ALMOST CONTACT STRUCTURES INDUCED ON HYPERSURFACES IN COMPLEX AND ALMOST COMPLEX SPACES

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Contents

Introduction.

A structure on an n-dimensional differentiable manifold given by a non-null tensor field f of type (1, 1) of a constant rank r and satisfying $f^3+f=0$ is called an *f-structure*. If $n=r$, then an f-structure gives an almost complex structure of the manifold and $n=r$ is necessarily even. If the manifold is orientable and $n-1=r$, then an f-structure gives an almost contact structure of the manifold and n is necessarily odd and r even.

A submanifold in an almost complex space admits an f-structure if we can choose a distribution along the submanifold which is invariant by the almost complex operator or, whose transform by the almost complex operator is contained in the tangent space of the submanifold.

For an orientable hypersurface in an almost complex space, we can choose a vector field whose transform by the almost complex operator belongs to the tangent space of the hypersurface, and consequently an orientable hypersurface in an almost complex space admits an f-structure which is an almost contact structure.

The purpose of the present paper is to study almost contact structures induced in this way on hypersurfaces in complex and almost complex spaces.

The number between brackets refer to the Bibliography at the end of the paper.

§1. *f*-structure.

Let there be given, in an n-dimensional differentiable manifold V of class C^∞,

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a non-null tensor field f of type (1, 1) and of class C^∞ satisfying

$$f^3+f=0. \tag{1.1}$$

We call such a structure an *f-structure* of rank r when the rank of f is constant everywhere and is equal to r, where r is necessarily even [14, 15].

If we put

$$l=-f^2, \qquad m=f^2+1,$$

we have

$$l+m=1, \quad l^2=l, \quad m^2=m, \quad lm=ml=0,$$

where 1 denotes the unit tensor. These equations show that the operators l and m applied to the tangent space at each point of the manifold are complementary projection operators. Thus, there exist in the manifold complementary distributions L and M corresponding to the projection operators l and m respectively. When the rank of f is equal to r, L is r-dimensional and M $(n-r)$-dimensional.

Let $f_b{}^a$ be components of an f-structure f of rank r.[1] Then its Nijenhuis tensor $N_{cb}{}^a$ is by definition

$$N_{cb}{}^a=f_c{}^e\nabla_e f_b{}^a-f_b{}^e\nabla_e f_c{}^a-(\nabla_c f_b{}^e-\nabla_b f_c{}^e)f_e{}^a, \tag{1.2}$$

where ∇_e denotes covariant differentiation with respect to a symmetric linear connection. The Nijenhuis tensor $N_{cb}{}^a$ does not depend on the symmetric connection involved.

In a recent paper [6], we have proved

THEOREM A. *A necessary and sufficient condition for the distribution L to be integrable is that*

$$N_{fe}{}^d l_c{}^f l_b{}^e m_d{}^a=0, \quad \text{or} \quad N_{cb}{}^d m_d{}^a=0, \tag{1.3}$$

$l_b{}^a$ and $m_b{}^a$ being components of the projection tensors l and m respectively.

Suppose that the distribution L is integrable and take an arbitrary vector field v which is tangent to an integral manifold of L. Then the vector field fv is tangent to the same integral manifold. Since we have

$$f^2=-l, \tag{1.4}$$

if we define an operator f' by

$$f'v=fv$$

in each tangent space of each integral manifold of L, f' is an almost complex structure in each integral manifold of L. When the distribution L is integrable and the almost complex structure f' induced from f on each integral manifold of L is also integrable, then we say that the f-structure is *partially integrable.* We have proved in [6] the following

THEOKEM B. *A necessary and sufficient condition for an f-structure to be partially integrable is that*

[1] The indices a, b, c, d, e, f run over the range $\{1, 2, \cdots, n\}$.

$$N_{fe}{}^{a} l_c{}^{f} l_b{}^{e} = 0. \tag{1.5}$$

We suppose now that there exists in each coordinate neighborhood a coordinate system in which an f-structure f has numerical components

$$(f_b{}^a) = \begin{pmatrix} 0 & -1_m & 0 \\ 1_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where 1_m denotes the $m \times m$ unit matrix, $r=2m$ being the rank of f. In this case, we say that the f-structure is *integrable.* We have proved also in [6] the following

THEOREM C. *A necessary and sufficient condition for an f-structure to be integrable is that*

$$N_{cb}{}^{a} = 0. \tag{1.6}$$

§2. Almost contact structure.

Let there be given, in an n-dimensional differentiable manifold V of class C^∞, a tensor field $f_b{}^a$ of type (1, 1), a contravariant vector field f^a and a covariant vector field f_b, all of which are of differentiability class C^∞. We suppose that they satisfy the conditions:

$$\begin{gathered} f_c{}^b f_b{}^a = -\delta_c^a + f_c f^a, \qquad f_b{}^a f^b = 0, \\ f_b{}^a f_a = 0, \qquad f^a f_a = 1. \end{gathered} \tag{2.1}$$

We call the set $(f_b{}^a, f^a, f_b)$ satisfying (2. 1) an *almost contact structure* (See for example [9]). It follows easily from (2. 1) that the rank of $f_b{}^a$ is $n-1$ everywhere. Furthermore, (2. 1) implies that $f_b{}^a$ satisfies (1. 1), i.e., that the tensor field $f_b{}^a$ is an f-structure of rank $n-1$, where n is necessarily odd.

