ON THE EXISTENCE OF ANALYTIC MAPPINGS BETWEEN TWO ULTRAHYPERELLIPTIC SURFACES

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§1. **Introduction.** Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions having no zero other than an infinite number of simple zeros. Then one of the authors [6], [7] established the following perfect condition for the existence of analytic mappings from R into S.

THEOREM **A.** *If there exists an analytic mapping φ from R into S, then there exists a pair of two entire functions $h(z)$ and $f(z)$ satisfying an equation*

$$f(z)^2G(z)=g\circ h(z)$$

and vice versa.

Let $\mathfrak{M}(R)$ be a family of non-constant meromorphic functions on R. Let f be a member of $\mathfrak{M}(R)$. Let $P(f)$ be the number of Picard's exceptional values of f, which we say α a Picard's value of f when α is not taken by f on R. Let $P(R)$ be a quantity defined by

$$\sup_{f\in\mathfrak{M}(R)}P(f)$$

(cf. [4]). Let $P(S)$ be the corresponding quantity attached to S.

In the present paper we shall give a perfect condition for the existence of analytic mappings from R into S in a case of $P(R)=P(S)=4$, which is more direct than theorem A, and shall give some characterizations of the surfaces R with $P(R)=3$ by the forms of defining functions G.

By a characterization, which was given in [5], of R with $P(R)=4$ by $G(z)$ we can put

$$(1.1)\qquad F(z)^2G(z)=(e^{H(z)}-\gamma)(e^{H(z)}-\delta),\qquad H(z)\not\equiv\text{const.},\qquad H(0)=0,\qquad \gamma\delta(\gamma-\delta)\neq 0$$

with two suitable entire functions F and H and two constants γ and δ. Similarly if $P(S)=4$, we can put

$$(1.2)\qquad f(w)^2g(w)=(e^{L(w)}-\gamma')(e^{L(w)}-\delta'),\qquad L(w)\not\equiv\text{const.},\qquad L(0)=0,\qquad \gamma'\delta'(\gamma'-\delta')\neq 0$$

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with two suitable entire functions f and L and two constants γ' and δ'. Then our result is:

THEOREM **B.** *Let R and S be two ultrahyperelliptic surfaces defined by* (1. 1) *and* (1. 2), *respectively. Then there exists an analytic mapping φ from R into S if and only if there exists an entire function $h(z)$ such that either*

(a) $$L\circ h(z)-L\circ h(0)=H(z),\qquad \gamma e^{L\circ h(0)}=\gamma',\qquad \delta e^{L\circ h(0)}=\delta'$$

or

(b) $$L\circ h(z)-L\circ h(0)=-H(z),\qquad \gamma\gamma'=e^{L\circ h(0)},\qquad \delta\delta'=e^{L\circ h(0)}.$$

A proof of this theorem B will be given in § 3.

Next if the surface R satisfies $P(R)=3$, then its defining function $G(z)$ satisfies

$$\begin{gathered}F^2G=1-2\beta_1e^{H_1}-2\beta_2e^{H_2}+\beta_1^2e^{2H_1}-2\beta_1\beta_2e^{H_1+H_2}+\beta_2^2e^{2H_2},\\ H_1(z)\not\equiv\text{const.},\qquad H_2(z)\not\equiv\text{const.},\qquad H_1(0)=H_2(0)=0,\qquad \beta_1\beta_2\neq 0\end{gathered}\tag{1. 3}$$

with three suitable entire functions F, H_1 and H_2 and two constants β_1 and β_2.

For completeness we shall give here a brief exposition of this fact. Since $P(R)=3$, there exists a two-valued entire algebroid function $\tilde{f}$ of z which is regular on R and whose defining equation is

$$F(z,\tilde{f})\equiv\tilde{f}^2-2f_1(z)\tilde{f}+f_1(z)^2-f_2(z)^2G(z)=0$$

with two single-valued entire functions $f_1(z)$ and $f_2(z)$ of z. Further we may assume that 0, 1 and ∞ are three exceptional values of $\tilde{f}$. Then, by Rémoundos' reasoning [8] pp. 25–27, we have three possibilities

$$\begin{bmatrix}F(z,0)\\F(z,1)\end{bmatrix}=\begin{bmatrix}c\\ \beta e^H\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}\beta e^H\\ c\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}\beta_1e^{H_1}\\ \beta_2e^{H_2}\end{bmatrix}$$

where c, β, β_1 and β_2 are non-zero constants and H, H_1 and H_2 are non-constant entire functions of z satisfying $H(0)=H_1(0)=H_2(0)=0$. However we may put aside the first and the second cases. In fact, from the equations

$$\begin{cases}f_1^2-f_2^2G=c,\\1-2f_1+f_1^2-f_2^2G=\beta e^H\end{cases}\quad\text{or}\quad\begin{cases}f_1^2-f_2^2G=\beta e^H,\\1-2f_1+f_1^2-f_2^2G=c,\end{cases}$$

we have

$$4f_2^2G=(\beta e^H-(1+\sqrt{c})^2)(\beta e^H-(1-\sqrt{c})^2).$$

By the characterization of surfaces with four Picard's values, we have $P(R)=4$. This is a contradiction. Therefore if $P(R)=3$, then we have the third case and obtain a representation

$$4f_2^2G=1-2\beta_1e^{H_1}-2\beta_2e^{H_2}+\beta_1^2e^{2H_1}-2\beta_1\beta_2e^{H_1+H_2}+\beta_2^2e^{2H_2}.$$

Conversely, the surface R defined by (1. 3) satisfies $P(R)\geqq 3$. In fact,

$$\tilde{f}(z)=\frac{1}{2}(1+\beta_1 e^{H_1}-\beta_2 e^{H_2})+\frac{i}{2}\sqrt{\tilde{G}(z)}$$

is an entire function on R which is two-valued for z, where

$$\tilde{G}(z)=1-2\beta_1 e^{H_1}-2\beta_2 e^{H_2}+\beta_1{}^2 e^{2H_2}-2\beta_1\beta_2 e^{H_1+H_2}-\beta_2{}^2 e^{2H_2}.$$

Then

$$F(z,\tilde{f})\equiv\tilde{f}^2-(1+\beta_1 e^{H_1}-\beta_2 e^{H_2})\tilde{f}+\beta_1 e^{H_1}$$

satisfies

$$F(z,0)=\beta_1 e^{H_1} \qquad \text{and} \qquad F(z,1)=\beta_2 e^{H_2}.$$

This shows that $\tilde{f}\neq 0$, 1 and ∞ on R.

However we did not succeed to determine all ultrahyperelliptic surfaces R with $P(R)=3$ (cf. [5], [7]). Here we shall show the following

THEOREM C. *The surface R defined by* (1. 3) *satisfies* $P(R)=3$ *if*

$$m(r,e^{H_1})=o(m(r,e^{H_2}))$$

outside a set of finite logarithmic measure.

A proof of this theorem C will be given in § 4.

Further in § 5 we shall determine all the surfaces R with $P(R)=3$ when H_1 and H_2 in (1. 3) are polynomials of degree 1 (Theorem D), and in § 6 we shall determine all the surfaces R with $P(R)=3$ when H_1 and H_2 in (1. 3) are polynomials of degree 2 (Theorem E).

§ 2. Lemmas. We need some preparatory lemmas in order to prove our results introduced in § 1. The notations T, m, N, N_1 and $\bar{N}$ are used in the sense of Nevanlinna [3]. The notation $N_2(r;a,f)$ is the N-function of simple a-points of f. The following lemma is a generalization of Borel's theorem [1] and its proof depends essentially on Nevanlinna's formulation [2].

LEMMA 1. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *be meromorphic functions and let* $g_1(z),\cdots,g_n(z)$ *be entire functions. Further suppose that*

$$T(r,a_j)=o\left(\sum_{\nu=1}^{n} m(r,e^{g_\nu})\right), \qquad j=0,1,\cdots,n,$$

holds outside a set of finite logarithmic measure. If an identity

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}=a_0(z) \tag{2. 1}$$

holds, then we have an identity

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z)e^{g_\nu(z)}=0 \tag{2. 2}$$

with the exception of a case for which all the constants c_ν *reduce to zero.*

Proof. Let $G_\nu(z)$ be $a_\nu(z)e^{g_\nu(z)}$. Then we have

$$\sum_{\nu=1}^{n} G_\nu(z)=a_0(z). \tag{2. 1'}$$

By differentiating both sides of (2. 1′), we have

$$\sum_{\nu=1}^{n} G_\nu^{(\mu)}(z)=a_0^{(\mu)}(z); \tag{2. 3}$$

$$\sum_{\nu=1}^{n} G_\nu(z)\frac{G_\nu^{(\mu)}(z)}{G_\nu(z)}=a_0^{(\mu)}(z), \qquad \mu=1, \cdots, n-1. \tag{2. 3'}$$

On the other hand, we have

$$G_\nu^{(\mu)}(z)=P_\mu(a_\nu, a_\nu', \cdots, a_\nu^{(\mu)}, g_\nu', \cdots, g_\nu^{(\mu)})e^{g_\nu(z)}$$

with a suitable polynomial P_μ of indicated functions $a_\nu, a_\nu', \cdots, a_\nu^{(\mu)}, g_\nu', \cdots, g_\nu^{(\mu)}$. Thus we have

$$T\left(r, \frac{G_\nu^{(\mu)}}{G_\nu}\right)\leqq O(T(r, a_\nu)+T(r, g_\nu))=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right)$$

outside a set of finite logarithmic measure. Suppose that the determinant of the simultaneous equations (2. 1′) and (2. 3′) $\Delta\neqq 0$. By solving (2. 3′) with respect to G_j, $j=1, \cdots, n$, we have

$$G_j=\frac{\Delta_j}{\Delta}$$

where

$$\Delta=\begin{vmatrix} 1 & \cdots & 1 \\ \frac{G_1'}{G_1} & \cdots & \frac{G_n'}{G_n} \\ \multicolumn{3}{c}{\cdots\cdots\cdots} \\ \frac{G_1^{(n-1)}}{G_1} & \cdots & \frac{G_n^{(n-1)}}{G_n} \end{vmatrix}, \qquad \Delta_j=\begin{vmatrix} 1 & \cdots & 1 & a_0 & 1 & \cdots & 1 \\ \frac{G_1'}{G_1} & \cdots & \frac{G_{j-1}'}{G_{j-1}} & a_0' & \frac{G_{j+1}'}{G_{j+1}} & \cdots & \frac{G_n'}{G_n} \\ \multicolumn{3}{c}{\cdots\cdots\cdots} & \cdots\cdots & \multicolumn{3}{c}{\cdots\cdots\cdots} \\ \frac{G_1^{(n-1)}}{G_1} & \cdots & \frac{G_{j-1}^{(n-1)}}{G_{j-1}} & a_0^{(n-1)} & \frac{G_{j+1}^{(n-1)}}{G_{j+1}} & \cdots & \frac{G_n^{(n-1)}}{G_n} \end{vmatrix}.$$

Since $T(r, G_\nu^{(\mu)}/G_\nu)=o(\sum_{\nu=1}^{n} m(r, e^{g_\nu}))$, we have

$$T(r, \Delta)=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right), \qquad T(r, \Delta_j)=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right), \qquad j=1, \cdots, n,$$

outside a set of finite logarithmic measure. Thus we have

$$\begin{aligned} m(r, e^{g_\nu})&=T(r, e^{g_\nu})\leqq T(r, a_\nu)+T(r, G_\nu) \\ &\leqq T(r, a_\nu)+T(r, \Delta)+T(r, \Delta_\nu)=o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right), \end{aligned}$$

and hence

$$\sum_{\nu=1}^{n} m(r, e^{g_\nu}) = o\left(\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right)$$

outside a set of finite logarithmic measure. This is a contradiction. Consequently we obtain $\Delta \equiv 0$ and we complete the proof.

