at each point of V, v^{a} and w^{a} being arbitrary vector fields satisfying the conditions $f^{x}{}_{a}v^{a}=0$, $f^{x}{}_{a}w^{a}=0$, if and only if $S_{cb}{}^{x}\equiv 0$, that is, if and only if $L_{cb}{}^{x}\equiv L_{bc}{}^{x}$. (The condition $S_{cb}{}^{x}\equiv 0$ does not depend on the induced connections $\Gamma_{c}{}^{a}{}_{b}$ and $\Gamma_{c}{}^{x}{}_{y}$ involved).*

THEOREM 3. 3. *For an f-surface V in an almost complex space W, we have $(N_{ji}{}^{h}B_{c}{}^{j}B_{b}{}^{i})w^{c}v^{b}=0$, v^{a} and w^{a} being arbitrary vector fields satisfying the conditions $f^{x}{}_{a}v^{a}=0$, $f^{x}{}_{a}w^{a}=0$, if and only if $S_{cb}{}^{a}\equiv 0$. (The condition $S_{cb}{}^{a}\equiv 0$ does not depend on the induced connections $\Gamma_{c}{}^{a}{}_{b}$ and $\Gamma_{c}{}^{x}{}_{y}$ involved).*

§ 4. *f*-surfaces in a complex space.

Let W be a complex space with a complex structure $F_{i}{}^{h}$. Then, the Nijenhuis tensor $N_{ji}{}^{h}$ of $F_i{}^h$ vanishes identically. As is well known, there exists a symmetric linear connection $\Gamma_j{}^h{}_i$ such that (Cf. Yano [20])

$$\nabla_j F_i{}^h = 0. \tag{4.1}$$

In the present paper, by a *complex space* we mean a space admitting a complex structure $F_i{}^h$ and a symmetric connection $\Gamma_j{}^h{}_i$ satisfying (4. 1).

Let there be given an f-surface V in a complex space W. Then, taking account of (4. 1), we have from (3. 17)

$$\tag{4.2} \begin{cases} \nabla_c f_b{}^a + h_{cb}{}^y f_y{}^a - h_c{}^a{}_y f^y{}_b = 0, \\ \nabla_c f^x{}_b + h_{ce}{}^x f_b{}^e = 0, \\ \nabla_c f_y{}^a - h_c{}^e{}_y f_e{}^a = 0, \\ \nabla_c f_\beta{}^\alpha = 0 \end{cases}$$

and

$$\tag{4.3} \begin{cases} h_c{}^e{}_y f^x{}_e - h_{ce}{}^x f_y{}^e = 0, \\ h_{cb}{}^\alpha - h_{cb}{}^\gamma f_\gamma{}^\alpha + h_c{}^\alpha{}_y f^y{}_b = 0, \\ h_{ce}{}^\alpha f_y{}^e + h_c{}^\beta{}_y f_\beta{}^\alpha = 0, \\ h_c{}^e{}_\beta f_e{}^a - h_c{}^a{}_\gamma f_\beta{}^\gamma - h_c{}^y{}_\beta f_y{}^a = 0, \\ h_c{}^e{}_\beta f^x{}_e - h_c{}^x{}_\gamma f_\beta{}^\gamma = 0. \end{cases}$$

On the other hand, since $N_{ji}{}^h = 0$, we find from (3. 20), (3. 21), (3. 22) and (3. 23)

$$\tag{4.4} \begin{cases} S_{cb}{}^a = f^x{}_c (h_b{}^e{}_x f_e{}^a - f_b{}^e h_e{}^a{}_x) - f^x{}_b (h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x), \\ S_{cb}{}^x = f^y{}_c H_{by}{}^x - f^y{}_b H_{cy}{}^x, \\ S_{cy}{}^a = -(h_c{}^a{}_y + h_e{}^d{}_y f_c{}^e f_d{}^a) + f^z{}_c (h_e{}^a{}_z f_y{}^e), \\ S_{cy}{}^x = -f_c{}^e H_{ey}{}^x, \\ S_{zy}{}^a = 0 \end{cases}$$

respectively, where, taking account of the first equation of (4. 3), we have put

$$H_{by}{}^x = h_{be}{}^x f_y{}^e = h_b{}^e{}_y f^x{}_e.$$

From the first equation of (4. 4), we have

PROPOSITION 4. 1. *For any f-surface in a complex space, we have*

$$S_{cb}{}^a \equiv 0.$$

From the expression (2. 4) for $L_{cb}{}^x$ and the second equation of (4. 2), we find

$$L_{cb}{}^{x}=-(h_{cb}{}^{x}+h_{ed}{}^{x}f_{c}{}^{e}f_{b}{}^{d})+f^{y}{}_{c}H_{by}{}^{x}. \tag{4.5}$$

Thus we have

PROPOSITION 4.2. *For any f-surface in a complex space, we have the expression* (4.5) *for* $L_{cb}{}^{x}$ *and* $L_{cb}{}^{x}\equiv L_{bc}{}^{x}$ (Cf. Hermann [1]).

We have from the second and the third equations of (4.2)

$$\begin{cases} h_{cb}{}^{x}=(\nabla_{c}f^{x}{}_{e})f_{b}{}^{e}+H_{cy}{}^{x}f^{y}{}_{b}, \\ h_{c}{}^{a}{}_{y}=-(\nabla_{c}f_{y}{}^{e})f_{e}{}^{a}+H_{cy}{}^{x}f_{x}{}^{a} \end{cases} \tag{4.7}$$

respectively. If we take account of $h_{cb}{}^{x}=h_{bc}{}^{x}$, we have directly from the first equation of (4.7)

$$(\nabla_{c}f^{x}{}_{e})f_{b}{}^{e}-(\nabla_{b}f^{x}{}_{e})f_{c}{}^{e}+H_{cy}{}^{x}f^{y}{}_{b}-H_{by}{}^{x}f^{y}{}_{c}=0. \tag{4.8}$$

Substituting (4.7) into the first equation of (4.2), we obtain

$$\nabla_{c}f_{b}{}^{a}+(\nabla_{c}f^{x}{}_{e})f_{b}{}^{e}f_{x}{}^{a}+(\nabla_{c}f_{y}{}^{e})f^{y}{}_{b}f_{e}{}^{a}=0. \tag{4.9}$$

We have moreover from (4.2)

$$(\nabla_{c}f^{x}{}_{e})f_{y}{}^{e}=0, \qquad (\nabla_{c}f_{y}{}^{e})f^{x}{}_{e}=0. \tag{4.10}$$

Taking account of (4.10), we see from (4.7) and (4.9) that the following three conditions are equivalent to each other:

(a′) $\nabla_{c}f_{b}{}^{a}=0.$

(b′) $\nabla_{c}f^{x}{}_{b}=0$ and $\nabla_{c}f_{y}{}^{a}=0.$

(c′) $h_{cb}{}^{x}=H_{cy}{}^{x}f^{y}{}_{b}$ and $h_{c}{}^{a}{}_{y}=H_{cy}{}^{x}f_{x}{}^{a}.$

When the condition (c′) is satisfied, we find, taking account of $h_{cb}{}^{x}=h_{bc}{}^{x}$,

$$H_{cy}{}^{x}=f^{z}{}_{c}\lambda_{zy}{}^{x},$$

where

$$\lambda_{zy}{}^{x}=\lambda_{yz}{}^{x}.$$

Thus we have

THEOREM 4.1. *For an f-surface in a complex space, the following three conditions are equivalent to each other:*

(a) $\nabla_{c}f_{b}{}^{a}=0.$

(b) $\nabla_{c}f^{x}{}_{b}=0$ *and* $\nabla_{c}f_{y}{}^{a}=0.$

(c) $h_{cb}{}^{x}=f^{z}{}_{c}f^{y}{}_{b}\lambda_{zy}{}^{x}$ *and* $h_{c}{}^{a}{}_{y}=f^{z}{}_{c}f_{x}{}^{a}\lambda_{zy}{}^{x},$

$\lambda_{zy}{}^x$ being a certain tensor field such that $\lambda_{zy}{}^x = \lambda_{yz}{}^x$. When one of these conditions is satisfied, the induced f-structure $f_b{}^a$ is integrable and $S_{cb}{}^a = 0$ is satisfied.

