CERTAIN INFINITESIMAL TRANSFORMATION OF NORMAL CONTACT METRIC MANIFOLD

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Introduction.

In the previous paper [2], the author studied infinitesimal conformal and projective transformations of normal contact metric manifold. In the present paper, we study certain infinitesimal transformation of normal contact metric manifold and prove the following

THEOREM 1. *In a normal contact metric manifold, any curvature-preserving infinitesimal transformation is necessarily an infinitesimal isometry.*

If the above theorem is proved, we have obviously the following

THEOREM 2. *In a normal contact metric manifold, an infinitesimal affine transformation is necessarily an infinitesimal isometry.*

THEOREM 3. *In a normal contact metric manifold, an infinitesimal homothetic transformation is necessarily an infinitesimal isometry.*

1. Normal contact metric manifold.

Let M be a differentiable manifold of dimension $2n+1$. If there is defined on M a differentiable 1-form η having the property that

$$(1.1)\qquad \eta\wedge\overbrace{d\eta\wedge\cdots\wedge d\eta}^{n}\neq 0,$$

then, M and η are respectively called a contact manifold and a contact form on M.

Now we put $\phi=d\eta$, that is

$$2\phi_{ji}=\partial_j\eta_i-\partial_i\eta_j,$$

where η_i, ϕ_{ji} denote the components of the form ϕ and η respectively. We denote by g_{ji} the Riemannian metric tensor such that $\phi_i{}^h=g^{hr}\phi_{ir}$, $\eta^i=g^{ir}\eta_r$ satisfy the following properties:

$$(1.2)\qquad g_{ji}\eta^j\eta^i=1,$$

$$(1.3)\qquad \phi_j{}^i\eta_i=0,$$

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(1. 4) $$\phi_j{}^i\phi_k{}^j=-\delta_k^i+\eta^i\eta_k,$$

from which we have

(1. 5) $$\phi_{ji}\phi^{ji}=2n.$$

We call a contact manifold M with such a Riemannian metric a contact metric manifold. A contact metric manifold is said to be *normal* if the tensor $N_{ji}{}^h$ defined by

(1. 6) $$N_{ji}{}^h=\phi_j{}^r(\partial_r\phi_i{}^h-\partial_i\phi_r{}^h)-\phi_i{}^r(\partial_r\phi_j{}^h-\partial_j\phi_r{}^h)+\eta_j\partial_i\eta^h-\eta_i\partial_j\eta^h$$

vanishes identically.

In a normal contact metric manifold the following identities are well known.[1]

(1. 7) $$\nabla_j\eta_i=\phi_{ji},$$

(1. 8) $$\nabla_j\phi_{ih}=\eta_i g_{jh}-\eta_h g_{ji}.$$

From these identities we have

(1. 9) $$\eta_r R_{kji}{}^r=\eta_k g_{ji}-\eta_j g_{ki},$$

applying Ricci's identity to η_i.

2. Infinitesimal transformations.

Let $\underset{v}{\mathcal{L}}$ be a notation of Lie derivatives with respect to an infinitesimal transformation v^i. In a Riemannian manifold, if an infinitesimal transformation v^i satisfies

(2. 1) $$\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}\equiv\nabla_j\nabla_i v^h+R_{kji}{}^h v^k=0,$$

we call it an infinitesimal affine transformation. In a Riemannian manifold the identity

(2. 2) $$\underset{v}{\mathcal{L}}R_{kji}{}^h=\nabla_k\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}-\nabla_j\underset{v}{\mathcal{L}}\left\{ {h \atop ki} \right\}$$ [2]

being valid, an infinitesimal affine transformation is obviously curvature-preserving.

If an infinitesimal transformation v^i satisfies

$$\underset{v}{\mathcal{L}}g_{ji}\equiv\nabla_j v_i+\nabla_i v_j=2cg_{ji},\qquad c=\text{const.},$$

we call it an infinitesimal homothetic transformation. Furthermore if $c=0$, it is called an infinitesimal isometry. Since the identity

(2. 3) $$\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}=\frac{1}{2}g^{hr}\left(\nabla_j\underset{v}{\mathcal{L}}g_{ri}+\nabla_i\underset{v}{\mathcal{L}}g_{jr}-\nabla_r\underset{v}{\mathcal{L}}g_{ji}\right)$$

is valid, we know that an infinitesimal homothetic transformation is necessarily an infinitesimal affine transformation and consequently is curvature-preserving.