By lemma 1 we can immediately conclude:

LEMMA 2. *Let $a_1(z), \cdots, a_n(z)$ be meromorphic functions and let $g(z)$ be an entire function. Further suppose that*

$$T(r, a_j) = o(m(r, e^g)), \qquad j = 1, \cdots, n,$$

holds outside a set of finite logarithmic measure. Then an identity of the following form

$$\sum_{\nu=1}^{n} a_\nu(z) e^{\nu g(z)} \equiv 0$$

is impossible unless $a_1(z) \equiv \cdots \equiv a_n(z) \equiv 0$.

Further we shall prove the following lemma which is fundamental to the proofs of theorem B and theorem C.

LEMMA 3. *Let $a_1(z), \cdots, a_9(z)$ be meromorphic functions and let $H(z)$ and $L(z)$ be entire functions. Further suppose that*

$$T(r, a_j) = o(m(r, e^H)), \qquad j = 1, \cdots, 9$$

and

$$m(r, e^H) \sim m(r, e^L)$$

hold outside a set of finite logarithmic measure. Then an identity of the following form

(2. 4) $$a_1 e^{2L+2H} + a_2 e^{2L+H} + a_3 e^{L+2H} + a_4 e^{2L} + a_5 e^{2H} + a_6 e^{L+H} + a_7 e^L + a_8 e^H + a_9 = 0$$

is impossible, if the product of $a_1, \cdots, a_9$ does not vanish identically.

Proof. If (2. 4) holds, then by lemma 1 we have

$$c_1 a_1 e^{2L+2H} + c_2 a_2 e^{2L+H} + c_3 a_3 e^{L+2H} + c_4 a_4 e^{2L} + c_5 a_5 e^{2H} + c_6 a_6 e^{L+H} + c_7 a_7 e^L + c_8 a_8 e^H = 0$$

with suitable constants c_i, so that

(2. 5) $$c_1 a_1 e^{2L+H} + c_2 a_2 e^{2L} + c_3 a_3 e^{L+H} + c_4 a_4 e^{2L-H} + c_5 a_5 e^H + c_6 a_6 e^L + c_7 a_7 e^{L-H} + c_8 a_8 = 0.$$

If all $c_i c_j$ $(i \neq j; i, j = 1, \cdots, 7)$ are zero, then (2. 5) reduces to the one of the following identities:

(i) $\quad c_1 a_1 e^{2L+H} + c_8 a_8 = 0, \quad$ i.e. $\quad e^H = c(a_8/a_1) e^{-2L};$

(ii) $\quad c_2 a_2 e^{2L} + c_8 a_8 = 0, \quad$ i.e. $\quad e^L = c\sqrt{a_8/a_2};$

(iii) $\quad c_3 a_3 e^{L+H} + c_8 a_8 = 0, \quad$ i.e. $\quad e^H = c(a_8/a_3) e^{-L};$

(iv) $c_4a_4e^{2L-H}+c_8a_8=0$, i.e. $e^H=c(a_4/a_8)e^{2L}$;

(v) $c_5a_5e^H+c_8a_8=0$, i.e. $e^H=ca_8/a_5$;

(vi) $c_6a_6e^L+c_8a_8=0$, i.e. $e^L=ca_8/a_6$;

(vii) $c_7a_7e^{L-H}+c_8a_8=0$, i.e. $e^H=c(a_7/a_8)e^L$

with a non-zero constant c.

If at least one of c_ic_j $(i\neq j; i, j=1, \cdots, 7)$ is not zero, then by lemma 1, which is applicable to (2. 5) in this case, we have

$$c_1'a_1e^{2L+H}+c_2'a_2e^{2L}+c_3'a_3e^{L+H}+c_4'a_4e^{2L-H}+c_5'a_5e^H+c_6'a_6e^L+c_7'a_7e^{L-H}=0$$

with suitable constants c_i', so that

(2. 6) $$c_1'a_1e^{L+2H}+c_2'a_2e^{L+H}+c_3'a_3e^{2H}+c_4'a_4e^L+c_5'a_5e^{-L+2H}+c_6'a_6e^H+c_7'a_7=0.$$

If all $c_i'c_j'$ $(i\neq j, i, j=1, \cdots, 6)$ are zero, then (2. 6) reduces to the one of the following identities:

(viii) $c_1'a_1e^{L+2H}+c_7'a_7=0$, i.e. $e^L=c(a_7/a_1)e^{-2H}$;

(ix) $c_2'a_2e^{L+H}+c_7'a_7=0$, i.e. $e^H=c(a_7/a_2)e^{-L}$;

(x) $c_3'a_3e^{2H}+c_7'a_7=0$, i.e. $e^H=c\sqrt{a_7/a_3}$;

(xi) $c_4'a_4e^L+c_7'a_7=0$, i.e. $e^L=ca_7/a_4$;

(xii) $c_5'a_5e^{-L+2H}+c_7'a_7=0$, i.e. $e^L=c(a_5/a_7)e^{2H}$;

(xiii) $c_6'a_6e^H+c_7'a_7=0$, i.e. $e^H=ca_7/a_6$

with a non-zero constant c.

If at least one of $c_i'c_j'$ $(i\neq j; i, j=1, \cdots, 6)$ is not zero, then by using lemma 1 to (2. 6) we have

$$c_1''a_1e^{L+2H}+c_2''a_2e^{L+H}+c_3''a_3e^{2H}+c_4''a_4e^L+c_5''a_5e^{-L+2H}+c_6''a_6e^H=0$$

with suitable constants c_i'', so that

(2. 7) $$c_1''a_1+c_2''a_2e^{-H}+c_3''a_3e^{-L}+c_4''a_4e^{-2H}+c_5''a_5e^{-2L}+c_6''a_6e^{-L-H}=0.$$

If all $c_i''c_j''$ $(i\neq j; i, j=2, \cdots, 6)$ are zero, then (2. 7) reduces to the one of the following identities:

(xiv) $c_1''a_1+c_2''a_2e^{-H}=0$, i.e. $e^H=ca_2/a_1$;

(xv) $c_1''a_1+c_3''a_3e^{-L}=0$, i.e. $e^L=ca_3/a_1$;

(xvi) $c_1''a_1+c_4''a_4e^{-2H}=0$, i.e. $e^H=c\sqrt{a_4/a_1}$;

(xvii) $c_1''a_1+c_5''a_5e^{-2L}=0$, i.e. $e^L=c\sqrt{a_5/a_1}$;

(xviii) $c_1''a_1+c_6''a_6e^{-L-H}=0$, i.e. $e^H=c(a_6/a_1)e^{-L}$

with a non-zero constant c.

If at least one of $c_i''c_j''$ $(i\neq j; i, j=2, \cdots, 6)$ is not zero, then by using lemma 1 to (2. 7) we have

$$d_2a_2e^{-H}+d_3a_3e^{-L}+d_4a_4e^{-2H}+d_5a_5e^{-2L}+d_6a_6e^{-L-H}=0$$

with suitable constants d_j, so that

(2. 8) $$d_2a_2+d_3a_3e^{-L+H}+d_4a_4e^{-H}+d_5a_5e^{-2L+H}+d_6a_6e^{-L}=0.$$

If all d_id_j $(i\neq j;\ i, j=3, \cdots, 6)$ are zero, then (2. 8) reduces to the one of the following identities:

(xix) $d_2a_2+d_3a_3e^{-L+H}=0$, i.e. $e^H=c(a_2/a_3)e^L$;

(xx) $d_2a_2+d_4a_4e^{-H}=0$, i.e. $e^H=ca_4/a_2$;

(xxi) $d_2a_2+d_5a_5e^{-2L+H}=0$, i.e. $e^H=c(a_2/a_5)e^{2L}$;

(xxii) $d_2a_2+d_6a_6e^{-L}=0$, i.e. $e^L=ca_6/a_2$

with a non-zero constant c.

If at least one of d_id_j $(i\neq j;\ i, j=3, \cdots, 6)$ is not zero, then by using lemma 1 to (2. 8) we have

$$d_3'a_3e^{-L+H}+d_4'a_4e^{-H}+d_5'a_5e^{-2L+H}+d_6'a_6e^{-L}=0$$

with suitable constants d_i', so that

(2. 9) $$d_3'a_3+d_4'a_4e^{L-2H}+d_5'a_5e^{-L}+d_6'a_6e^{-H}=0.$$

If all $d_i'd_j'$ $(i\neq j;\ i, j=4, 5, 6)$ are zero, then (2. 9) reduces to the one of the following identities:

(xxiii) $d_3'a_3+d_4'a_4e^{L-2H}=0$, i.e. $e^L=c(a_3/a_4)e^{2H}$;

(xxiv) $d_3'a_3+d_5'a_5e^{-L}=0$, i.e. $e^L=ca_5/a_3$;

(xxv) $d_3'a_3+d_6'a_6e^{-H}=0$, i.e. $e^H=ca_6/a_3$

with a non-zero constant c.