If we take account of (4. 2), we have from the definition (1. 2) of $N_{cb}{}^a$

$$N_{cb}{}^a = f^y{}_c(h_b{}^e{}_y f_e{}^a - f_b{}^e h_e{}^a{}_y) - f^y{}_b(h_c{}^e{}_y f_e{}^a - f_c{}^e h_e{}^a{}_y) + (h_{ce}{}^y f_b{}^e - h_{be}{}^y f_c{}^e) f_y{}^a. \tag{4. 11}$$

Therefore, if $N_{cb}{}^a = 0$, then we have from (4. 11)

$$\begin{cases} h_{ce}{}^x f_b{}^e - h_{be}{}^x f_c{}^e = f^y{}_c f_b{}^e H_{ey}{}^x - f^y{}_b f_c{}^e H_{ey}{}^x, \\ h_c{}^e{}_y f_e{}^a - f_c{}^e h_e{}^a{}_y = f^z{}_c(h_e{}^d{}_z f_y{}^e) f_d{}^a - f_b{}^e H_{ey}{}^x f_x{}^a. \end{cases} \tag{4. 12}$$

Conversely, if (4. 12) are satisfied, we have $N_{cb}{}^a = 0$. If we take account of the expression (4. 4) for $S_{cb}{}^a$, we see that the second equation of (4. 12) is equivalent to

$$S_{cb}{}^a = f^z{}_c f^y{}_b(h_e{}^d{}_z f_y{}^e - h_e{}^d{}_y f_z{}^e) f_d{}^a + (f^y{}_c S_{by}{}^x - f^y{}_b S_{cy}{}^x) f_x{}^a.$$

Thus, taking account of the expression (4. 5) for $L_{cb}{}^x$, we have

PROPOSITION 4. 3. *For an f-surface in a complex space, the induced f-structure is integrable, if and only if one of the following conditions* (a) *and* (b) *is satisfied·*

(a)
$$\begin{cases} h_{ce}{}^x f_b{}^e - h_{be}{}^x f_c{}^e \equiv 0, \\ h_c{}^e{}_y f_e{}^a - f_c{}^e h_e{}^a{}_y = f^z{}_c(h_e{}^d{}_z f_y{}^e) f_d{}^a - f_c{}^e H_{ey}{}^x f_x{}^a. \end{cases}$$

(b)
$$\begin{cases} L_{cb}{}^x \equiv 0, \\ S_{cb}{}^a = f^z{}_c f^y{}_b(h_e{}^a{}_z f_y{}^e - h_e{}^a{}_y f_z{}^e) + (f^y{}_c S_{by}{}^x - f^y{}_b S_{cy}{}^x) f_x{}^a. \end{cases}$$

If we take account of (4. 5) and (4. 11), we see that the following four conditions are equivalent to each other:

$$N_{cb}{}^a \equiv 0, \qquad N_{cb}{}^a f^x{}_a \equiv 0, \qquad L_{cb}{}^x \equiv 0,$$
$$h_{cb}{}^x + h_{ed}{}^x f_c{}^e f_b{}^d \equiv 0.$$

Thus, by virtue of Theorems B and D stated in § 1, we have

PROPOSITION 4. 4. *For an f-surface in a complex space, the following four conditions are equivalent to each other:*

(a) *The distribution L is integrable* ($N_{cb}{}^a f^x{}_a = 0$).

(b) *The induced f-structure is partially integrable* ($N_{cb}{}^a \equiv 0$).

(c) $L_{cb}{}^x \equiv 0$.

(d) $h_{cb}{}^x + h_{ed}{}^x f_c{}^e f_b{}^d \equiv 0$

We now suppose that the distribution M is integrable. Then, by means of

Theorem A stated in § 1, we have

(4. 13) $$f_z{}^c f_y{}^b N_{cb}{}^e f_e{}^a = 0,$$

which implies

(4. 14) $$(f_z{}^e h_e{}^a{}_y - f_y{}^e h_e{}^a{}_z) - (f_z{}^e H_{ey}{}^x - f_y{}^e H_{ez}{}^x) f_x{}^a = 0.$$

Conversely, if (4. 14) is satisfied, we have (4. 13) and consequently we see that the distribution M is integrable. Thus we have

PROPOSITION 4. 5. *For an f-surface in a complex space, the distribution M is integrable, if and only if*

$$(f_z{}^e h_e{}^a{}_y - f_y{}^e h_e{}^a{}_z) - (f_z{}^e H_{ey}{}^x - f_y{}^e H_{ez}{}^x) f_x{}^a = 0.$$

We suppose next that $S_{cb}{}^a = 0$ is satisfied. Then, we have by means of Proposition 2. 1

$$S_{cy}{}^x = 0,$$

which is equivalent to the condition

(4. 15) $$H_{cy}{}^x = f^z{}_c \lambda_{zy}{}^x,$$

where

$$\lambda_{zy}{}^x = \lambda_{yz}{}^x = f_z{}^c f_y{}^b h_{cb}{}^x.$$

Taking account of (4. 4), we see that $S_{cb}{}^a = 0$ is equivalent to the conditions

(4. 16) $$\begin{cases} h_c{}^e{}_y f_e{}^a - f_c{}^e h_e{}^a{}_y = f^z{}_c (h_e{}^d{}_z f_y{}^e) f_d{}^a, \\ (h_e{}^d{}_z f_y{}^e - h_e{}^d{}_y f_z{}^e) f_d{}^a = 0, \end{cases}$$

which are equivalent to

(4. 17) $$h_c{}^a{}_y + h_e{}^d{}_y f_c{}^e f_d{}^a = f^z{}_c (h_e{}^a{}_z f_y{}^e).$$

Taking account of (4. 15) and substituting (4. 7) in the first equation of (4. 16), we find

(4. 18) $$\nabla_c f_y{}^a + f_c{}^e f_d{}^a (\nabla_e f_y{}^d) - f^z{}_c f_y{}^e (\nabla_e f_z{}^a) = 0.$$

Conversely, if we substitute the third equation of (4. 2) in (4. 18), we obtain the first equation of (4. 16). Transvecting (4. 18) with $f_z{}^c$, we find

$$f_z{}^e \nabla_e f_y{}^a - f_y{}^e \nabla_e f_z{}^a = 0,$$

which implies together with (4. 2) the second equation of (4. 16). Thus we have

PROPOSITION 4. 6. *For an f-surface in a complex space, the following four conditions are equivalent to each other:*

(a) $S_{cb}{}^{a}=0.$

(b) $\begin{cases} h_{c}{}^{e}{}_{y}f_{e}{}^{a}-f_{c}{}^{e}h_{e}{}^{a}{}_{y}=f^{z}{}_{c}(h_{e}{}^{d}{}_{z}f_{y}{}^{e})f_{d}{}^{a}, \\ (h_{e}{}^{d}{}_{z}f_{y}{}^{e}-h_{e}{}^{d}{}_{y}f_{z}{}^{e})f_{d}{}^{a}=0. \end{cases}$

(c) $h_{c}{}^{a}{}_{y}+f_{c}{}^{e}f_{d}{}^{a}h_{e}{}^{d}{}_{y}=f^{z}{}_{c}(h_{e}{}^{a}{}_{z}f_{y}{}^{e}).$

(d) $\begin{cases} \nabla_{c}f_{y}{}^{a}+f_{c}{}^{e}f_{d}{}^{a}(\nabla_{e}f_{y}{}^{d})-f^{z}{}_{c}f_{y}{}^{e}(\nabla_{e}f_{z}{}^{a})=0, \\ H_{cy}{}^{x}=f^{z}{}_{c}\lambda_{zy}{}^{x} \qquad (\lambda_{zy}{}^{x}=\lambda_{yz}{}^{x}). \end{cases}$

We have also from Propositions 4. 3, 4. 4 and 4. 6

THEOREM 4. 2. *If, for an f-surface in a complex space, the condion $S_{cb}{}^{a}=0$ is satisfied, then the following three conditions are equivalent to each other:*

(a) *The induced f-structure is integrable* ($N_{cb}{}^{a}=0$).

(b) *The induced f-structure is partially integrable* ($N_{cb}{}^{a}\equiv 0$).

(c) *The distribution L is integrable* ($N_{cb}{}^{a}f^{x}{}_{a}=0$).

§ 5. Normal f-surfaces in a complex space.

Let there be given an f-surface V in a complex space W. We suppose now that the induced f-structure $f_{b}{}^{a}$ is normal with respect to the connection $\Gamma_{c}{}^{x}{}_{y}$ induced on the vector bundle $M(V)$. Such an f-surface is said to be *normal*. Thus, from Theorem F stated in § 2, we see that an f-surface is normal if and only if we have

$$S_{cb}{}^{a}=0 \qquad \text{and} \qquad R_{dcy}{}^{x}=0, \tag{5. 1}$$

where $S_{cb}{}^{a}$ is the tensor field defined by (2. 5) and

$$R_{dcy}{}^{x}=\partial_{d}\Gamma_{c}{}^{x}{}_{y}-\partial_{c}\Gamma_{d}{}^{x}{}_{y}+\Gamma_{d}{}^{x}{}_{z}\Gamma_{c}{}^{z}{}_{y}-\Gamma_{c}{}^{x}{}_{z}\Gamma_{d}{}^{z}{}_{y} \tag{5. 2}$$

is the curvature tensor of the induced connection $\Gamma_{c}{}^{x}{}_{y}$.

On the other hand, as was seen in § 3, there exists a bundle isomorphism F^{*}: $N(V)\rightarrow M(V)$, which is defined by (3. 11), and the connection induced in $N(V)$ has the same components $\Gamma_{c}{}^{x}{}_{y}$ as that induced in $M(V)$. Therefore, if the f-surface V is normal, the connection induced in $N(V)$ has vanishing curvature tensor because of (5. 1). Thus, if the f-surface is normal, the structure group of the vector bundle $N(V)$ is reducible to a discrete group.