1) Sasaki and Hatakeyama [3]. Okumura [1].

2) Yano [4].

3. Proof of Theorem 1.

To prove Theorem 1, we begin with the

LEMMA 1. *Let v^i be a curvature-preserving infinitesimal transformation of normal contact metric manifold. Then v^i satisfies, for a suitable scalar function ρ*

$$\underset{v}{\mathcal{L}} g_{ji}=\rho\eta_j\eta_i. \tag{3. 1}$$

Proof. Taking the Lie derivatives of both side of (1. 9), we have

$$R_{kji}{}^h\underset{v}{\mathcal{L}}\eta_h=g_{ji}\underset{v}{\mathcal{L}}\eta_k+\eta_k\underset{v}{\mathcal{L}}g_{ji}-g_{ki}\underset{v}{\mathcal{L}}\eta_j-\eta_j\underset{v}{\mathcal{L}}g_{ki}.$$

Transvecting the above equation with η^k and making use of (1. 9), we get

$$(\eta^h g_{ji}-\eta_i\delta_j^h)\underset{v}{\mathcal{L}}\eta_h=g_{ji}\eta^k\underset{v}{\mathcal{L}}\eta_k+\underset{v}{\mathcal{L}}g_{ji}-\eta_i\underset{v}{\mathcal{L}}\eta_j-\eta_j\eta^k\underset{v}{\mathcal{L}}g_{ki},$$

from which

$$\underset{v}{\mathcal{L}}g_{ji}=\eta_j\eta^r\underset{v}{\mathcal{L}}g_{ri}. \tag{3. 2}$$

Since the left hand side of (3. 2) is a symmetric tensor, the vector $\eta^r\underset{v}{\mathcal{L}}g_{ri}$ must be proportional to η_i. Thus we get (3. 1). This completes the proof.

Using this lemma, we now prove Theorem 1. From the assumption of Theorem 1, the infinitesimal transformation is curvature-preserving and so we have

$$\underset{v}{\mathcal{L}}R_{kji}{}^h=0.$$

Consequently we have

$$\phi^{kj}\phi_h{}^i\underset{v}{\mathcal{L}}R_{kji}{}^h=2\rho\phi^{kj}\phi_h{}^i\nabla_k\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}=0, \tag{3. 3}$$

because of (2. 2).

On the other hand, by means of (1. 7), (2. 3) and (3. 1), it follows that

$$\underset{v}{\mathcal{L}}\left\{ {h \atop ji} \right\}=\frac{1}{2}g^{hr}[\nabla_j\rho\eta^h\eta_i+\nabla_i\rho\eta_j\eta^h-\nabla^h\rho\eta_j\eta_i$$
$$+\rho(\phi_j{}^h\eta_i+\phi_{ji}\eta^h+\phi_{ij}\eta^h+\eta_j\phi_i{}^h+\phi_j{}^h\eta_i+\eta_j\phi_i{}^h)].$$

Substituting this into (3. 3) and making use of (1. 3), (1. 4) and (1. 7), we get

$$\begin{aligned}\phi^{kj}\phi_h{}^i\underset{v}{\mathcal{L}}R_{kji}{}^h&=2\rho\phi^{kj}\phi_h{}^i(\phi_j{}^h\phi_{ki}+\phi_{kj}\phi_i{}^h)\\&=2\rho[(-\delta_j^i+\eta^i\eta_j)(\delta_i^j-\eta^j\eta_i)-4n^2]\\&=-4n(2n+1)\rho=0.\end{aligned} \tag{3. 4}$$

Consequently we have

$$\rho=0. \tag{3. 5}$$

Substituting (3. 5) into (3. 1), we have

$$\underset{v}{\mathcal{L}}g_{ji}=0.$$

Thus the infinitesimal transformation v^i is an infinitesimal isometry. This completes the proof.

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