If at least one of $d_i'd_j'$ $(i\neq j;\ i, j=4, 5, 6)$ is not zero, then by using lemma 1 to (2. 9) we have

$$d_4''a_4e^{L-2H}+d_5''a_5e^{-L}+d_6''a_6e^{-H}=0$$

with suitable constants d_i'', so that

(xxvi) $$d_4''a_4e^{2(L-H)}+d_6''a_6{}^{L-H}+d_5''a_5=0.$$

All the relations (i), $\cdots$, (xxvi) give contradictions. In fact, since we have

$$T(r, a_j)=o(m(r, e^H))=o(m(r, e^L)), \qquad j=1, \cdots, 9,$$

the cases (ii), (v), (vi), (x), (xi), (xiii), (xiv), (xv), (xvi), (xvii), (xx), (xxii), (xxiv) and (xxv) are all absurd. The cases (i), (iv), (viii), (xii), (xxi) and (xxiii) contradict that $m(r, e^H)\sim m(r, e^L)$. If (vii) or (xix) holds, we can put

$$e^H=ae^L, \qquad T(r, a)=o(m(r, e^L))$$

with a meromorphic function a, and hence the equation (2. 4) has the form

$$a_1a^2e^{4L}+(a_2+a_3a)ae^{3L}+(a_4+a_5a^2+a_6a)e^{2L}+(a_7+a_8a)e^L+a_9=0.$$

This is absurd by lemma 2. Similarly if (iii) or (ix) or (xviii) holds, by putting $e^H=ae^{-L}$, (2.4) reduces to the form

$$a_4e^{4L}+(a_2a+a_7)e^{3L}+(a_1a^2+a_6a+a_9)e^{2L}+(a_3a+a_8)ae^L+a_6a=0,$$

which is also a contradiction by lemma 1. In the remaining case (xxvi) we have

$$m(r, e^{L-H})=o(m(r, e^H)).$$

Then, by writing (2.4) in the form of

$$(a_1e^{2L-2H})e^{4H}+(a_2e^{2L-2H}+a_3e^{L-H})e^{3H}$$
$$+(a_4e^{2L-2H}+a_5+a_6e^{L-H})e^{2H}+(a_7e^{L-H}+a_8)e^H+a_9=0,$$

we can apply lemma 2. Then we arrive at a contradiction.

Thus we have completed our proof of this lemma.

Let $L(z)$ be an entire function. Then almost all zeros of $e^{L(z)}-\alpha$ are simple zeros (cf. [5]). Involving this fact we obtain:

LEMMA 4. *Under an assumption on the growth of an entire function* g

$$m(r, g)=o(m(r, e^L))$$

outside a set of finite logarithmic measure, we have

$$N_2(r; 0, e^L-g)\sim m(r, e^L)$$

and

$$N_1(r; 0, e^L-g)=o(m(r, e^L))$$

outside a set of finite logarithmic measure.

Proof. Let φ be a meromorphic function defined by

$$\frac{e^L-g}{-g}.$$

Then we have

$$N(r; \infty, \varphi)=N(r; 0, g)\leqq m(r, g)=o(m(r, e^L)),$$
$$N(r; 1, \varphi)=N(r; 0, e^L)=0,$$
$$N(r; \infty, \varphi')\leqq 2N(r; 0, g)=o(m(r, e^L)),$$
$$T(r, \varphi)\leqq T(r, e^L)+T(r, g)+T(r, 1/g)+O(1)\leqq m(r, e^L)+o(m(r, e^L))$$

and

$$m(r, e^L)\leqq m(r, e^L-g)+m(r, g)+O(1)\leqq T(r, \varphi)+o(m(r, e^L))$$

outside a set of finite logarithmic measure. By the second fundamental theorem for φ, we have

$$T(r, \varphi)\leqq N(r; 0, \varphi)+N(r; \infty, \varphi)+N(r; 1, \varphi)-N_1(r, \varphi)+O(\log (rT(r, \varphi)))$$

outside a set of finite logarithmic measure. Since

$$N_1(r, \varphi)=N(r; \infty, 1/\varphi')+2N(r; \infty, \varphi)-N(r; \infty, \varphi')$$
$$=N(r; 0, \varphi')+o(m(r, e^L)),$$

we have

$$T(r, \varphi)\leqq N(r; 0, \varphi)-N(r; 0, \varphi')+O(\log(rm(r, e^L)))+o(m(r, e^L))$$
$$\leqq \bar{N}(r; 0, \varphi)+o(m(r, e^L))$$
$$=N_2(r; 0, \varphi)+\bar{N}_1(r; 0, \varphi)+o(m(r, e^L))$$
$$=N_2(r; 0, e^L-g)+\bar{N}_1(r; 0, e^L-g)+o(m(r, e^L))$$

outside a set of finite logarithmic measure. On the other hand we have

$$N_2(r; 0, e^L-g)+N_1(r; 0, e^L-g)+\bar{N}_1(r; 0, e^L-g)$$
$$=N(r; 0, e^L-g)=N(r; 0, \varphi)\leqq T(r, \varphi)+O(1)$$
$$\leqq N_2(r; 0, e^L-g)+\bar{N}_1(r; 0, e^L-g)+o(m(r, e^L))$$

outside a set of finite logarithmic measure. Thus we obtain

$$\bar{N}_1(r; 0, e^L-g)\leqq N_1(r; 0, e^L-g)=o(m(r, e^L))$$

and

$$N_2(r; 0, e^L-g)=m(r, e^L)+o(m(r, e^L))$$

outside a set of finite logarithmic measure, since $T(r, \varphi)=m(r, e^L)+o(m(r, e^L))$. These imply the desired result.

§3. **Proof of Theorem B.** The sufficiency part is evident by theorem A. In fact, since $G(z)$ has no multiple zero, we have

$$\left\{\frac{F(z)}{f\circ h(z)}\right\}^2 G(z)=g\circ h(z)$$

where $F(z)/f\circ h(z)$ is an entire function.

In order to prove the necessity part, using again theorem A it suffices to consider an equation of the following form:

$$f^{*2}(e^{L\circ h}-\gamma')(e^{L\circ h}-\delta')=(e^H-\gamma)(e^H-\delta) \tag{3.1}$$

where f^* is a meromorphic function which has poles and zeros at most at the multiple zeros of $(e^{L\circ h}-\gamma')(e^{L\circ h}-\delta')$ and $(e^H-\gamma)(e^H-\delta)$, respectively. By lemma 4 we have

$$2m(r, e^{L\circ h})\sim N_2(r; 0, (e^{L\circ h}-\gamma')(e^{L\circ h}-\delta'))$$
$$\sim N_2(r; 0, (e^H-\gamma)(e^H-\delta))\sim 2m(r, e^H)$$

outside a set of finite logarithmic measure, so that

$$m(r, e^{L\circ h})\sim m(r, e^H). \tag{3.2}$$

Further we have

$$T(r,f^*)=O(T(r,e^{L\circ h})+T(r,e^H))$$

and

$$\begin{aligned}&N(r;0,f^*)+N(r;\infty,f^*)\\ &\leqq N_1(r;0,(e^{L\circ h}-\gamma')(e^{L\circ h}-\delta'))+N_1(r;0,(e^H-\gamma)(e^H-\delta))\\ &\leqq N(r;0,(L\circ h)')+N(r;0,H')\leqq m(r,(L\circ h)')+m(r,H')+O(1)\\ &\leqq m(r,L\circ h)+m(r,H)+O(\log(rm(r,L\circ h)m(r,H)))\\ &=o(m(r,e^{L\circ h})+m(r,e^H))\end{aligned}$$

outside a set of finite logarithmic measure. Thus we have

$$\begin{aligned}T(r,f^{*\prime}/f^*)&=m(r,f^{*\prime}/f^*)+N(r;\infty,f^{*\prime}/f^*)\\ &\leqq O(\log(rm(r,f^*)))+N(r;0,f^*)+N(r;\infty,f^*)\\ &=o(m(r,e^{L\circ h})+m(r,e^H))\end{aligned}\tag{3. 3}$$

outside a set of finite logarithic measure.

By differentiation of both sides of (3. 1), we have

$$f^{*2}\left[2\frac{f^{*\prime}}{f^*}(e^{2L\circ h}-(\gamma'+\delta')e^{L\circ h}+\gamma'\delta')+(L\circ h)'(2e^{2L\circ h}-(\gamma'+\delta')e^{L\circ h})\right]=H'(2e^{2H}-(\gamma+\delta)e^H),$$

and again by using (3. 1)

$$\left[2\frac{f^{*\prime}}{f^*}(e^{2L\circ h}-(\gamma'+\delta')e^{L\circ h}+\gamma'\delta')+(L\circ h)'(2e^{2L\circ h}-(\gamma'+\delta')e^{L\circ h})\right][e^{2H}-(\gamma+\delta)e^H+\gamma\delta]$$
$$=H'(2e^{2H}-(\gamma+\delta)e^H)(e^{2L\circ h}-(\gamma'+\delta')e^{L\circ h}+\gamma'\delta'),$$

so that we have

$$\begin{aligned}&a_1e^{2L\circ h+2H}+(\gamma+\delta)a_2e^{2L\circ h+H}+(\gamma'+\delta')a_3e^{L\circ h+2H}+a_4e^{2L\circ h}+a_5e^{2H}\\ &\quad+(\gamma+\delta)(\gamma'+\delta')a_6e^{L\circ h+H}+(\gamma'+\delta')a_7e^{L\circ h}+(\gamma+\delta)a_8e^H+a_9=0\end{aligned}\tag{3. 4}$$

where

$$a_1=2(f^{*\prime}/f^*+(L\circ h)'-H'),\qquad a_2=-(2f^{*\prime}/f^*+2(L\circ h)'-H'),$$
$$a_3=-(2f^{*\prime}/f^*+(L\circ h)'-2H'),\qquad a_4=2\gamma\delta(f^{*\prime}/f^*+(L\circ h)'),$$
$$a_5=2\gamma'\delta'(f^{*\prime}/f^*-H'),\qquad a_6=2f^{*\prime}/f^*+(L\circ h)'-H',$$
$$a_7=-\gamma\delta(2f^{*\prime}/f^*+(L\circ h)'),\qquad a_8=-\gamma'\delta'(2f^{*\prime}/f^*-H')$$

and

$$a_9=2\gamma\delta\gamma'\delta'f^{*\prime}/f^*.$$

By (3. 2) and (3. 3) we can apply lemma 3 to (3. 4). Thus we can deduce that

$$(\gamma+\delta)(\gamma'+\delta')a_1a_2a_3a_4a_5a_6a_7a_8$$

vanishes identically.

If $a_1\equiv 0$, then we have

$$f^*=ce^{-L\circ h+H}$$

with a non-zero constant c, and (3. 1) reduces to the equation

$$(c^2-1)e^{2L\circ h+2H}+(\gamma+\delta)e^{2L\circ h+H}-c^2(\gamma'+\delta')e^{L\circ h+2H}-\gamma\delta e^{2L\circ h}+c^2\gamma'\delta' e^{2H}=0.$$

Hence by Borel's theorem [1] this is impossible unless $L\circ h(z)-L\circ h(0)=H(z)$, which is the desired result (a) in our theorem.

If $a_2\equiv 0$, then we have

$$f^{*2}=ce^{-2L\circ h+H}$$

with a non-zero constant c, and (3. 1) reduces to the equation

$$e^{2L\circ h+2H}-(\gamma+\delta+c)e^{2L\circ h+H}+\gamma\delta e^{2L\circ h}+c(\gamma'+\delta')e^{L\circ h+H}-c\gamma'\delta' e^{H}=0.$$

However this is a contradiction by Borel's theorem. Similarly if $a_3\equiv 0$, we arrive at a contradiction.