When the vector bundle $N(V)$ admits a locally flat connection $\Gamma_c{}^x{}_y$, there exists in each coordinate neighborhood U of the f-surface V an ordered set $\{C_x{}^h\}$ of normal vector fields $C_x{}^h$ spanning the fibre N_P of $N(V)$ at each point P of U, such that all of components $\Gamma_c{}^x{}_y$ of the induced connection vanish with respect to $\{C_x{}^h\}$. Such an ordered set of local normal vector fields $C_x{}^h$ is called an *adapted normal frame* in $N(V)$. In each coordinate neighborhood U, an adapted normal frame is determined up to transformations with constant coefficients, that is, for an adapted normal frame $\{C_x{}^h\}$

$$C_{x'}{}^h = A_{x'}^x C_x{}^h$$

determine another adapted frame $C_{x'}{}^h$ if and only if $(A_{x'}^x)$ is a non-singular, constant matrix in U. Summing up, if we take account of Theorem F, we have

THEOREM 5. 1. *In a complex space, a necessary and sufficient condition for an f-surface to be normal is that it satisfy the conditions*

$$R_{dcy}{}^x = 0 \qquad \textit{and} \qquad S_{cb}{}^a = 0.$$

When an f-surface V is normal, the following facts (a) *and* (b) *are valid:*

(a) *The structure group of the vector bundle N(V) is reduced to a discrete group. If the f-surface is simply connected, the vector bundle N(V) is a product bundle.*

(b) *All of the five tensor fields S's vanish identically, i.e.*

$$S_{cb}{}^a = (f_c{}^e \partial_e f_b{}^a - f_b{}^e \partial_e f_c{}^a) - (\partial_c f_b{}^e - \partial_b f_c{}^e) f_e{}^a + (\partial_c f^x{}_b - \partial_b f^x{}_c) f_x{}^a = 0,$$

$$S_{cb}{}^x = f_c{}^e (\partial_e f^x{}_b - \partial_b f^x{}_e) - f_b{}^e (\partial_e f^x{}_c - \partial_c f^x{}_e) = 0,$$

$$S_{cy}{}^a = f_y{}^e \partial_e f_c{}^a - f_c{}^e \partial_e f_y{}^a + f_e{}^a \partial_c f_y{}^e = 0,$$

$$S_{cy}{}^x = f_y{}^e (\partial_e f^x{}_c - \partial_c f^x{}_e) = 0,$$

$$S_{zy}{}^a = f_z{}^e \partial_e f_y{}^a - f_y{}^e \partial_e f_z{}^a = 0$$

with respect to an adapted normal frame $\{C_x{}^h\}$ *in each coordinate neighborhood U of the f-surface.*

Theorem 4. 2 and Proposition 2. 6 imply immediately

THEOREM 5. 2. *For a normal f-surface in a complex space, the following three conditions are equivalent to each other:*

(a) *The induced f-structure is integrable.*

(b) *The induced f-structure is partially integrable.*

(c) *The distribution $\underline{L}$ is integrable.*

If an f-surface is normal in a complex space, then the distribution M is integrable.

§ 6. f-surfaces in a locally flat complex space.

We consider in this section f-surfaces in a complex space W which is locally flat, i.e. whose curvature tensor vanishes identically. If we suppose that the holomorphic extension $T_P^H(V)$ of tangent space $T_P(V)$ coincides with the tangent space $T_P(W)$ of the enveloping space W at each point P of V, i.e. if

$$T_P^H(V)=T_P(W), \tag{6. 1}$$

we obtain

$$\nabla_d\nabla_c B_b{}^h-\nabla_c\nabla_d B_b{}^h=R_{dcb}{}^a B_a{}^h, \tag{6. 2}$$

$$\nabla_d\nabla_c C_y{}^h-\nabla_c\nabla_d C_y{}^h=R_{dcy}{}^x C_x{}^h \tag{6. 3}$$

as a consequence of locally flatness of the enveloping space W, where

$$R_{dcb}{}^a=\partial_d\Gamma_c{}^a{}_b-\partial_c\Gamma_d{}^a{}_b+\Gamma_d{}^a{}_e\Gamma_c{}^e{}_b-\Gamma_c{}^a{}_e\Gamma_d{}^e{}_b,$$

$$R_{dcy}{}^x=\partial_d\Gamma_c{}^x{}_y-\partial_d\Gamma_c{}^x{}_y+\Gamma_d{}^x{}_z\Gamma_c{}^z{}_y-\Gamma_c{}^x{}_z\Gamma_d{}^z{}_y$$

are curvature tensors of the induced connections $\Gamma_c{}^a{}_b$ and $\Gamma_c{}^x{}_y$ respectively. Substituting the first and the second equations of (3. 16) with vanishing $h_{cb}{}^\alpha$ and $h_{cy}{}^\alpha$ into (6. 2) and (6. 3), we find

$$\begin{cases} R_{dcb}{}^a=h_d{}^a{}_x h_{cb}{}^x-h_c{}^a{}_x h_{db}{}^x, \\ R_{dcy}{}^x=h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y, \\ \nabla_d h_{cb}{}^x-\nabla_c h_{db}{}^x=0, \\ \nabla_d h_c{}^a{}_y-\nabla_c h_d{}^a{}_y=0. \end{cases} \tag{6. 4}$$

We suppose that the f-surface satisfies the condition (6. 1) and has the following properties

$$h_{cb}{}^x=f^z{}_c f^y{}_b \lambda_{zy}{}^x, \qquad h_c{}^a{}_y=f^z{}_c f_x{}^a \lambda_{zy}{}^x, \tag{6. 5}$$

where

$$\lambda_{zy}{}^x=\lambda_{yz}{}^x.$$

Substituting (6. 5) in (6. 4), we obtain

$$\begin{cases} R_{dcb}{}^a=f^u{}_d f^z{}_c f^y{}_b f_x{}^a(\lambda_{uv}{}^x\lambda_{zy}{}^v-\lambda_{zv}{}^x\lambda_{uy}{}^v), \\ R_{dcy}{}^x=f^u{}_d f^z{}_c(\lambda_{uv}{}^x\lambda_{zy}{}^v-\lambda_{zv}{}^x\lambda_{uy}{}^v) \end{cases} \tag{6. 6}$$

and

$$(6.7)\qquad \begin{cases} f_u{}^e\nabla_e\lambda_{zy}{}^x - f_z{}^e\nabla_e\lambda_{uy}{}^x = 0, \\ \nabla_b\lambda_{zy}{}^x \equiv 0. \end{cases}$$

The both equations of (6. 6) imply

$$(6.8)\qquad R_{dcb}{}^a = R_{acy}{}^z f^y{}_b f_z{}^a.$$

Thus we have

PROPOSITION 6. 1. *If, in a locally flat complex space, an f-surface satisfies the conditions* (6. 1) *and one of three conditions* (a), (b) *and* (c) *mentioned in Theorem* 4. 1, *then it has the properties* (6. 6), (6. 7) *and* (6. 8).

We suppose next that a normal f-surface V of a locally flat complex space W satisfies the conditions (6. 1) and (6. 5). Then, we have from Theorem 5. 1

$$R_{dcy}{}^x = 0,$$

which implies together with (6. 8)

$$R_{dcb}{}^a = 0,$$

i.e. that the induced connection $\Gamma_c{}^a{}_b$ of V is locally flat. Thus we have

THEOREM 6. 1. *If, in a locally flat complex space W, a normal f-surface V satisfies one of three conditions* (a), (b) *and* (c) *mentioned in Theorem* 4. 1, *and, if the holomorphic extension* $T^H_P(V)$ *of tangent space of V coincides with the tangent space* $T_P(W)$ *of the enveloping space W at each point of V, then the induced connection* $\Gamma_c{}^a{}_b$ *of V is locally flat and the equations*

$$(6.9)\qquad \begin{cases} h_{cb}{}^x = f^z{}_c f^y{}_b \lambda_{zy}{}^x, \quad h_c{}^a{}_y = f^z{}_c f_x{}^a \lambda_{zy}{}^x, \\ \lambda_{uv}{}^x \lambda_{zy}{}^v - \lambda_{zv}{}^x \lambda_{uy}{}^v = 0, \\ f_u{}^e \nabla_e \lambda_{zy}{}^x - f_z{}^e \nabla_e \lambda_{uy}{}^x = 0, \\ \nabla_b \lambda_{zy}{}^x \equiv 0 \end{cases}$$

are valid, where $\lambda_{zy}{}^x = \lambda_{yz}{}^x$.

If an f-surface in a locally flat complex space satisfies the condition (6. 5), then by virtue of Proposition 4. 6 we have

$$S_{cb}{}^a = 0,$$

since the condition (c) of Proposition 4. 6 is valid as a consequence of (6. 5). We suppose that the f-surface satisfies the condition (6. 1) and is locally flat. Then, taking account of (6. 8), we have

$$R_{dcy}{}^x = 0$$

because of $R_{dcb}{}^{a}=0$. Thus we have

THEOREM 6. 2. *If, in a locally flat complex space, W an f-surface V satisfies one of three conditions* (a), (b) *and* (c) *mentioned in Theorem* 4. 1, *and, if the surface V is locally flat and the holomorphic extension* $T^H_P(V)$ *of tangent space of V coincides with the tangent space* $T_P(W)$ *of the enveloping space W at each point of V, then the surface V is normal.*

Coming back to the general case, in a complex space W of $N=2n$ dimensions, we take a submanifold V of n dimensions and suppose that $T_P(V)\cap F(T_P(V))=\{0\}$ at each point P of V. We call such a submanifold V an *antiholomorphic surface* of W. Denoting by $B_a{}^h$ n local tangent vector fields defined by (3. 5), we see that n vector fields

$$C_b{}^h=F_i{}^h B_b{}^i \tag{6. 10}$$

span the space $F(T_P(V))$. If we put

$$\nabla_c B_b{}^h=h_{cb}{}^e C_e{}^h, \tag{6. 11}$$

then we have

$$\nabla_c C_b{}^h=-h_{cb}{}^e B_e{}^h \tag{6. 12}$$

by means of (6. 10). Then, $h_{cb}{}^a$ is called the *second fundamental tensor* of the antiholomorphic surface V.

We now consider a normal f-surface V in an N-dimensional space CA^s of s complex numbers $(z^1, z^2, \cdots, z^s)$, $2s$ being equal to N. If we put

$$z^\lambda=x^\lambda+\sqrt{-1}\,y^\lambda \qquad (\lambda=1, 2, \cdots, s),$$

then (x^λ, y^λ) are cartesian coordinates in CA^s. Then, taking account of Theorem 5. 1, we see that the connection $\Gamma_c{}^x{}_y$ induced in the vector bundle $N(V)$ is of zero curvature, i.e. that there exists in each coordinate neighborhood of V an adapted normal frame $\{C_x{}^h\}$ with respect to which the components $\Gamma_c{}^x{}_y$ of the induced connection vanish identically. If we assume moreover that the f-surface V satisfies the condition (6. 1) and one of three conditions mentioned in Theorem 4. 1, we see from Theorem 6. 1 that the connection $\Gamma_c{}^a{}_b$ induced in the f-surface V is locally flat, i.e. that there exist in each coordinate neighborhood of V local coordinates with respect to which the components $\Gamma_c{}^a{}_b$ of the induced connection of V vanishes identically. Thus we have

$$\nabla_c f_b{}^a=\partial_c f_b{}^a, \qquad \nabla_c f_y{}^a=\partial_c f_y{}^a, \qquad \nabla_c f^x{}_b=\partial_c f^x{}_b$$

with respect to such local coordinates and such an adapted normal frame.

We have from Theorem 4. 1

$$(6.13)\qquad \begin{cases} \partial_c f_b{}^a = \nabla_c f_b{}^a = 0, & \\ \partial_c f_y{}^a = \nabla_c f_y{}^a = 0, & \partial_c f^x{}_b = \nabla_c f^x{}_b = 0, \\ h_{cb}{}^x = f^z{}_c f^y{}_b \lambda_{zy}{}^x, & h_c{}^a{}_y = f^z{}_c f_x{}^a \lambda_{zy}{}^x, \end{cases}$$

where $\lambda_{zy}{}^x = \lambda_{yz}{}^x$ satisfy (6. 9). Taking account of (6. 13), we see that each integral manifold of the distribution L, which is spanned locally in the enveloping space CA^s by vector fields $f_b{}^a B_a{}^h$, is a complex plane $CA^{\frac{r}{2}}$, r being the rank of the induced f-structure. Thus we may assume that each integral manifold of L is expressed in CA^s by linear equations

$$z^{\frac{r}{2}+1} = \text{const.}, \cdots, z^s = \text{const.}$$

Next, taking account of (6. 13), we see that the holomorphic extension $M^H = M + F(M)$ of the distribution M is parallel along the f-surface V, since M^H is spanned locally in the space CA^s by vector fields $f_y{}^a B_a{}^h$ and $C_x{}^h$. Thus we may assume that the distribution M^H is, along the f-surface V, parallel to the complex plane defined by linear equations

$$z^1 = 0, \qquad z^2 = 0, \cdots, z^{\frac{r}{2}} = 0.$$

Therefore, each integral manifold of the distribution M is an antiholomorphic submanifold in a complex plane defined by linear equations

$$z^1 = \text{const.}, \ z^2 = \text{const.}, \cdots, z^{\frac{r}{2}} = \text{const.}$$

Summing up, we find that the base submanifold of the given f-surface V is conjugate to a portion of a submanifold $\widetilde{V}$ under the group of all affine transformations operating on CA^s and preserving the complex structure of CA^s, where the f-submanifold $\widetilde{V}$ is defined by equations of the form

$$(6.14)\qquad x^u = x^u(\zeta^x), \qquad y^u = y^u(\zeta^x) \qquad \left(u = \frac{r}{2} + 1, \cdots, s\right),$$

where ζ^x are parameters of $\widetilde{V}$ and r is the rank of the induced f-structure. The equations (6. 14) determine, in the subspace $CA^{s-\frac{r}{2}}$ defined in CA^s by $z^1 = z^2 = \cdots = z^{\frac{r}{2}} = 0$, an antiholomorphic surface whose second fundamental tensor coincides with $\lambda_{zy}{}^x$, where $\lambda_{zy}{}^x$ are constant along the distribution L and satisfy the second and the third equations of (6. 9). Thus, the induced connection of the antiholomorphic surface defined in $CA^{s-\frac{r}{2}}$ by (6. 14) should be locally flat. Thus we have

THEOREM 6. 3. *If, in the N-dimensional space CA^s of s complex numbers $z^\lambda = x^\lambda + \sqrt{-1} y^\lambda$ ($\lambda = 1, 2, \cdots, s$; $N = 2s$), there exists a normal f-surface V satisfying one of the conditions* (a), (b) *and* (c) *mentioned in Theorem* 4. 1, *and, if the holo-*

morphic extension $T_P^H(V)$ *of tangent space of* V *coincides with tangent space of the enveloping space* CA^s, *then the base submanifold of the given f-surface* V *is conjugate to an f-submanifold* $\widetilde{V}$ *defined by* (6. 14) *under the group of all affine transformations operating on* CA^s *and preserving the complex structure of* CA^s, *where the antiholomorphic surface defined in* $CA^{s-\frac{r}{2}}$ *by* (6. 14) *is locally flat.*

§ 7. *f*-surfaces in an almost Hermitian space.

We consider an *almost Hermitian space* W of differentiability class C^∞ with an almost Hermitian structure (F_i^h, G_{ji}) of class C^∞, where F_i^h is an almost complex structure and G_{ji} a positive definite Riemannian metric such that

$$F_j^t F_i^s G_{ts} = G_{ji}. \tag{7. 1}$$

The tensor field

$$F_{ji} = F_j^h G_{hi} \tag{7. 2}$$

is skew symmetric. If the Riemannian connection $\{{}_j{}^h{}_i\}$ determined by G_{ji} satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{7. 3}$$

then the space is called an *almost Kählerian space*. If, moreover, the Nijenhuis tensor $N_{ji}{}^h$ defined by (3. 19) vanishes identically, the almost Hermitian space is called a *Kählerian space*. A necessary and sufficient condition for an almost Hermitian space to be Kählerian is given by

$$\nabla_j F_i^h = 0 \tag{7. 4}$$

(Cf. Yano [20]).

We now consider an f-submanifold V in an almost Hermitian space W. Then, there exists uniquely a subspace N_P in the holomorphic extension $T_P^H(V)$ of tangent space $T_P(V)$ at each point P of V such that N_P is orthogonal to $T_P(V)$ and $F(N_P) \subset T_P(V)$, and N_P is $(n-r)$-dimensional if dim $H_P = r$. Furthermore, there exists uniquely a subspace $\bar{N}_P$ of $N-2n+r$ dimensions in each tangent space $T_P(W)$ such that $F(\bar{N}_P) = \bar{N}_P$ and $\bar{N}_P$ is orthogonal to $T_P^H(V)$ at each point P of V. Thus we have an f-surface $\{V, N(V), \bar{N}(V)\}$ corresponding uniquely to the given f-submanifold V and denote it simply by V.

We follow notations introduced in § 3. Then, local vector fields $C_y{}^h$ are orthogonal to $B_b{}^h$ and $C_\beta{}^h$, and $C_\beta{}^h$ are orthogonal to $B_b{}^h$ and $C_y{}^h$. Therefore we find

$$\begin{cases} G_{ji} B_b{}^j C_y{}^i = 0, \qquad G_{ji} B_b{}^j C_\beta{}^i = 0, \\ \qquad G_{ji} C_x{}^j C_\beta{}^i = 0. \end{cases} \tag{7. 5}$$

If we put

$$\begin{cases} \qquad g_{cb} = G_{ji} B_c{}^j B_b{}^i, \\ g_{zy} = G_{ji} C_z{}^j C_y{}^i, \qquad g_{\gamma\beta} = G_{ji} C_\gamma{}^j C_\beta{}^i, \end{cases} \tag{7. 6}$$

then g_{cb} is a Riemannian metric in V, which is called the *induced Riemannian metric* of V, and g_{zy} and $g_{\gamma\beta}$ are metric tensors in the vector bundles $N(V)$ and $\bar{N}(V)$ respectively. The metrics g_{zy} and $g_{\gamma\beta}$ are called the *induced metrics* of $N(V)$ and $\bar{N}(V)$ respectively.

Taking account of (3. 6), (7. 5) and (7. 6), we obtain

$$B^a{}_i = g^{ab} G_{ih} B_b{}^h, \qquad C^x{}_i = g^{xy} G_{ih} C_y{}^h,$$

$$C^\alpha{}_i = g^{\alpha\beta} G_{ih} C_\beta{}^h,$$

where g^{ab}, g^{xy} and $g^{\alpha\beta}$ are respectively defined by

$$g^{ae} g_{eb} = \delta_b^a, \qquad g^{xu} g_{uy} = \delta_y^x, \qquad g^{\alpha\gamma} g_{\gamma\beta} = \delta_\beta^\alpha.$$

It is well known that the induced connection

$$\Gamma_c{}^a{}_b = (\partial_c B_a{}^h + B_c{}^j B_b{}^i \{{}_j{}^h{}_i\}) B^a{}_h$$

defined by (3. 12) coincides with the Riemannian connection $\{{}_c{}^a{}_b\}$ determined by the induced metric g_{cb} of V. Thus we have

$$\nabla_d g_{cb} = 0.$$

Similarly, the induced connections

$$\Gamma_c{}^x{}_y = (\partial_c C_y{}^h + B_c{}^j C_y{}^i \{{}_j{}^h{}_i\}) C^x{}_h,$$

$$\Gamma_c{}^\alpha{}_\beta = (\partial_c C_\beta{}^h + B_c{}^j C_\beta{}^i \{{}_j{}^h{}_i\}) C^\alpha{}_h$$

defined by (3. 13) and (3. 14) respectively have the following properties

$$\nabla_c g_{zy} = \partial_c g_{zy} - \Gamma_c{}^x{}_z g_{xy} - \Gamma_c{}^x{}_y g_{zx} = 0,$$

$$\nabla_c g_{\gamma\beta} = \partial_c g_{\gamma\beta} - \Gamma_c{}^\delta{}_\gamma g_{\delta\beta} - \Gamma_c{}^\delta{}_\beta g_{\gamma\delta} = 0.$$