If $a_4\equiv 0$, then we have

$$f^{*}=ce^{-L\circ h}$$

with a non-zero constant c, and (3. 1) reduces to the equation

$$e^{2L\circ h+2H}-(\gamma+\delta)e^{2L\circ h+H}+(\gamma\delta-c^2)e^{2L\circ h}+c^2(\gamma'+\delta')e^{L\circ h}-c^2=0.$$

Hence by Borel's theorem we have the desired result (b) in our theorem. If $a_5\equiv 0$, then we have similarly the desired result (b) in our theorem.

If $a_6\equiv 0$, then we have

$$f^{*2}=ce^{-L\circ h+H}$$

with a non-zero constant c, and (3. 1) reduces to the equation

$$ce^{2L\circ h+H}-e^{L\circ h+2H}+((\gamma+\delta)-c(\gamma'+\delta'))e^{L\circ h+H}-\gamma\delta e^{L\circ h}+c\gamma'\delta' e^{H}=0.$$

Hence by Borel's theorem we have the result (a) and the result (b) in our theorem according as $L\circ h-H\equiv$const. and $L\circ h+H\equiv$const., respectively.

If $a_7\equiv 0$, then we have

$$f^{*2}=ce^{-L\circ h}$$

with a non-zero constant c, and (3. 1) reduces to the equation

$$ce^{2L\circ h}-e^{L\circ h+2H}+(\gamma+\delta)e^{L\circ h+2H}-(\gamma\delta+c(\gamma'+\delta'))e^{L\circ h}+c\gamma'\delta'=0.$$

However this is a contradiction by Borel's theorem. Similarly if $a_8\equiv 0$, we arrive at a contradiction.

If $\gamma+\delta=0$, then we have

(3. 5) $$a_1e^{2L\circ h+2H}+(\gamma'+\delta')a_3e^{L\circ h+2H}+a_4e^{2L\circ h}+a_5e^{2H}+(\gamma'+\delta')a_7e^{L\circ h}+a_9=0.$$

We may assume that $a_1a_3a_4a_5a_7\not\equiv 0$. Then by using lemma 1 to (3. 5) we have

$$c_1a_1e^{2L\circ h+2H}+c_3a_3e^{L\circ h+2H}+c_4a_4e^{2L\circ h}+c_5a_5e^{2H}+c_7a_7e^{L\circ h}=0$$

with suitable constants c_i, so that

(3. 6) $$c_1a_1e^{L\circ h+2H}+c_3a_3e^{2H}+c_4a_4e^{L\circ h}+c_5a_5e^{-L\circ h+2H}+c_7a_7=0.$$

If all c_ic_j $(i\neq j; i, j=1, 3, 4, 5)$ are zero, then (3. 6) reduces to the one of the following identities:

(i) $c_1a_1e^{L\circ h+2H}+c_7a_7=0,$ i.e. $e^{L\circ h}=c(a_7/a_1)e^{-2H};$

(ii) $c_3a_3e^{2H}+c_7a_7=0,$ i.e. $e^{H}=c\sqrt{a_7/a_3};$

(iii) $c_4a_4e^{L\circ h}+c_7a_7=0,$ i.e. $e^{L\circ h}=ca_7/a_4;$

(iv) $c_5a_5e^{-L\circ h+2H}+c_7a_7=0,$ i.e. $e^{L\circ h}=c(a_5/a_7)e^{2H}$

with a non-zero constant c.

If at least one of c_ic_j $(i\neq j; i, j=1, 3, 4, 5)$ is not zero, then by using lemma 1 to (3. 6) we have

$$c_1'a_1e^{L\circ h+2H}+c_3'a_3e^{2H}+c_4'a_4e^{L\circ h}+c_5'a_5e^{-L\circ h+2H}=0$$

with suitable constants c_i', so that

(3. 7) $$c_1'a_1e^{2H}+c_3'a_3e^{-L\circ h+2H}+c_5'a_5e^{-2L\circ h+2H}+c_4'a_4=0.$$

If all $c_i'c_j'$ $(i\neq j; i, j=1, 3, 5)$ are zero, then (3. 7) reduces to the one of the following identities:

(v) $c_1'a_1e^{2H}+c_4'a_4=0,$ i.e. $e^{H}=c\sqrt{a_4/a_1};$

(vi) $c_3'a_3e^{-L\circ h+2H}+c_4'a_4=0,$ i.e. $e^{L\circ h}=c(a_3/a_4)e^{2H};$

(vii) $c_5'a_5e^{-2L\circ h+2H}+c_4'a_4=0,$ i.e. $e^{L\circ h}=c\sqrt{a_5/a_4}\,e^{H}$

with a non-zero constant c.

If at least one of $c_i'c_j'$ $(i\neq j; i, j=1, 3, 5)$ is not zero, then by using lemma 1 to (3. 7) we have

$$c_1''a_1e^{2H}+c_3''a_3e^{-L\circ h+2H}+c_5''a_5e^{-2L\circ h+2H}=0$$

with suitable constants c_i'', so that

(viii) $$c_1''a_1e^{2L\circ h}+c_3''a_3e^{L\circ h}+c_5''a_5=0.$$

These relations (i), ⋯, (viii) lead to a contradiction. Indeed, the cases (ii), (iii) and (v) are evidently untenable. The cases (i), (iv) and (vi) contradict (3. 2). If the relation (vii) holds, then (3. 5) reduces to the identity

$$a_1a_4e^{4L\circ h}+(\gamma'+\delta')a_3a_4e^{3L\circ h}+(1+c)a_4a_5e^{2L\circ h}+c(\gamma'+\delta')a_5a_7e^{L\circ h}+ca_5a_9=0.$$

However this is a contradiction by lemma 2. The case (viii) also contradicts lemma 2.

If $\gamma'+\delta'=0$, then we have similarly a contradiction unless $a_1a_2a_4a_5a_8\equiv0$.

Thus we have completed our proof.

§ 4. **Proof of Theorem C.** In order to prove theorem C, it suffices to show the impossibility of an identity of the form

(4. 1) $$f^{*2}(e^{H}-\gamma)(e^{H}-\delta)=\tilde{G};$$

$$\tilde{G}=1-2\beta_1e^{H_1}-2\beta_2e^{H_2}+\beta_1^2e^{2H_1}-2\beta_1\beta_2e^{H_1+H_2}+\beta_2^2e^{2H_2},$$

$$\beta_1\beta_2\gamma\delta(\gamma-\delta)\neq0,\qquad H(0)=H_1(0)=H_2(0)=0$$

provided that

$$m(r, e^{H_1}) = o(m(r, e^{H_2})) \tag{4. 2}$$

outside a set of finite logarithmic measure, where H, H_1 and H_2 are non-constant entire functions and γ, δ, β_1 and β_2 are constants and f^* is a meromorphic function which has poles and zeros at most at the multiple zeros of $(e^H-\gamma)(e^H-\delta)$ and $\tilde{G}$, respectively.

We put

$$\tilde{G} = F_1 \cdot F_2 \cdot F_3 \cdot F_4;$$

$$F_1 = 1 + \sqrt{\beta_1} e^{H_1/2} + \sqrt{\beta_2} e^{H_2/2}, \qquad F_2 = 1 + \sqrt{\beta_1} e^{H_1/2} - \sqrt{\beta_2} e^{H_2/2},$$

$$F_3 = 1 - \sqrt{\beta_1} e^{H_1/2} + \sqrt{\beta_2} e^{H_2/2}, \qquad F_4 = 1 - \sqrt{\beta_1} e^{H_1/2} - \sqrt{\beta_2} e^{H_2/2}.$$

Since F_i and F_j $(i \neq j; i, j = 1, 2, 3, 4)$ have no common zero and

$$m(r, e^{H_1/2}) = o(m(r, e^{H_2/2}))$$

outside a set of finite logarithmic measure, by lemma 4 we have

$$N_2(r; 0, \tilde{G}) = \sum_{i=1}^{4} N_2(r; 0, F_i) \sim 4m(r, e^{H_2/2}) \sim 2m(r, e^{H_2})$$

and

$$N_1(r; 0, \tilde{G}) = \sum_{i=1}^{4} N_1(r; 0, F_i) = o(m(r, e^{H_2}))$$

outside a set of finite logarithmic measure. Thus we have

$$2m(r, e^H) \sim N_2(r; 0, (e^H-\gamma)(e^H-\delta)) \sim N_2(r; 0, \tilde{G}) \sim 2m(r, e^{H_2})$$

outside a set of finite logarithmic measure, so that

$$m(r, e^H) \sim m(r, e^{H_2}). \tag{4. 3}$$

Further clearly we have

$$T(r, f^*) = O(T(r, e^H) + T(r, e^{H_1}) + T(r, e^{H_2})).$$

Thus we have

$$\begin{aligned} T(r, f^{*\prime}/f^*) &= m(r, f^{*\prime}/f^*) + N(r; \infty, f^{*\prime}/f^*) \\ &\leq O(\log (rm(r, f^*))) + N(r; 0, f^*) + N(r; \infty, f^*) \\ &= o(m(r, e^H) + m(r, e^{H_2})) \end{aligned}$$

outside a set of finite logarithmic measure, so that

$$T(r, f^{*\prime}/f^*) = o(m(r, e^H)) = o(m(r, e^{H_2})). \tag{4. 4}$$

By differentiation of both sides of the equation (4. 1) we have

$$f^{*2}\left[2\frac{f^{*\prime}}{f^*}(e^{2H}-(\gamma+\delta)e^H+\gamma\delta) + H'(2e^{2H}-(\gamma+\delta)e^H)\right] = \tilde{G}'$$

$$\equiv -2\beta_1 H_1' e^{H_1} - 2\beta_2 H_2' e^{H_2} + 2\beta_1^2 H_1' e^{2H_1} - 2\beta_1\beta_2 (H_1'+H_2') e^{H_1+H_2} + 2\beta_2^2 H_2' e^{2H_2},$$

and, using (4. 1), we finally have

$$a_1e^{2H+2H_2}+a_2e^{2H+H_2}+(\gamma+\delta)a_3e^{H+2H_2}+a_4e^{2H}+a_5e^{2H_2} +(\gamma+\delta)a_6e^{H+H_2}+(\gamma+\delta)a_7e^H+a_8e^{H_2}+a_9=0, \tag{4. 5}$$

where

$$a_1=2\beta_2^2(f^{*\prime}/f^*+H'-H_2'),$$
$$a_2=-2\beta_2[(2f^{*\prime}/f^*+2H'-H_2')(\beta_1e^{H_1}+1)-\beta_1H_1'e^{H_1}],$$
$$a_3=-\beta_2^2(2f^{*\prime}/f^*+H'-2H_2'),$$
$$a_4=2(f^{*\prime}/f^*+H')(\beta_1^2e^{2H_1}-2\beta_1e^{H_1}+1)-2H_1'(\beta_1^2e^{2H_1}-\beta_1e^{H_1}),$$
$$a_5=2\beta_2^2\gamma\delta(f^{*\prime}/f^*-H_2'),$$
$$a_6=2\beta_2[(2f^{*\prime}/f^*+H'-H_2')(\beta_1e^{H_1}+1)-\beta_1H_1'e^{H_1}],$$
$$a_7=-(2f^{*\prime}/f^*+H')(\beta_1^2e^{2H_1}-2\beta_1e^{H_1}+1)+2H_1'(\beta_1^2e^{2H_1}-\beta_1e^{H_1}),$$
$$a_8=-2\beta_2\gamma\delta[(2f^{*\prime}/f^*-H_2')(\beta_1e^{H_1}+1)-\beta_1H_1'e^{H_1}],$$
$$a_9=2\gamma\delta[(f^{*\prime}/f^*)(\beta_1^2e^{2H_1}-2\beta_1e^{H_1}+1)-H_1'(\beta_1^2e^{2H_1}-\beta_1e^{H_1})].$$

If $(\gamma+\delta)a_1a_2a_3a_4a_5a_6a_7a_8a_9\not\equiv 0$, then by (4. 2), (4. 3) and (4. 4) the identity (4. 5) contradicts lemma 3.