Transvecting the second equation of (7. 1) with $B_c{}^j B_b{}^i$ and taking account of (3. 8) and (7. 5), we find

$$f_c{}^e f_b{}^d g_{ed} + f^y{}_c f^x{}_b g_{yx} = g_{cb}. \tag{7. 7}$$

Transvecting the second equations of (7. 1) with $B_c{}^j C_y{}^i$ and taking account of (3. 8) and (7. 5), we find

$$f_c{}^e f_y{}^d g_{ed} = 0. \tag{7. 8}$$

Finally, if we transvect (7. 1) with $C_y{}^j C_x{}^i$ and take account of (3. 8), we find

$$f_y{}^c f_x{}^b g_{cb} = g_{yx}. \tag{7. 9}$$

The equation (7. 7) shows that $f_b{}^a$ and g_{cb} form an (f, g)-structure in V, which is called the *induced (f, g)-structure* of V. An f-surface with such an induced (f, g)-

structure is called a *metric f-surface.*

In our metric case, the second fundamental tensors h's appearing in equations (3. 16) have the following properties

$$h_c{}^a{}_y = h_{cb}{}^x g^{ba} g_{xy}, \qquad h_c{}^a{}_\beta = h_{cb}{}^\alpha g^{ba} g_{\alpha\beta},$$
$$h_c{}^x{}_\beta = h_{cy}{}^\alpha g^{yx} g_{\alpha\beta}. \tag{7. 10}$$

Transvecting the first equation of (3. 17) with $B^a{}_h$, we obtain

$$\nabla_c f_b{}^a = h_c{}^a{}_y f^y{}_b - h_{cb}{}^y f_y{}^a + (\nabla_j F_i{}^h) B_c{}^j B_b{}^i B^a{}_h$$

or equivalently

$$\nabla_c f_{ba} = h_{ca}{}^y f_{yb} - h_{cb}{}^y f_{ya} + (\nabla_j F_{ih}) B_c{}^j B_b{}^i B_a{}^h$$

because of (7. 10), where we have put

$$f_{yb} = f^x{}_b g_{xy} = f_y{}^a g_{ab}.$$

Then we have

$$\nabla_c f_{ba} + \nabla_b f_{ac} + \nabla_a f_{cb} = (\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}) B_c{}^j B_b{}^i B_a{}^h \tag{7. 12}$$

by virtue of $h_{cb}{}^x = h_{bc}{}^x$. Thus we have

PROPOSITION 7. 1. *For a metric f-surface in an almost Kählerian space, the form $f_{cb} d\eta^c \wedge d\eta^b$ is closed* (Cf. Nakagawa [5]).

§ 8. Metric *f*-surfaces in a Kählerian space.

We assume that the enveloping space is Kählerian. Then, by virtue of $\nabla_j F_i{}^h = 0$, we find from (4. 2)

$$\begin{cases} \nabla_c f_b{}^a + h_{cb}{}^y f_y{}^a - h_c{}^a{}_y f^y{}_b = 0, \\ \nabla_c f^x{}_b + h_{ce}{}^x f_b{}^e = 0, \end{cases} \tag{8. 1}$$

the second equation of which is equivalent to

$$\nabla_c f_y{}^a - h_c{}^e{}_y f_e{}^a = 0.$$

If we take account of the definition (1. 2) of the Njienhuis tensor $N_{cb}{}^a$, we have by means of Proposition 7. 1

$$N_{ed}{}^a m_c{}^e m_b{}^d = -(\nabla_e f_{df} + \nabla_d f_{fe}) f^{fa} m_c{}^e m_b{}^d = (\nabla_f f_{ed}) f^{fa} m_c{}^e m_b{}^d,$$

which implies together with (4. 9)

$$N_{ed}{}^a m_c{}^e m_b{}^d = 0.$$

Thus, taking account of theorem A stated in §1, we have

THEOREM 8.1. *For a metric f-surface in a Kählerian space, the distribution M is integrable* (Cf. Nakagawa [5]).

Let there be given a metric f-surface in a Kählerian space. Then, as a consequence of the condition (a) mentioned in Proposition 4.3 and $h_{cb}{}^{x}=h_{bc}{}^{x}$, the induced f-structure is integrable if and only if

$$h_{cb}{}^{x}=m_{c}{}^{e}h_{eb}{}^{x}+m_{b}{}^{d}h_{cd}{}^{x}-m_{c}{}^{e}m_{b}{}^{d}h_{ed}{}^{x}, \tag{8.2}$$

or equivalently

$$h_{c}{}^{a}{}_{y}=m_{c}{}^{e}h_{e}{}^{a}{}_{y}+m_{d}{}^{a}h_{c}{}^{d}{}_{y}-m_{c}{}^{e}m_{d}{}^{a}h_{e}{}^{d}{}_{y},$$

where $m_{b}{}^{a}=f^{x}{}_{b}f_{x}{}^{a}$. On the other hand, the first equation of (4.3) implies

$$h_{ce}{}^{x}f_{x}{}^{a}m_{b}{}^{e}=h_{c}{}^{e}{}_{y}f^{y}{}_{b}m_{e}{}^{a}.$$

Taking account of this equation, we see that (8.2) is equivalent to the condition

$$\nabla_{c}f^{x}{}_{b}=-m_{c}{}^{e}f_{b}{}^{d}h_{ed}{}^{x} \tag{8.3}$$

as a consequence of (4.2) and (4.7) and we have

$$\nabla_{c}f_{b}{}^{a}=m_{c}{}^{e}(h_{e}{}^{a}{}_{y}f^{y}{}_{b}-h_{eb}{}^{y}f_{y}{}^{a}) \tag{8.4}$$

by means of the first equation of (4.2). Summing up, we have

PROPOSITION 8.1. *For a metric f-surface in a Kählerian space, a necessary and sufficient condition for the induced f-structure to be integrable is that one of the conditions* (8.2) *and* (8.3) *is satisfied. When the induced f-structure is integrable, the equation* (8.4) *is satisfied.*

It follows from the second equation of (4.4) that

$$H_{cy}{}^{x}=h_{ce}{}^{x}f_{y}{}^{e}=f^{z}{}_{c}\lambda_{zy}{}^{x}$$

is satisfied if and only if $S_{cb}{}^{x}=0$. Since the third expression of (4.4) for $S_{cy}{}^{a}$ can be written as

$$S_{cz}{}^{e}g_{eb}g^{zy}=-(h_{cb}{}^{y}-f_{c}{}^{e}f_{b}{}^{d}h_{ed}{}^{y})+f^{z}{}_{c}H_{bz}{}^{y},$$

we have the equation $H_{cy}{}^{x}=f^{z}{}_{c}\lambda_{zy}{}^{x}$ above if and only if

$$S_{cy}{}^{e}g_{eb}=S_{by}{}^{e}g_{ec}.$$

Thus, taking account of the fourth expression of (4.4) for $S_{cy}{}^{x}$, we have

PROPOSITION 8.2. *For a metric f-surface in a Kählerian space, the condition*

$$H_{cy}{}^{x}=h_{ce}{}^{x}f_{y}{}^{e}=f^{z}{}_{c}\lambda_{zy}{}^{x}$$

with $\lambda_{zy}{}^x = \lambda_{yz}{}^x$ *is equivalent to one of the following three conditions:*

$$S_{cb}{}^x = 0, \qquad S_{cy}{}^e g_{eb} = S_{by}{}^e g_{ec}, \qquad S_{cy}{}^x = 0.$$

Differentiating $m_b{}^a = f^x{}_b f_x{}^a$ covariantly, then we find

$$\nabla_c m_b{}^a = -h_{ce}{}^x (f_b{}^e f_x{}^a + f^{ae} f_{xb})$$

by means of (8. 1), where $f^{ae} = g^{ad} f_d{}^e$, $f_{xb} = g_{be} f_x{}^e$. We thus find

$$m_c{}^d \nabla_d m_b{}^a = -m_c{}^d h_{de}{}^x (f_b{}^e f_x{}^a + f^{ae} f_{xb}),$$

which implies that the condition

$$m_c{}^d \nabla_d m_b{}^a = 0$$

is equivalent to the condition

$$H_{cy}{}^x = f^z{}_c \lambda_{zy}{}^x.$$

On the other hand, the condition $m_c{}^d \nabla_d m_b{}^a = 0$ is equivalent to the condition that *the distribution M is flat*, i.e. that, if we translate any vector belonging to M parallelly along M, the translated vector belongs always to M (Cf. Walker [14], [15] and Yano [16]). Thus we have

THEOREM 8. 2. *For a metric f-surface in a Kählerian space, the distribution M is flat if and only if one of the three conditions mentioned in Proposition* 8. 2 *is satisfied* (Cf. Nakagawa [5]).

We shall now study metric f-surfaces satisfying $S_{cb}{}^a = 0$. We know that all of other S's vanish if $S_{cb}{}^a = 0$. Thus, if $S_{cb}{}^a = 0$, we have from (4. 4)

$$f^x{}_c (h_b{}^e{}_x f_e{}^a - f_b{}^e h_e{}^a{}_x) - f^x{}_b (h_c{}^e{}_x f_e{}^a - f_c{}^e h_e{}^a{}_x) = 0,$$

$$f^y{}_c H_{by}{}^x - f^y{}_b H_{cy}{}^x = 0,$$

$$h_c{}^a{}_y + h_e{}^d{}_y f_c{}^e f_d{}^a = f^z{}_c h_e{}^a{}_z f_y{}^e,$$

$$f_c{}^e H_{ey}{}^x = 0,$$

from which we obtain

$$h_{cb}{}^x - h_{ed}{}^x f_c{}^e f_b{}^d = m_c{}^e m_b{}^d h_{ed}{}^x.$$

Conversely, if the equation above is satisfied, we have $H_{cy}{}^x = f^z{}_c \lambda_{zy}{}^x$ and $h_c{}^e{}_y f_e{}^a - f_c{}^e h_e{}^a{}_y = 0$, which implies $S_{cb}{}^a = 0$. Thus we have

PROPOSITION 8. 3. *A necessary and sufficient condition for a metric f-surface in a Kählerian space to have vanishing* $S_{cb}{}^a$ *is that*

$$h_{cb}{}^x - f_c{}^e f_b{}^d h_{ed}{}^x = m_c{}^e m_b{}^d h_{ed}{}^x, \tag{8. 5}$$

or equivalently

$$h_c{}^a{}_y+f_c{}^e f_d{}^a h_e{}^d{}_y=m_c{}^e m_d{}^a h_e{}^d{}_y.$$

If (8. 5) is satisfied, we have $H_{cy}{}^x=f^z{}_c\lambda_{zy}{}^x$ and consequently the expression (4. 5) for $L_{cb}{}^x$ reduces to

$$L_{cb}{}^x=-(h_{cb}{}^x+f_c{}^e f_b{}^d h_{ed}{}^x)+m_c{}^e m_b{}^d h_{ed}{}^x. \tag{8. 6}$$

Taking account of (8. 5), we have from (8. 6)

$$L_{cb}{}^x=-2f_c{}^e f_b{}^d h_{ed}{}^x. \tag{8. 7}$$

Consequently, if $L_{cb}{}^x$ has this form, then $S_{cb}{}^x=L_{cb}{}^x-L_{bc}{}^x=0$ and consequently we have

$$H_{cy}{}^x=f^z{}_c\lambda_{zy}{}^x$$

by means of Proposition 8. 2. Therefore, taking account of (4. 5) and (8. 7), we obtain

$$h_{cb}{}^x-f_c{}^e f_b{}^d h_{ed}{}^x=m_c{}^e m_b{}^d h_{ed}{}^x,$$

which implies $S_{cb}{}^a=0$ by virtue of Proposition 8. 3. Thus we have

PROPOSITION 8. 4. *In a Kählerian space, a necessary and sufficient condition for a metric f-surface to have vanishing $S_{cb}{}^a$ is that*

$$L_{cb}{}^x=-2f_c{}^e f_b{}^d h_{ed}{}^x.$$

The condition $S_{cy}{}^a=0$ is equivalent to (4. 17) because of (4. 4). Thus we have

PROPOSITION 8. 5. *For a metric f-surface in a Kählerian space, the two conditions*

$$S_{cb}{}^a=0, \qquad S_{cy}{}^a=0$$

are equivalent to each other.

The equation (8. 5) is equivalent to

$$h_{ce}{}^x f_b{}^e+h_{be}{}^x f_c{}^e=0,$$

which is, by virtue of the second equation of (8. 1), equivalent to

$$\nabla_c f^x{}_b+\nabla_b f^x{}_c=0. \tag{8. 8}$$

On putting $m_{ba}=m_b{}^e g_{ea}$, we have

$$m_{ba}=f^y{}_b f^x{}_a g_{yx}$$

and hence

$$\begin{aligned}&\nabla_c m_{ba}+\nabla_b m_{ac}+\nabla_a m_{cb}\\&\quad=(\nabla_c f^y{}_b+\nabla_b f^y{}_c)f^x{}_a g_{yx}\\&\quad+(\nabla_b f^y{}_a+\nabla_a f^y{}_b)f^x{}_c g_{yx}+(\nabla_a f^y{}_c+\nabla_c f^y{}_a)f^x{}_b g_{yx}.\end{aligned}$$

Therefore, the condition

$$\nabla_c m_{ba}+\nabla_b m_{ac}+\nabla_a m_{cb}=0 \tag{8.9}$$

is equivalent to (8. 8). Thus we have

PROPOSITION 8. 6. *For a metric f-surface in a Kählerian space, the condition* $S_{cb}{}^a=0$ *is valid if and only if one of the two conditions* (8. 8) *and* (8. 9) *is satisfied.*

From Propositions 8. 3, 8. 4, 8. 5 and 8. 6, we have

THEOREM 8. 3. *For a metric f-surface in a Kählerian space, the following six conditions are equivalent to each other:*

(a) $S_{cb}{}^a=0.$

(b) $S_{cy}{}^a=0.$

(c) $h_{cb}{}^x-f_c{}^e f_b{}^d h_{ed}{}^x=m_c{}^e m_b{}^d h_{ed}{}^x.$

(d) $L_{cb}{}^x=-2f_c{}^e f_b{}^d h_{ed}{}^x.$

(e) $\nabla_c f^x{}_b+\nabla_b f^x{}_c=0.$

(f) $\nabla_c m_{ba}+\nabla_b m_{ac}+\nabla_a m_{cb}=0.$

We now assume that the condition (e) given in Theorem 8. 3 is satisfied. Then vector field $v_b=v_x f^x{}_b$ satisfies

$$\nabla_c v_b+\nabla_b v_c=0,$$

if $\nabla_c v_x=0$ is valid.

Next, if we take an arbitrary geodesic $\eta^a=\eta^a(s)$ in a metric f-surface of a Kählerian space, s being the arc-length of the geodesic, then the condition (f) given in Theorem 8. 3 is equivalent to the condition

$$\frac{d}{ds}\left(m_{cb}\frac{d\eta^c}{ds}\,\frac{d\eta^b}{ds}\right)=0,$$

that is, that the function $m_{cb}(d\eta^c/ds)(d\eta^b/ds)$ is constant along any geodesic. In such a case, we say that $m_{cb}(d\eta^c/ds)(d\eta^b/ds)$ is a *first quadratic integral* of the system of geodesics (T. Y. Thomas [13]). Thus we have from Theorem 8. 3

THEOREM 8. 4. *For a metric f-surface in a Kählerian space, a necessary and sufficient condition for* $S_{cb}{}^a$ *to vanish is that the system of geodesics of the metric* *f-surface has a first quadratic integral* $m_{cb}(d\eta^c/ds)(d\eta^b/ds)$. *When* $S_{cb}{}^a=0$ *is satisfied, vector field* $v_b=v_x f^x{}_b$ *satisfies*

$$\nabla_c v_b+\nabla_b v_c=0,$$

i.e. $v^a=g^{ab}v_b$ *is a Killing vector field if* $\nabla_c v_x=0$.

We assume now that the tensor field $f_c{}^a$ is a Killing tensor, i.e. that it satisfies the condition (Yano [19])

$$\nabla_c f_b{}^a+\nabla_b f_c{}^a=0.$$

Then, substituting the first equation of (8. 1) in the equation above, we have

$$2h_{cb}{}^y f_y{}^a-h_c{}^a{}_y f^y{}_b-h_c{}^a{}_y f^y{}_c=0,$$

from which it follows

$$h_{cb}{}^x=f^z{}_c f^y{}_b \lambda_{zy}{}^x$$

and consequently

$$\nabla_c f_b{}^a=0, \qquad \nabla_c f^x{}_b=0.$$

The converse being evident, if we take account of Theorem 4. 1, we have

PROPOSITION 8. 7. *For a metric f-surface in a Kählerian space, a necessary and sufficient condition for* $f_b{}^a$ *to be a Killing tensor is that one of the following four conditions is satisfied:*

(a) $\quad h_{cb}{}^x=f^z{}_c f^y{}_b\lambda_{zy}{}^x, \qquad h_c{}^a{}_y=f^z{}_c f_x{}^a\lambda_{zy}{}^x.$

(b) $\quad h_{cb}{}^x=f^z{}_c f^y{}_b\lambda_{zy}{}^x, \qquad \lambda_{zy}{}^u g_{ux}=\lambda_{zx}{}^u g_{uy}.$

(c) $\quad \nabla_c f_b{}^a=0.$

(d) $\quad \nabla_c f^x{}_b=0$ (*or equivalently* $\nabla_c f_y{}^a=0$),

where $\lambda_{zy}{}^x=\lambda_{yz}{}^x$.