If $a_1\equiv 0$, then we have

$$f^*=ce^{-H+H_2}$$

with a non-zero constant c, and (4. 1) reduces to the equation

$$c^2e^{2H_2}-c^2(\gamma+\delta)e^{-H+2H_2}+c^2\gamma\delta e^{-2H+2H_2}=\tilde{G},$$

which contradicts Borel's theorem.

If $a_2\equiv 0$, then we have

$$f^{*2}=c(\beta_1e^{H_1}+1)e^{-2H+H_2}$$

with a non-zero constant c. However this equation is absurd, since the right hand term has simple zeros.

If $a_3\equiv 0$, then we have

$$f^{*2}=ce^{-H+H_2}$$

with a non-zero constant c, and (4. 1) reduces to the equation

$$ce^{H+H_2}-c(\gamma+\delta)e^{H_2}+c\gamma\delta e^{-H+H_2}=\tilde{G},$$

which contradicts Borel's theorem.

If $a_4\equiv 0$, then we have

$$f^*=c(\beta_1e^{H_1}-1)e^{-H}$$

with a non-zero constant c. Then (4. 1) reduces to the equation

$$c^2(\beta_1e^{H_1}-1)^2(e^H-\gamma)(e^H-\delta)=e^{2H}\tilde{G}.$$

However by Borel's theorem we arrive at a contradiction.

If $a_5 \equiv 0$, then we have

$$f^* = ce^{H_2}$$

with a non-zero constant c. Then (4. 1) reduces to the equation

$$c^2 e^{2H+2H_2} - c^2(\gamma+\delta)e^{H+2H_2} + c^2\gamma\delta e^{2H_2} = \tilde{G}.$$

This is also untenable by Borel's theorem.

If $a_6 \equiv 0$, then we have

$$f^{*2} = c(\beta_1 e^{H_1}+1)e^{-H+H_2}$$

with a non-zero constant c. However this equation is absurd, since the right hand term has simple zeros.

If $a_7 \equiv 0$, then we have

$$f^{*2} = c(\beta_1 e^{H_1}-1)^2 e^{-H}$$

with a non-zero constant c. Then (4. 1) reduces to the equation

$$c(\beta_1 e^{H_1}-1)^2(e^H-\gamma)(e^H-\delta) = e^H\tilde{G}.$$

However by Borel's theorem we arrive at a contradiction.

If $a_8 \equiv 0$, then we have

$$f^{*2} = c(\beta_1 e^{H_1}+1)e^{H_2}$$

with a non-zero constant c. However this equation is absurd, since the right hand term has simple zeros.

If $a_9 \equiv 0$, then we have

$$f^* = c(\beta_1 e^{H_1}-1)$$

with a non-zero constant c. Then (4. 1) reduces to the equation

$$c^2(\beta_1 e^{H_1}-1)^2(e^H-\gamma)(e^H-\delta) = \tilde{G}.$$

However by Borel's theorem we arrive at a contradiction.

If $\gamma+\delta=0$, then (4. 5) reduces to the equation

$$a_1 e^{2H+2H_2} + a_2 e^{2H+H_2} + a_4 e^{2H} + a_5 e^{2H_2} + a_8 e^{H_2} + a_9 = 0.$$

In this case we may assume that $a_1a_2a_4a_5a_8a_9 \neq 0$. However we arrive at a contradiction by a similar argument as in (3. 5).

Thus we have the desired result.

§5. Let R be an ultrahyperelliptic surface defined by an equation

$$y^2 = G(x),$$

where

$$G(x) = 1 - 2\beta_1 e^{\alpha_1 x} - 2\beta_2 e^{\alpha_2 x} + \beta_1^2 e^{2\alpha_1 x} - 2\beta_1\beta_2 e^{(\alpha_1+\alpha_2)x} + \beta_2^2 e^{2\alpha_2 x},$$

$$\beta_1\beta_2\alpha_1\alpha_2 \neq 0.$$

If R satisfies $P(R)=4$, then by the argument explained in § 1 we have

$$g(z)^2G(z)=f(z)^2(e^{H(z)}-\gamma)(e^{H(z)}-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0, \qquad H(0)=0$$

for suitable entire functions g, f and H. Here we may assume that $g(z)$ and $f(z)$ have no common zero. By the lemma given in [5], we have

$$N_2(r; 0, e^H-\gamma)\sim N_2(r; 0, e^H-\delta)\sim m(r, e^H)$$

outside a set of finite logarithmic measure. Since all simple zeros of $(e^H-\gamma)(e^H-\delta)$ are the zeros of $G(z)$, we have

$$2m(r, e^H)\sim N_2(r; 0, (e^H-\gamma)(e^H-\delta))\leqq N_2(r; 0, G)\leqq m(r, G)$$

outside a set of finite logarithmic measure. If H is a transcendental entire function or a polynomial of degree greater than one, then we have

$$\rho_G=\overline{\lim_{r\to\infty}}\frac{\log m(r, G)}{\log r}\geqq\overline{\lim_{r\to\infty}}\frac{\log m(r, e^H)}{\log r}\geqq 2,$$

which is absurd, since $\rho_G=1$. Thus H must have the form αz. Then we have

$$f(z)^2(e^{\alpha z}-\gamma)(e^{\alpha z}-\delta)=g(z)^2G(z).$$

Let z_n be

$$\frac{1}{\alpha}\log\gamma+\frac{1}{\alpha}2n\pi i, \qquad n=0, \pm 1, \cdots,$$

then these are simple zeros of $e^{\alpha z}-\gamma$. Therefore $G(z_n)=0$. Let

$$u=e^{(\alpha_1/\alpha)2\pi i}, \qquad v=e^{(\alpha_2/\alpha)2\pi i}, \qquad A=\beta_1 e^{(\alpha_1/\alpha)\log\gamma}=\beta_1\gamma^{(\alpha_1/\alpha)}, \qquad B=\beta_2 e^{(\alpha_2/\alpha)\log\gamma}=\beta_2\gamma^{(\alpha_2/\alpha)}.$$

Then for all integers n we have

$$0=G(z_n)=1-2Au^n-2Bv^n+A^2u^{2n}-2ABu^nv^n+B^2v^{2n}.$$

By the lemma given in [4] we have

$$u=1 \quad\text{and}\quad v=1.$$

This implies that

$$\alpha_1=p_1\alpha \quad\text{and}\quad \alpha_2=p_2\alpha$$

for some suitable non-zero integers p_1 and p_2.

Putting $e^{\alpha z/2}=\chi$, we have

$$G(z)=F(e^{\alpha z/2});$$
$$F(\chi)=1-2\beta_1\chi^{2p_1}-2\beta_2\chi^{2p_2}+\beta_1^2\chi^{4p_1}-2\beta_1\beta_2\chi^{2(p_1+p_2)}+\beta_1^2\chi^{4p_2}.$$

Since $e^{\alpha z/2}-\chi_0$, $\chi_0\neq 0$ has no zero other than an infinite number of simple zeros and $e^{\alpha z/2}$ has no zero, every multiple zero of $G(z)$ occurs at a suitable multiple zero of $F(\chi)$ and vice versa. Thus $F(\chi)$ has only four simple zeros $\sqrt{\gamma}$, $-\sqrt{\gamma}$, $\sqrt{\delta}$ and $-\sqrt{\delta}$.

In the first place we assume that $0<p_1<p_2$. Evidently we have

$$F(\chi)=F_1(\chi)\cdot F_2(\chi)\cdot F_3(\chi)\cdot F_4(\chi);$$

$$F_1(\chi)=1-\sqrt{\beta_1}\chi^{p_1}-\sqrt{\beta_2}\chi^{p_2},\qquad F_2(\chi)=1-\sqrt{\beta_1}\chi^{p_1}+\sqrt{\beta_2}\chi^{p_2},$$
$$F_3(\chi)=1+\sqrt{\beta_1}\chi^{p_1}-\sqrt{\beta_2}\chi^{p_2},\qquad F_4(\chi)=1+\sqrt{\beta_1}\chi^{p_1}+\sqrt{\beta_2}\chi^{p_2}.$$

Since no two members of F_1, F_2, F_3 and F_4 have common zero, we may seek for all the multiple zeros of each function $F_j(j=1,2,3,4)$. Then, since there is no triple zero in each factor F_j, every multiple zero is a double zero. From the equations

$$\begin{cases}F_1(\chi)=0,\\ F_1'(\chi)=0,\end{cases}\quad \begin{cases}F_2(\chi)=0,\\ F_2'(\chi)=0,\end{cases}\quad \begin{cases}F_3(\chi)=0,\\ F_3'(\chi)=0,\end{cases}\quad \begin{cases}F_4(\chi)=0,\\ F_4'(\chi)=0,\end{cases}$$

we have

$$\begin{cases}\chi^{p_1}=X,\\ \chi^{p_2}=Y,\end{cases}\quad \begin{cases}\chi^{p_1}=X,\\ \chi^{p_2}=-Y,\end{cases}\quad \begin{cases}\chi^{p_1}=-X,\\ \chi^{p_2}=Y,\end{cases}\quad \begin{cases}\chi^{p_1}=-X,\\ \chi^{p_2}=-Y,\end{cases}$$

respectively, where $X=p_2/(p_2-p_1)\sqrt{\beta_1}$ and $Y=p_1/(p_1-p_2)\sqrt{\beta_2}$. Thus every double zero is a common point between p_1-th roots of X and p_2-th roots of Y or that of X and of $-Y$ or that of $-X$ and of Y or that of $-X$ and of $-Y$, respectively. Let $E(l,p)$ be the set of $|l|^{1/p}e^{(\arg l)/p+2n\pi i/p}$, $n=0,1,\cdots,p-1$. If $E(X,p_1)\cap E(Y,p_2)\neq\phi$, then there are d common points of $E(X,p_1)$ and $E(Y,p_2)$, where d is the greatest common measure of p_1 and p_2.