We next assume that the tensor field $f_{cb}=f_c{}^e g_{eb}$ is harmonic. Since we had in Proposition 7. 1

$$\nabla_c f_{ba}+\nabla_b f_{ac}+\nabla_a f_{cb}=0,$$

the condition for f_{cb} to be harmonic is equivalent to the condition

$$g^{cb}\nabla_c f_b{}^a=-g^{cb}(h_{cb}{}^x f_x{}^a-h_c{}^a{}_y f^y{}_b)=0,$$

from which we have

$$h_{ab}{}^x f_x{}^b=f^y{}_a\lambda_y, \qquad \lambda_y=g^{cb}h_{cb}{}^x g_{xy}.$$

The converse being evident, we have

PROPOSITION 8. 8. *For a metric f-surface in a Kählerian space, a necessary and sufficient condition for f_{cb} to be harmonic is that*

$$h_{cb}{}^x f_x{}^b = f^y{}_c \lambda_y, \qquad \lambda_y = g^{cb} h_{cb}{}^x g_{xy}.$$

We now suppose

$$L_{cb}{}^x = 0 \tag{8. 10}$$

in a metric f-surface in a Kählerian space. Then, from the definition (2. 4) of $L_{cb}{}^x$ and (4. 10), we find

$$f_c{}^e L_{eb}{}^x = -(\nabla_c f^x{}_b - \nabla_b f^x{}_c) + f^y{}_c f_y{}^e \nabla_e f^x{}_b = 0,$$

from which we have

$$f_y{}^b \nabla_b f^x{}_c = 0$$

and consequently

$$\nabla_c f^x{}_b - \nabla_b f^x{}_c = 0. \tag{8. 11}$$

Therefore, the two conditions (8. 10) and (8. 11) are equivalent to each other. Consequently, it follows from Proposition 8. 2 that the condition (8. 10) is equivalent to one of three conditions:

$$S_{cb}{}^x = 0, \qquad S_{cy}{}^x = 0, \qquad H_{cy}{}^x = f^z{}_c \lambda_{zy}{}^x.$$

By means of (4. 2), the condition (8. 11) is equivalent to

$$h_{ce}{}^x f_b{}^e - h_{be}{}^x f_c{}^e = 0,$$

from which we find

$$h_{cb}{}^x + f_c{}^e f_b{}^d h_{ed}{}^x = f^z{}_c f^y{}_b \lambda_{zy}{}^x,$$

$$h_c{}^a{}_y - f_c{}^e f_d{}^a h_e{}^d{}_y = f^z{}_c f_x{}^a \lambda_z{}^x{}_y,$$

$$h_c{}^e{}_y f_e{}^a + f_c{}^e h_e{}^d{}_y = 0,$$

$\lambda_z{}^x{}_y$ being defined by $\lambda_z{}^x{}_y = \lambda_{zv}{}^u g^{vx} g_{uy}$, and consequently $S_{cb}{}^a$ and $S_{cy}{}^a$ take respectively the form

$$\begin{cases} S_{cb}{}^a = 2(f^x{}_c h_b{}^e{}_x - f^x{}_b h_c{}^e{}_x) f_e{}^a, \\ S_{cy}{}^a = -2 f_c{}^e f_d{}^a h_e{}^d{}_y + f^z{}_c f_x{}^a (\lambda^x{}_{yz} - \lambda^x{}_{zy}) \end{cases} \tag{8. 12}$$

because of (4. 4), $\lambda^x{}_{zy}$ being defined by $\lambda^x{}_{zy} = \lambda_{vz}{}^u g^{xv} g_{uy}$. Thus we have

PROPOSITION 8. 9. *For a metric f-surface in a Kählerian space, a necessary* *and sufficient condition for* $L_{cb}{}^x$ *to vanish is that*

$$\nabla_c f^x{}_b - \nabla_b f^x{}_c = 0.$$

If this condition is satisfied, the tensors $S_{cb}{}^a$ *and* $S_{cy}{}^a$ *are given by* (8. 12) *and*

$$S_{cb}{}^x = 0, \qquad S_{cy}{}^x = 0.$$

Let $v^a = v^x f_x{}^a$ be a vector field belonging to the distribution M and satisfying $\nabla_c v^x = 0$. Then we find

$$\begin{aligned}\nabla_a v^a &= (\nabla_a f_x{}^a) v^x \\ &= (f_d{}^e h_e{}^d{}_x) v^x = 0\end{aligned}$$

by means of (8. 1), because $f_d{}^e h_e{}^d{}_x = f^{ed} h_{ed}{}^y g_{yx} = 0$. On putting $v_b = g_{ba} v^a$, we obtain

$$\nabla_c v_b - \nabla_b v_c = (\nabla_c f^x{}_b - \nabla_b f^x{}_c) v_x,$$

v_x being defined by $v_x = g_{xy} v^y$. Thus we have from Proposition 8. 9

PROPOSITION 8. 10. *If, in a metric f-surface of a Kählerian space,* $L_{cb}{}^x = 0$ *is valid, then any vector field* $v^a = v^x f_x{}^a$ *belonging to the distribution M and satisfying* $\nabla_c v^x = 0$ *is harmonic.*

We assume now that $N_{cb}{}^a = 0$ and $S_{cb}{}^a = 0$ are satisfied in a metric f-surface of a Kählerian space. Then, $S_{cb}{}^a = 0$ implies $S_{cy}{}^x = 0$, from which we find

$$h_{cb}{}^x f_y{}^b = f^z{}_c \lambda_{zu}{}^x.$$

Therefore, taking account of Theorem 4. 1 and Proposition 8. 1, we have

$$\text{(a)} \qquad h_{cb}{}^x = f^z{}_c f^y{}_b \lambda_{zy}{}^x, \qquad \lambda_{zy}{}^u g_{ux} = \lambda_{zx}{}^u g_{uy}.$$

$$\text{(8. 13)} \qquad \text{(b)} \qquad \nabla_c f_b{}^a = 0.$$

$$\text{(c)} \qquad \nabla_c f^x{}_b = 0.$$

respectively from (8. 2), (8. 4) and (8. 3). Thus we have

THEOREM 8. 5. *A necessary and sufficient condition that the induced f-structure of a metric f-surface in a Kählerian space is integrable and* $S_{cb}{}^a = 0$ *is satisfied is that one of the three conditions* (a), (b) *and* (c) *stated in* (8. 13) *is satisfied.*

We next assume that the second fundamental tensor $h_{cb}{}^x$ of a metric f-surface in a Kählerian space has the form

$$\text{(8. 14)} \qquad h_{cb}{}^x \equiv A^x g_{cb}$$

and satisfies the condition

$$\text{(8. 15)} \qquad h_{cb}{}^x f_y{}^b \equiv 0,$$

where $f^x{}_a A_x$ is a certain vector field belonging to the distribution M. If this is the case, we say that the metric f-surface is *f-umbilic*. Then the tensor $h_{cb}{}^x$ has the form

$$h_{cb}{}^x = A^x g_{cb} + f^z{}_c f^y{}_b B_{zy}{}^x, \tag{8. 16}$$

$B_{zy}{}^x$ being a tensor field satisfying the condition

$$B_{zy}{}^x = B_{yz}{}^x.$$

Conversely, if the condition (8. 16) is satisfied, then the metric f-surface is f-umbilic.

Taking account of (4. 2) and (4. 5), we have from (8. 16)

$$L_{cb}{}^x = 2A^x(-g_{cb} + m_{cb}) \tag{8. 17}$$

and

$$\nabla_c f^x{}_b = A^x f_{cb}. \tag{8. 18}$$

Next, substituting (8. 16) in the first equation of (4. 4), we find

$$S_{cb}{}^a = 0.$$

Thus we have

PROPOSITION 8. 11. *When a metric f-surface in a Kählerian space is f-umbilic, the conditions*

$$L_{cb}{}^x = aA^x(-g_{cb} + m_{cb}),$$

$$\nabla_c f^x{}_b = A^x f_{cb}$$

are satisfied and moreover $S_{cb}{}^a = 0$ *is valid.*

If we assume that a metric f-surface in a Kählerian space satisfies the condition

$$\nabla_c f^x{}_b - \nabla_b f^x{}_c = 2A^x f_{cb},$$

or equivalently

$$L_{cb}{}^x = 2A^x(-g_{cb} + m_{cb})$$

(Cf. equation (2. 20)), then we have

$$h_{cb}{}^x + f_c{}^e f_b{}^d h_{ed}{}^x = 2A^x g_{cb} - 2A^x m_{cb} + m_c{}^e h_{eb}{}^x \tag{8. 19}$$

by virtue of (4. 5). Transvecting (8. 19) with $f_y{}^b$, we find

$$H_{cy}{}^x = h_{cb}{}^x f_y{}^b = f^z{}_c \lambda_{zy}{}^x,$$

which implies together with (8. 19)

$$h_{cb}{}^{x}+f_{c}{}^{e}f_{b}{}^{d}h_{ed}{}^{x}=2A^{x}g_{cb}+f^{z}{}_{c}f^{y}{}_{b}(\lambda_{zy}{}^{x}-2g_{zy}A^{x}). \tag{8. 20}$$

If we assume moreover that $S_{cb}{}^{a}=0$ is satisfied in the metric f-surface, then we have from Proposition 8. 3

$$h_{cb}{}^{x}-f_{c}{}^{e}f_{b}{}^{d}h_{ed}{}^{x}=f^{z}{}_{c}f^{y}{}_{b}\lambda_{zy}{}^{x}. \tag{8. 21}$$

Adding two equations (8. 20) and (8. 21), we obtain

$$h_{cb}{}^{x}=A^{x}g_{cb}+f^{z}{}_{c}f^{y}{}_{b}B_{zy}{}^{x},$$

where

$$B_{zy}{}^{x}=\lambda_{zy}{}^{x}-g_{zy}A^{x}.$$

Thus, taking account of Proposition 8. 11, we have

THEOREM 8. 6. *A necessary and sufficient condition for a metric f-surface in a Kählerian space to be f-umbilic is that two conditions*

$$S_{cb}{}^{a}=0 \qquad \textit{and} \qquad \nabla_{c}f^{x}{}_{b}-\nabla_{b}f^{x}{}_{c}=A^{x}f_{cb}$$

are satisfied.

§ 9. Metric f-surfaces in a Fubini space.