If there is no double zero in $F(\chi)$, then we have $4p_2=4$, that is

$$0<p_1<p_2=1.$$

This is untenable. Therefore we may without loss of generality assume that $E(X,p_1)\cap E(Y,p_2)\neq\phi$.

If $E(-X,p_1)\cap E(Y,p_2)=\phi$ and $E(X,p_1)\cap E(-Y,p_2)=\phi$ and $E(-X,p_1)\cap E(-Y,p_2)=\phi$, then we have $4p_2-2d=4$, that is,

$$2p_1<2p_2=2+d\leqq 2+p_1.$$

This implies that

$$p_1=d=1.$$

Thus we have $2p_2=3$. This is untenable.

If $E(-X,p_1)\cap E(Y,p_2)\neq\phi$ but $E(X,p_1)\cap E(-Y,p_2)=\phi$ and $E(-X,p_1)\cap E(-Y,p_2)=\phi$, then $E(-X,p_1)\cap E(Y,p_2)$ contains just d points and hence we have

$$4p_2-4d=4,\qquad \text{i.e.}\qquad p_2=1+d.$$

Therefore we have

$$p_1<p_2=1+d\leqq 1+p_1.$$

Thus we have $d=1$, $p_1=1$ and $p_2=2$. Then $\beta_1^2=16\beta_2$ holds.

If further $E(X,p_1)\cap E(-Y,p_2)\neq\phi$, then $E(-X,p_1)\cap E(-Y,p_2)\neq\phi$ and these two sets contain just d points, respectively. Thus we have

$$2d\leqq p_1\qquad\text{and}\qquad 4p_2-8d=4.$$

And hence we have

$$p_1<p_2=1+2d\leqq 1+p_1.$$

This implies that $d=1$, $p_1=2$ and $p_2=3$. This is untenable, since $E(-X,2)\cap E(-Y,3)=\phi$.

Next we assume that $p_1<0<p_2$. Then, putting $p_1=-q_1$, we get

$$F(\chi)\frac{\chi^{4q_1}}{\beta_1^2}=1-2\frac{\beta_2}{\beta_1}\chi^{2(p_2+q_1)}-2\frac{1}{\beta_1}\chi^{2q_1}+\frac{\beta_2^2}{\beta_1^2}\chi^{4(p_2+q_1)}-2\frac{\beta_2}{\beta_1^2}\chi^{2p_2+4q_1}+\frac{1}{\beta_1^2}\chi^{4q_1}.$$

Since $0<q_1<p_2+q_1$, we can make use of the above result. Then we have

$$p_2+q_1=2q_1=2.$$

This implies that $p_2=q_1=1$ and hence $p_1=-1$, $p_2=1$ and $16\beta_1\beta_2=1$.

If $p_2<p_1<0$, we put $p_1=-q_1$ and $p_2=-q_2$. Then we have

$$F(\chi)\frac{\chi^{4q_2}}{\beta_2^2}=1-2\frac{1}{\beta_2}\chi^{2q_2}-2\frac{\beta_1}{\beta_2}\chi^{2(q_2-q_1)}+\frac{1}{\beta_2^2}\chi^{4q_2}-2\frac{\beta_1}{\beta_2^2}\chi^{4q_2-2q_1}+\frac{\beta_1^2}{\beta_2^2}\chi^{4(q_2-q_1)}.$$

Since $0<q_2-q_1<q_2$, we can again apply the above fact. Then we have

$$q_2-q_1=1,\qquad q_2=2\quad\text{and}\quad \beta_1^2=16\beta_2.$$

And hence we obtain

$$p_1=-1,\qquad p_2=-2\quad\text{and}\quad \beta_1^2=16\beta_2.$$

In the last case we assume that $p_1=p_2$. Then we have

$$\begin{aligned}G(z)&=1-2\beta_1e^{\alpha p_1z}-2\beta_2e^{\alpha p_1z}+\beta_1^2e^{2\alpha p_1z}-2\beta_1\beta_2e^{2\alpha p_1z}+\beta_2^2e^{2\alpha p_1z}\\&=1-2(\beta_1+\beta_2)e^{\alpha p_1z}+(\beta_1-\beta_2)^2e^{2\alpha p_1z}\\&=(1-Me^{\alpha p_1z})(1-Ne^{\alpha p_1z})\end{aligned}$$

where $M=(\sqrt{\beta_1}+\sqrt{\beta_2})^2$ and $N=(\sqrt{\beta_1}-\sqrt{\beta_2})^2$. This implies that

$$g(z)^2G(z)=g(z)^2MN\left(e^{\alpha p_1z}-\frac{1}{M}\right)\left(e^{\alpha p_1z}-\frac{1}{N}\right),\qquad MN(M-N)\neq 0,$$

if we assume that $\beta_1\neq\beta_2$. Further $p_1=1$ holds. If $\beta_1=\beta_2$, then either M or N is equal to zero but one of them does not vanish. Thus $G(z)$ has a form $A(e^{\alpha p_1z}-1/A)$. This implies that $p_1=2$.

Thus we have all the possible cases for which $P(R)=4$, which can be listed as follows:

(1) $p_1=2,\ p_2=1,\ \beta_2^2=16\beta_1$; (2) $p_1=1,\ p_2=2,\ \beta_1^2=16\beta_2$;

(3) $p_1=1,\ p_2=-1,\ 16\beta_1\beta_2=1$; (4) $p_1=-1,\ p_2=1,\ 16\beta_1\beta_2=1$;

(5) $p_1=-2,\ p_2=-1,\ \beta_2^2=16\beta_1$; (6) $p_1=-1,\ p_2=-2,\ \beta_1^2=16\beta_2$;

(7) $p_1=p_2=1$, β_1,β_2 are free but $\beta_1\neq\beta_2$; (8) $p_1=p_2=2$ and $\beta_1=\beta_2$.

Evidently in the first six cases we have

$$G(z)=f(z)^2(e^{\alpha z}-\gamma)(e^{\alpha z}-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0$$

with a suitable entire function f and two suitable constants γ and δ. Therefore $P(R)$ is equal to 4 for all eight cases.

Summing up these results, we have

THEOREM **D.** *Let R be an ultrahyperelliptic surface defined by an equation*

$$y^2=1-2\beta_1 e^{\alpha_1 x}-2\beta_2 e^{\alpha_2 x}+\beta_1^2 e^{2\alpha_1 x}-2\beta_1\beta_2 e^{(\alpha_1+\alpha_2)x}+\beta_2^2 e^{2\alpha_2 x},$$

with $\beta_1\beta_2\alpha_1\alpha_2\neq 0$. Then $P(R)$ is equal to 3 excepting the following four cases:

(1) $\alpha_1=2\alpha_2$, $\beta_2^2=16\beta_1$; (2) $\alpha_2=2\alpha_1$, $\beta_1^2=16\beta_2$;

(3) $\alpha_1=-\alpha_2$, $16\beta_1\beta_2=1$; (4) $\alpha_1=\alpha_2$, β_1, β_2 *are free*

for which we have $P(R)=4$.

§ **6.** Let R be an ultrahyperelliptic surface defined by an equation

$$y^2=G(x);$$

$$G(x)=1-2\beta_1 e^{\alpha_1 x^2+\gamma_1 x}-2\beta_2 e^{\alpha_2 x^2+\gamma_2 x}+\beta_1^2 e^{2\alpha_1 x^2+2\gamma_1 x}$$
$$-2\beta_1\beta_2 e^{(\alpha_1+\alpha_2)x^2+(\gamma_1+\gamma_2)x}+\beta_2^2 e^{2\alpha_2 x^2+2\gamma_2 x}, \qquad \beta_1\beta_2(|\alpha_1|+|\alpha_2|)\neq 0.$$

Then it is evident that R satisfies $P(R)\geqq 3$.

Suppose that $P(R)=4$. By the argument explained in § 1, we have

$$g(z)^2G(z)=f(z)^2(e^{H(z)}-\gamma)(e^{H(z)}-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0 \tag{6. 1}$$

with suitable entire functions g, f and H. Further by the same argument as in § 5, $H(z)$ has the form $\alpha z^2+\beta z$, because

$$\rho_G=\overline{\lim_{r\to\infty}}\frac{\log m(r, G)}{\log r}=2.$$

Then we have two possibilities $\alpha\neq 0$ and $\alpha=0$.

I. Case of $\alpha\neq 0$. Replacing z by $z-\beta/2\alpha$, (6. 1) reduces to an equation

$$g(z-\beta/2\alpha)^2\tilde{G}(z)=\tilde{f}(z)^2(e^{\alpha z^2}-\tilde{\gamma}))e^{\alpha z^2}-\tilde{\delta}); \tag{6. 2}$$

$$\tilde{G}(z)=1-2\tilde{\beta}_1 e^{\alpha_1 z^2+\tilde{\gamma}_1 z}-2\tilde{\beta}_2 e^{\alpha_2 z^2+\tilde{\gamma}_2 z}+\tilde{\beta}_1^2 e^{2\alpha_1 z^2+2\tilde{\gamma}_1 z}-2\tilde{\beta}_1\tilde{\beta}_2 e^{(\alpha_1+\alpha_2)z^2+(\tilde{\gamma}_1+\tilde{\gamma}_2)z}+\tilde{\beta}_2^2 e^{2\alpha_2 z^2+2\tilde{\gamma}_2 z},$$

$$\tilde{f}(z)=e^{\beta^2/4\alpha}f(z-\beta^2/2\alpha), \qquad \tilde{\gamma}=\gamma/e^{\beta^2/4\alpha}, \qquad \tilde{\delta}=\delta/e^{\beta^2/4\alpha},$$

$$\tilde{\beta}_1=\beta_1 e^{\alpha_1\beta^2/4\alpha^2-\gamma_1\beta/2\alpha}, \qquad \tilde{\beta}_2=\beta_2 e^{\alpha_2\beta^2/4\alpha^2-\gamma_2\beta/2\alpha},$$

$$\tilde{\gamma}_1=-\alpha_1\beta/\alpha+\gamma_1, \qquad \tilde{\gamma}_2=-\alpha_2\beta/\alpha+\gamma_2.$$

Let z_n and $\tilde{z}_n$; $n=0, \pm 1, \cdots$ be

$$\left(\frac{1}{\alpha}\log\tilde{\gamma}+\frac{1}{\alpha}2n\pi i\right)^{1/2} \quad \text{and} \quad -\left(\frac{1}{\alpha}\log\tilde{\gamma}+\frac{1}{\alpha}2n\pi i\right)^{1/2}$$

respectively. Then these are simple zeros of $e^{\alpha z^2}-\tilde{\gamma}$. Therefore $\tilde{G}(z_n)=0$ and $\tilde{G}(\tilde{z}_n)=0$.

Putting

$$X_0=e^{2\alpha_1\pi i/\alpha}, \qquad Y_0=e^{2\alpha_2\pi i/\alpha}, \qquad A=\tilde{\beta}_1 e^{\alpha_1 \log\tilde{\gamma}/\alpha}, \qquad B=\tilde{\beta}_2 e^{\alpha_2 \log\tilde{\gamma}/\alpha},$$

we have

$$F_n\equiv 1-2AX_0^n e^{\tilde{\gamma}_1 z_n}-2BY_0^n e^{\tilde{\gamma}_2 z_n}+A^2X_0^{2n}e^{2\tilde{\gamma}_1 z_n}$$
$$-2ABX_0^n Y_0^n e^{(\tilde{\gamma}_1+\tilde{\gamma}_2)z_n}+B^2Y_0^{2n}e^{2\tilde{\gamma}_2 z_n}=0$$

and

$$\tilde{F}_n\equiv 1-2AX_0^n e^{\tilde{\gamma}_1 \tilde{z}_n}-2BY_0^n e^{\tilde{\gamma}_2 \tilde{z}_n}+A^2X_0^{2n}e^{2\tilde{\gamma}_1 \tilde{z}_n}$$
$$-2ABX_0^n Y_0^n e^{(\tilde{\gamma}_1+\tilde{\gamma}_2)\tilde{z}_n}+B^2Y_0^{2n}e^{2\tilde{\gamma}_2 \tilde{z}_n}=0$$

for every integer n.

We use that if a constant χ satisfies $|\chi|<1$, then we have

$$\chi^n e^{czn}\to 0 \qquad \text{for } n\to+\infty$$

and

$$\chi^n e^{czn}\to\infty \qquad \text{for } n\to-\infty$$

with an arbitrary constant c.

At first we conclude that

$$|X_0|=1 \quad\text{and}\quad |Y_0|=1.$$

In fact, if $|X_0|<1$ and $|Y_0|<1$, then

$$\lim_{n\to\infty} F_n=1;$$

if $|X_0|>1$ and $|Y_0|>1$, then

$$\lim_{n\to-\infty} F_n=1;$$

and if $|X_0|>1$ and $|Y_0|\leqq 1$, or if $|X_0|\leqq 1$ and $|Y_0|>1$, then

$$\lim_{n\to\infty} F_n=\infty.$$

These all results lead to a contradiction, because $F_n=0$.

Next we show that

$$\tilde{\gamma}_1=0 \quad\text{and}\quad \tilde{\gamma}_2=0.$$

In fact, since

$$z_n=\left(\frac{1}{\alpha}(\log\gamma+2n\pi i)\right)^{1/2}=\sqrt{2n\pi}\left(\frac{i}{\alpha}\right)^{1/2}\left(1+O\left(\frac{1}{n}\right)\right)^{1/2}$$

and

$$\tilde{z}_n=-\left(\frac{1}{\alpha}(\log\gamma+2n\pi i)\right)^{1/2}=-\sqrt{2n\pi}\left(\frac{i}{\alpha}\right)^{1/2}\left(1+O\left(\frac{1}{n}\right)\right)^{1/2},$$

if $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_1]>0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_2]>0$, then

$$\lim_{n\to\infty} \tilde{F}_n=1;$$

if $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_1]<0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_2]<0$, then

$$\lim_{n\to\infty} F_n=1;$$

and if $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_1]>0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_2]\leqq 0$, or if $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_1]\leqq 0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_2]>0$, then

$$\lim_{n\to\infty} F_n=\infty.$$

These all results lead to a contradiction, because $F_n=\tilde{F}_n=0$. Thus we have $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_1]=0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}\tilde{\gamma}_2]=0$.

Similarly from relations

$$z_n=-\sqrt{-2n\pi}\left(\frac{i}{\alpha}\right)^{1/2} i\left(1+O\left(\frac{1}{n}\right)\right)^{1/2}$$

and

$$\tilde{z}_n=\sqrt{-2n\pi}\left(\frac{i}{\alpha}\right)^{1/2} i\left(1+O\left(\frac{1}{n}\right)\right)^{1/2}$$

we have $\mathrm{Re}\,[(i/\alpha)^{1/2}i\tilde{\gamma}_1]=0$ and $\mathrm{Re}\,[(i/\alpha)^{1/2}i\tilde{\gamma}_2]=0$. Consequently we have $(i/\alpha)^{1/2}\tilde{\gamma}_1=(i/\alpha)^{1/2}\tilde{\gamma}_2=0$, and hence $\tilde{\gamma}_1=\tilde{\gamma}_2=0$.

From $\tilde{\gamma}_1=\tilde{\gamma}_2=0$, we have

$$1-2AX_0^n-2BY_0^n+A^2X_0^{2n}-2ABX_0^nY_0^n+B^2Y_0^{2n}=0$$

for all integer n. By the lemma given in [4], we have

$$X_0=1 \quad\text{and}\quad Y_0=1.$$

If $\alpha_1\alpha_2\neq 0$, this implies that

$$\alpha_1=p_1\alpha \quad\text{and}\quad \alpha_2=p_2\alpha$$

for suitable non-zero integers p_1 and p_2.

Putting $e^{\alpha z^2/2}=\chi$, we have

$$\tilde{G}(z)=\tilde{F}(e^{\alpha z^2/2});$$
$$\tilde{F}(\chi)=1-2\tilde{\beta}_1\chi^{2p_1}-2\tilde{\beta}_2\chi^{2p_2}+\tilde{\beta}_1{}^2\chi^{4p_1}-2\tilde{\beta}_1\tilde{\beta}_2\chi^{2p_1+2p_2}+\tilde{\beta}_2{}^2\chi^{4p_2}.$$

Since $e^{\alpha z^2/2}-\chi_0$, $\chi_0\neq 0$, $z\neq 0$ has no zero other than an infinite number of simple zeros and $e^{\alpha z^2/2}$ has no zero, every multiple zero of $\tilde{G}(z)$, $z\neq 0$ occurs from a suitable multiple zero of $F(\chi)$ and vice versa. Thus $F(\chi)$ has just four simple zeros $\sqrt{\tilde{\gamma}}$, $-\sqrt{\tilde{\gamma}}$, $\sqrt{\tilde{\delta}}$ and $-\sqrt{\tilde{\delta}}$.

Then by the same argument as in § 5, we have all the possible cases for which $P(R)=4$, which can be listed as follows:

(1) $p_1=2$, $p_2=1$, $\tilde{\beta}_2{}^2=16\tilde{\beta}_1$; (2) $p_1=1$, $p_2=2$, $\tilde{\beta}_1{}^2=16\tilde{\beta}_2$;

(2) $p_1=1$, $p_2=-1$, $16\tilde{\beta}_1\tilde{\beta}_2=1$; (4) $p_1=-1$, $p_2=1$, $16\tilde{\beta}_1\tilde{\beta}_2=1$;

(5) $p_1=-2,\ p_2=-1,\ \tilde{\beta}_2^2=16\tilde{\beta}_1$; (6) $p_1=-1,\ p_2=-2,\ \tilde{\beta}_1^2=16\tilde{\beta}_2$;

(7) $p_1=p_2=1,\ \tilde{\beta}_1, \tilde{\beta}_2$ are free but $\tilde{\beta}_1\neq\tilde{\beta}_2$; (8) $p_1=p_2=2$ and $\tilde{\beta}_1=\tilde{\beta}_2$.

Evidently, in all the cases, we have

$$\tilde{G}(z)=\tilde{f}(z)^2(e^{\alpha z^2}-\tilde{\gamma})(e^{\alpha z^2}-\tilde{\delta})$$

with two suitable constants $\tilde{\gamma}$ and $\tilde{\delta}$ and a suitable entire function $\tilde{f}$. And hence $P(R)$ is equal to 4.

If $\alpha_1\alpha_2=0$, we have

$$\tilde{G}(z)=\tilde{\beta}_2^2(e^{\alpha_2 z^2}-M)(e^{\alpha_2 z^2}-N);$$

$$M=(\sqrt{\tilde{\beta}_1}+1)^2/\tilde{\beta}_2, \qquad N=(\sqrt{\tilde{\beta}_1}-1)^2/\tilde{\beta}_2$$

or

$$\tilde{G}(z)=\tilde{\beta}_1^2(e^{\alpha_1 z^2}-M)(e^{\alpha_1 z^2}-N);$$

$$M=(\sqrt{\tilde{\beta}_2}+1)^2/\tilde{\beta}_1, \qquad N=(\sqrt{\tilde{\beta}_2}-1)^2/\tilde{\beta}_1.$$

And hence $P(R)$ is equal to 4.

II. Case of $\alpha=0$. Then (6.1) reduces to an equation

$$g(z)^2G(z)=f(z)^2(e^{\beta z}-\gamma)(e^{\beta z}-\delta). \tag{6.3}$$

Let z_n, $n=0, \pm 1, \cdots$, be

$$\frac{1}{\beta}\log\gamma+\frac{1}{\beta}2n\pi i.$$

Then these are simple zeros of $e^{\beta z}-\gamma$. Therefore $G(z_n)=0$. Since

$$G(z)=G_1(z)\cdot G_2(z)\cdot G_3(z)\cdot G_4(z);$$

$$G_1(z)=1-\sqrt{\beta_1}e^{(\alpha_1 z^2+\gamma_1 z)/2}-\sqrt{\beta_2}e^{(\alpha_2 z^2+\gamma_2 z)/2},$$

$$G_2(z)=1-\sqrt{\beta_1}e^{(\alpha_1 z^2+\gamma_1 z)/2}+\sqrt{\beta_2}e^{(\alpha_2 z^2+\gamma_2 z)/2},$$

$$G_3(z)=1+\sqrt{\beta_1}e^{(\alpha_1 z^2+\gamma_1 z)/2}-\sqrt{\beta_2}e^{(\alpha_2 z^2+\gamma_2 z)/2},$$

$$G_4(z)=1+\sqrt{\beta_1}e^{(\alpha_1 z^2+\gamma_1 z)/2}+\sqrt{\beta_2}e^{(\alpha_2 z^2+\gamma_2 z)/2},$$

we have

$$1+k_n\sqrt{\beta_1}e^{(\alpha_1 z_n^2+\gamma_1 z_n)/2}+l_n\sqrt{\beta_2}e^{(\alpha_2 z_n^2+\gamma_2 z_n)/2}=0$$

where k_n and l_n are suitable constants and are equal to 1 or -1.