We suppose that the enveloping space W is a Fubini space. Then the curvature tensor of W is given by

$$K_{kjih}=k(G_{kh}G_{ji}-G_{jh}G_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}) \tag{9. 1}$$

with a constant k (Cf. Yano [20]). If, taking a metric f-surface V in a Fubini space W, we assume that the holomorphic extension $T^{H}_{P}(V)$ of tangent space of V coincides with the tangent space $T_{P}(W)$ of W at each point P belonging to V, we obtain the following equations of Gauss and Codazzi

$$\begin{aligned}
&K_{kjih}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}B_{a}{}^{h}\\
&=K_{dcba}-(h_{da}{}^{y}h_{cb}{}^{x}-h_{ca}{}^{y}h_{db}{}^{x})g_{xy},\\
&K_{kjih}B_{d}{}^{k}B_{c}{}^{j}B_{b}{}^{i}C_{x}{}^{h}=(\nabla_{d}h_{cb}{}^{y}-\nabla_{c}h_{db}{}^{y})g_{yx},\\
&K_{kjih}B_{d}{}^{k}B_{c}{}^{j}C_{y}{}^{i}C_{x}{}^{h}\\
&=K_{dcyx}-(h_{db}{}^{z}h_{c}{}^{b}{}_{y}-h_{cb}{}^{z}h_{d}{}^{b}{}_{y})g_{zx},
\end{aligned} \tag{9. 2}$$

where

$$K_{dcba}=K_{dcb}{}^{e}g_{ea}, \qquad K_{dcyx}=K_{dcy}{}^{z}g_{zx},$$

$K_{dcb}{}^{a}$ and $K_{dcy}{}^{x}$ being the curvature tensors of the induced connections $\Gamma_{c}{}^{a}{}_{b}$ and $\Gamma_{c}{}^{x}{}_{y}$ respectively.

Substituting (9. 1) in (9. 2), we find

$$(9.3)\qquad \begin{aligned} &k(g_{da}g_{cb}-g_{ca}g_{db}+f_{da}f_{cb}-f_{ca}f_{db}-2f_{dc}f_{ba})\\ &=K_{dcba}-(h_{da}{}^{y}h_{cb}{}^{x}-h_{ca}{}^{y}h_{db}{}^{x})g_{xy},\\ &k(f^{x}{}_{d}f_{cb}-f^{x}{}_{c}f_{db}-2f_{dc}f^{x}{}_{b})=\nabla_{d}h_{cb}{}^{x}-\nabla_{c}h_{db}{}^{x},\\ &k(f^{x}{}_{d}f_{yc}-f^{x}{}_{c}f_{yd})=K_{dcy}{}^{x}-(h_{db}{}^{x}h_{c}{}^{b}{}_{y}-h_{cb}{}^{x}h_{d}{}^{b}{}_{y}), \end{aligned}$$

f_{yb} being defined by $f_{yb}=f^{x}{}_{b}g_{xy}$.

If we now assume that for the metric f-surface $S_{cb}{}^{a}=0$ is valid and the induced f-structure $f_{b}{}^{a}$ is integrable ($N_{cb}{}^{a}=0$), then we have from Theorem 8. 5

$$(9.4)\qquad \nabla_{c}f^{x}{}_{b}=0,\qquad h_{cb}{}^{x}=f^{z}{}_{c}f^{y}{}_{b}\lambda_{zy}{}^{x}.$$

Thus, taking account of the well known formula

$$\nabla_{d}\nabla_{c}f^{x}{}_{b}-\nabla_{c}\nabla_{d}f^{x}{}_{b}=-K_{dcb}{}^{a}f^{x}{}_{a}+K_{dcy}{}^{x}f^{y}{}_{b},$$

then we have

$$(9.5)\qquad K_{dcb}{}^{a}f^{x}{}_{a}=K_{dcy}{}^{x}f^{y}{}_{b}$$

by means of the first equation of (9. 4). Transvecting (9. 5) with $f_{e}{}^{b}$, we find

$$K_{dcb}{}^{a}f^{x}{}_{a}f_{e}{}^{b}=0,$$

which implies together with the first equation of (9. 3) and the second equation of (9. 4)

$$k=0$$

and consequently $K_{kji}{}^{h}=0$. Therefore, the enveloping space W should be locally flat. Thus we have

THEOREM 9. 1. *In a Fubini space, which is not locally flat, there exists no metric f-surface such that $S_{cb}{}^{a}=0$, the induced f-structure $f_{b}{}^{a}$ is integrable and the holomorphic extension of tangent space of the f-surface coincides with the tangent space of the enveloping Fubini space at each point of the f-surface.*

Taking account of (9. 4), we have from Theorems 6. 2 and 6. 3

THEOREM 9. 2. *If, in a Euclidean space E^{N} of even dimensions with the natural Kählerian structure, there is given a normal metric f-surface V such that the induced f-structure $f_{b}{}^{a}$ is integrable and the holomorphic extension of tangent space of the surface coincides with the tangent space of the enveloping Euclidean space E^{N}, then the surface is conjugate to a portion of a submanifold $\widetilde{V}$ appearing* *in Theorem* 6. 3 *under the group of all motions operating on* E^N *and preserving the complex structure of* E^N *and the induced connection* $\{{}_c{}^a{}_b\}$ *is locally flat.*

We next suppose that a metric f-surface V in a Fubini space W is f-umbilic and the holomorphic extension $T^H_P(V)$ of tangent space of V coincides with the tangent space $T_P(W)$of W at each point of V. Then we have from (8. 16)

$$h_{cb}{}^x = A^x g_{cb} + f^z{}_c f^y{}_b B_{zy}{}^x, \qquad B_{zy}{}^x = B_{yz}{}^x. \tag{9. 6}$$

Substituting (9. 6) in the second equation of (9. 3), we find

$$\begin{aligned} &k(f^x{}_d f_{cb} - f^x{}_c f_{db} - 2 f_{dc} f^x{}_b) \\ &= \nabla_d A^x g_{cb} - \nabla_c A^x g_{db} \\ &\quad + [(\nabla_d f^v{}_c) f^u{}_b - (\nabla_c f^v{}_d) f^u{}_b + f^v{}_c(\nabla_d f^u{}_b) - f^v{}_d(\nabla_c f^u{}_b)] B_{vu}{}^x \\ &\quad + f^v{}_c f^u{}_b \nabla_d B_{vu}{}^x - f^v{}_d f^u{}_b \nabla_c B_{vu}{}^x. \end{aligned} \tag{9. 7}$$

Transvecting (9. 7) with $f^{dc} f_y{}^b$, we obtain

$$rk\delta^x_y = f^{dc}(\nabla_d f^z{}_c) B_{zy}{}^x,$$

and, substituting (9. 6) in the second equations of (4. 2),

$$f^{dc}(\nabla_d f^z{}_c) = -rA^z,$$

r being the rank of the induced f-structure $f_b{}^a$. From these two equations we have

$$k\delta^x_y = -A^z B_{zy}{}^x. \tag{9. 8}$$

On the other hand, substituting the first equations of (4. 2) in the identity

$$(\nabla_c f_b{}^a) f^x{}_a + f_b{}^a(\nabla_c f^x{}_a) = 0,$$

we find

$$h_c{}^a{}_y f^y{}_b f^x{}_a - h_{ce}{}^x f_y{}^e f^y{}_b = 0,$$

which implies together with (9. 6)

$$(n-r-1)A^x = B_{vu}{}^v g^{ux} - g^{vu} B_{vu}{}^x. \tag{9. 9}$$

Thus we have

PROPOSITION 9. 1. *If, in a Fubini space W, a metric f-surface is f-umbilic and the holomorphic extension* $T^H_P(V)$ *of tangent space of V coincides with the tangent space* $T_P(W)$ *of W at each point* P *of V, then equation* (9. 8) *and* (9. 9) *are valid.*

If we now assume that $B_{zy}{}^x$ appearing in (9. 6) has the form

$$B_{zy}{}^x = g_{zy} B^x,$$

i.e. that $h_{cb}{}^x$ has the form

$$h_{cb}{}^x = A^x h_{cb} + B^x m_{cb},$$

then (9. 8) and (9. 9) reduce respectively to

$$(9.10) \qquad \begin{aligned} &kg^{xy} = -A^x B^y, \\ &(n-r-1)(A^x + B^x) = 0. \end{aligned}$$

The equations (9. 10) imply, provided $n-r-1 \neq 0$,

$$k=0, \qquad A^x=0, \qquad B^x=0$$

and consequently

$$(9.11) \qquad h_{cb}{}^x = 0.$$

Therefore, the enveloping Fubini space W is necessarily locally flat and the surface V is totally geodesic. Thus we have

THEOREM 9. 3. *In a Fubini space W, which is not locally flat, there exists no metric f-surface V such that its second fundamental tensor $h_{cb}{}^x$ has the form*

$$h_{cb}{}^x = A^x g_{cb} + B^x m_{cb}$$

and the holomorphic extension $T_P^H(V)$ of tangent space of V coincides with the tangent space $T_P(W)$ of W at each point of V, if the rank of the induced f-structure is smaller than $n-1$, the surface V being n-dimensional.

THEOREM 9. 4. *When, in a locally flat Fubini space, there exists a metric f-surface satisfying the conditions mentioned in Theorem 9. 3, the second fundamental tensor $h_{cb}{}^x$ of the f-surface vanishes identically, i.e. the surface is totally geodesic, if the rank of the induced f-structure is smaller than $n-1$, the surface being n-dimensional.*

If, for a metric hypersurface in a Euclidean space of even dimensions $n+1$, the conditions mentioned in Theorem 9. 4 are satisfied, then the hypersurface is a portion of a hypersphere S^n. If this is the case, the induced f-structure is of rank $n-1$. (Cf. Kurita [4], Tashiro [10], [11], Tashiro and Tachibana [12], Yano and Ishihara [21]).

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