Putting

$$X_1=e^{-2\alpha_1\pi^2/\beta^2}, \qquad Y_1=e^{2\alpha_1\pi i(\log\gamma)/\beta^2+\gamma_1\pi i/\beta},$$

$$X_2=e^{-2\alpha_2\pi^2/\beta^2}, \qquad Y_2=e^{2\alpha_2\pi i(\log\gamma)/\beta^2+\gamma_2\pi i/\beta},$$

$$A=\sqrt{\beta_1}e^{\alpha_1(\log\gamma)^2/2\beta^2+\gamma_1(\log\gamma)/2\beta}$$

and

$$B=\sqrt{\beta_2}\, e^{\alpha_2(\log\gamma)^2/2\beta^2+\gamma_2(\log\gamma)/2\beta},$$

we have

$$F_n\equiv 1+k_nAX_1^{n^2}Y_1^n+l_nBX_2^{n^2}Y_2^n=0$$

for every integer n.

At first we conclude that

$$|X_1|=|X_2|=|Y_1|=|Y_2|=1.$$

In fact, by eliminating $X_2^{n^2}$ on relations

$$F_n\equiv 1+k_nAX_1^{n^2}Y_1^n+l_nBX_2^{n^2}Y_2^n=0$$

and

$$F_{2n}\equiv 1+k_{2n}AX_1^{4n^2}Y_1^{2n}+l_{2n}BX_2^{4n^2}Y_2^{2n}=0,$$

we have

$$\begin{aligned}\tilde{F}_n\equiv\left(\frac{k_{2n}B^3Y_2^{2n}}{l_{2n}A^3Y_1^{2n}}+1\right)\frac{A^4Y_1^{4n}}{B^4Y_2^{4n}}X_1^{4n^2}+4k_n^3\frac{A^3Y_1^{3n}}{B^4Y_2^{4n}}X_1^{3n^2}+6\frac{A^2Y_1^{2n}}{B^4Y_2^{4n}}X_1^{2n^2}\\ +4k_n\frac{AY_1^n}{B^4Y_2^{4n}}X_1^{n^2}+\left(1+\frac{B^3Y_2^{2n}}{l_{2n}}\right)\frac{1}{B^4Y_2^{4n}}=0\end{aligned}$$

for every integer n. Suppose first that $|X_1|>1$. If

$$\overline{\lim_{n\to\pm\infty}}\left|\frac{k_{2n}B^3Y_2^{2n}}{l_{2n}A^3Y_1^{2n}}+1\right|>0,$$

then

$$\overline{\lim_{n\to\pm\infty}}|\tilde{F}_n|=\infty.$$

This is a contradiction, because $\tilde{F}_n=0$. If

$$\lim_{n\to\pm\infty}\left(\frac{k_{2n}B^3Y_2^{2n}}{l_{2n}A^3Y_1^{2n}}+1\right)=0,$$

then, by noting that for two numbers α and β $\lim_{n\to\infty}\alpha\beta^n=1$ holds if and only if $\alpha=\beta=1$, we have

$$\frac{Y_2^4}{Y_1^4}=\frac{B^6}{A^6}=1.$$

Hence we have

$$K_n\equiv\frac{k_{2n}B^3Y_2^{2n}}{l_{2n}A^3Y_1^{2n}}+1=0$$

for almost all n, because $K_n=0$ or 2 for every n. Then we have

$$\lim_{n\to\infty}\tilde{F}_n=\infty.$$

This is a contradiction, because $\tilde{F}_n=0$.

Next suppose that $|X_1|<1$. If

$$\overline{\lim_{n\to\pm\infty}}\left|1+\frac{B^3Y_2^{2n}}{l_{2n}}\right|>0,$$

then

$$\overline{\lim_{n\to\pm\infty}}\left|\frac{\tilde{F}_n}{X_1^{4n^2}}\right|=\infty.$$

This is a contradiction, because $\tilde{F}_n=0$. If

$$\lim_{n\to\pm\infty}\left(1+\frac{B^3Y_2^{2n}}{l_{2n}}\right)=0,$$

then we have

$$Y_2^4=B^6=1.$$

Hence we have

$$\tilde{K}_n\equiv 1+\frac{B^3Y_2^{2n}}{l_{2n}}=0$$

for almost all n, because $\tilde{K}_n=0$ or 2 for every n. Then we have

$$\lim_{n\to\infty}\frac{\tilde{F}_n}{X_1^{4n^2}}=\infty.$$

This is a contradiction, because $\tilde{F}_n=0$. So we have $|X_1|=1$. Again, by eliminating $X_1^{n^2}$ on relations

$$F_n=0 \quad \text{and} \quad F_{2n}=0$$

and by reasoning similarly, we deduce $|X_2|=1$. Thus $|X_1|=|X_2|=1$ holds. Further if $|Y_1|\neq|Y_2|$, then

$$\lim_{n\to\infty}F_n=\infty \qquad \text{or} \qquad \lim_{n\to-\infty}F_n=\infty;$$

and if $|Y_1|=|Y_2|<1$ or $|Y_1|=|Y_2|>1$, then

$$\lim_{n\to\infty}F_n=1 \qquad \text{or} \qquad \lim_{n\to-\infty}F_n=1.$$

These results lead to a contradiction, because $F_n=0$. Consequently we have $|X_1|=|X_2|=|Y_1|=|Y_2|=1$.

Next from $|X_1|=|X_2|=|Y_1|=|Y_2|=1$ we show that $X_1^2=X_2^2=Y_1^2=Y_2^2=1$. In fact, let

$$A=|A|e^{ia\pi}, \qquad X_1=e^{ix_1\pi}, \qquad Y_1=e^{iy_1\pi},$$

$$B=|B|e^{ib\pi}, \qquad X_2=e^{ix_2\pi}, \qquad Y_2=e^{iy_2\pi};$$

a, b, x_1, x_2, y_1, y_2: real constants,

then

$$|B|=|BX_2^{n^2}Y_2^n|=|1+AX_1^{n^2}Y_1^n+2(AX_1^{n^2}Y_1^n)^{1/2}|$$
$$=1+2|A|^{1/2}\cos\left(\frac{n^2x_1+ny_1+a}{2}+\delta\right)\pi+|A|, \qquad \delta=0 \text{ or } 1.$$

Thus x_1 and y_1 must be integers, and hence $X_1^2=Y_1^2=1$. Similarly we have $X_2^2=Y_2^2=1$.

From the above fact, we can put

$$\frac{2\alpha_1\pi^2}{\beta^2}=p_1\pi i, \qquad \frac{2\alpha_1\pi i}{\beta^2}\log\gamma+\frac{\gamma_1\pi i}{\beta}=q_1\pi i,$$
$$\frac{2\alpha_2\pi^2}{\beta^2}=p_2\pi i, \qquad \frac{2\alpha_2\pi i}{\beta^2}\log\gamma+\frac{\gamma_2\pi i}{\beta}=q_2\pi i$$

where p_1, p_2, q_1 and q_2 are integers. And hence we have

$$(6.4) \qquad -p_1\log\gamma+\frac{\gamma_1\pi i}{\beta}=q_1\pi i, \qquad -p_2\log\gamma+\frac{\gamma_2\pi i}{\beta}=q_2\pi i.$$

On the other hand, by putting

$$z_n=\frac{1}{\beta}\log\delta+\frac{1}{\beta}2n\pi i$$

we must have

$$(6.5) \qquad -p_1\log\delta+\frac{\gamma_1\pi i}{\beta}=q_3\pi i, \qquad -p_2\log\delta+\frac{\gamma_2\pi i}{\beta}=q_4\pi i$$

with some suitable integers q_3 and q_4.

From (6.4) and (6.5) we have

$$\left(\frac{\delta}{\gamma}\right)^{2p_1}=\left(\frac{\delta}{\gamma}\right)^{2p_2}=1.$$

Since $\delta/\gamma\neq1$, p_1 and p_2 are zeros. This is a contradiction, because

$$ip_1=\frac{\alpha_1\pi^2}{\beta^2}, \qquad ip_2=\frac{\alpha_2\pi^2}{\beta^2}$$

and

$$|\alpha_1|+|\alpha_2|\neq0.$$

Consequently the case II does not occur.

Summing up the above results, we have:

THEOREM E. *Let R be an ultrahyperelliptic surface defined by an equation*

$$y^2=1-2\beta_1e^{\alpha_1x^2+\gamma_1x}-2\beta_2e^{\alpha_2x^2+\gamma_2x}+\beta_1^2e^{2\alpha_1x^2+2\gamma_1x}-2\beta_1\beta_2e^{(\alpha_1+\alpha_2)x^2+(\gamma_1+\gamma_2)x}+\beta_2^2e^{2\alpha_2x^2+2\gamma_2x};$$
$$\beta_1\beta_1\alpha_2\alpha_2\neq0.$$

Then $P(R)$ is equal to 3 excepting the following four cases:

(1) $\alpha_1=2\alpha_2$, $\gamma_1=2\gamma_2$, $\beta_2^2=16\beta_1$; (2) $\alpha_2=2\alpha_1$, $\gamma_2=2\gamma_1$, $\beta_1^2=16\beta_2$;

(3) $\alpha_1=-\alpha_2$, $\gamma_1=-\gamma_2$, $16\beta_1\beta_2=1$; (4) $\alpha_1=\alpha_2$, $\gamma_1=\gamma_2$, β_1, β_2 *are free*

for which we have $P(R)=4$.

§ 7. Remarks.

It should be remarked here that a conjecture stated in the problem (1) in [7] is not exact. By our theorems D and E we are obliged to add a more exceptional case $H_1=-H_2$, $16\beta_1\beta_2=1$ in the terminologies in [7]. To solve the problem in its most general form seems to be difficult and to be necessary any other method.

We can give a positive answer to the problem 3 in [7]. In fact, let R and S be two ultrahyperelliptic Riemann surfaces defined by

$$y^2=G(z)\equiv 81e^{4z^2}-72e^{3z^2}-2e^{2z^2}-8e^{z^2}+1$$

and

$$u^2=g(w)\equiv z(81e^{4z}-72e^{3z}-2e^{2z}-8e^{z}+1),$$

respectively. Then we have $P(R)=3$ and $P(S)=2$ by theorem D and theorem E (cf. [5]). Putting $f\equiv z$ and $h\equiv z^2$, we have an identity

$$f(z)^2G(z)=g\circ h(z).$$

Thus by theorem A there exists an analytic mapping from R into S.

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