Sasaki and Hatakeyama [10] have introduced following four tensor fields:

$$\begin{aligned} S_{cb}{}^a &= N_{cb}{}^a + (\nabla_c f_b - \nabla_b f_c) f^a, \\ S_{cb} &= L_{cb} - L_{bc}, \\ S_c{}^a &= f^e \nabla_e f_c{}^a + (\nabla_c f^e) f_e{}^a - (\nabla_e f^a) f_c{}^e, \\ S_c &= f^e (\nabla_e f_c - \nabla_c f_e), \end{aligned} \tag{2.2}$$

where $N_{cb}{}^a$ is the Nijenhuis tensor of $f_b{}^a$ defined by (1. 2), L_{cb} a tensor field defined by

$$L_{cb} = f_c{}^e (\nabla_e f_b - \nabla_b f_e), \tag{2.3}$$

and ∇_c denotes covariant differentiation with respect to a symmetric linear connection. All S's and L defined above does not depend on the symmetric connection involved. We call the tensor field L_{cb} the *Levi tensor* of the almost contact structure.

From the first equation of (2. 2), we find

(2. 4) $$S_{cb}{}^a f_a = (\nabla_c f_b - \nabla_b f_c) - (\nabla_f f_e - \nabla_e f_f) f_c{}^f f_b{}^e,$$

(2. 5) $$S_{cb}{}^a f^b = S_c{}^e f_e{}^a - S_c f^a,$$

from which

(2. 6) $$S_c = -S_{cb}{}^a f^b f_a.$$

From the definition of L_{cb}, we find

$$f^c L_{cb} = 0,$$

$$L_{cb} f^b = -f_c{}^e S_e,$$

(2. 7) $$f_c{}^f L_{fb} = -(\nabla_c f_b - \nabla_b f_c) + f_c S_b,$$

$$L_{ce} f_b{}^e = (\nabla_f f_e - \nabla_e f_f) f_c{}^f f_b{}^e,$$

$$L_{fe} f_c{}^f f_b{}^e = L_{bc} + f_c S_e f_b{}^e.$$

Substituting

$$\nabla_c f_b - \nabla_c f_b = -f_c{}^f L_{fb} + f_c S_b,$$

$$(\nabla_f f_e - \nabla_e f_f) f_c{}^f f_b{}^e = L_{ce} f_b{}^e$$

obtained respectively from the third and the fourth equations of (2. 7) into (2. 4), we find

(2. 8) $$S_{cb}{}^a f_a = -f_c{}^f L_{fb} - L_{ce} f_b{}^e + f_c S_b.$$

Using (2. 7) and (2. 8), we can easily verify

(2. 9) $$S_{cb} = S_{ce}{}^a f_b{}^e f_a - S_f f_c{}^f f_b,$$

from which

(2. 10) $$S_c = S_{fe} f_c{}^f f^e$$

by virtue of $S_c f^c = 0$. From the last equation of (2. 7), we find

(2. 11) $$S_{cb} + S_{fe} f_c{}^f f_b{}^e = f_c S_e f_b{}^e - f_b S_e f_c{}^e.$$

We next note that

$$S_c{}^a = \mathcal{L} f_c{}^a, \qquad S_c = \mathcal{L} f_c,$$

where $\mathcal{L}$ denotes the Lie differentiation [16] with respect to f^a. Taking Lie derivatives of (2. 1), we find

$$S_c{}^b f_b{}^a + f_c{}^b S_b{}^a = S_c f^a,$$

(2. 12) $$S_b{}^a f_a + f_b{}^a S_a = 0,$$

$$S_b{}^a f^b = 0,$$

$$S_a f^a = 0.$$

From (2. 5) and the first equation of (2. 12), we find

(2. 13) $$S_{cb}{}^a f^b = -f_c{}^e S_e{}^a,$$

from which

(2. 14) $$S_c{}^a = S_{fe}{}^a f_c{}^f f^e,$$

(2. 15) $$S_c = S_e{}^a f_c{}^e f_a.$$

From (2. 6), (2. 9) and (2. 14), we see [10] that

$$S_{cb}{}^a = 0 \quad \text{implies} \quad S_{cb}=0, \quad S_c{}^a=0, \quad S_c=0.$$

When the condition $S_{cb}{}^a=0$ is satisfied, the almost contact structure is said to be *normal* [10].

If, in a manifold with an almost contact structure $(f_b{}^a, f^a, f_b)$, a tensor field, say, $T_{cb}{}^a$ has components of the form

(2. 16) $$T_{cb}{}^a = f_c P_b{}^a + f_b Q_c{}^a$$

with certain tensor fields $P_b{}^a$ and $Q_b{}^a$, the tensor $T_{cb}{}^a$ is said to be *congruent to zero with respect to modulus* f_c. In such a case, the relation (2. 16) is expressed in a simplified form as

(2. 17) $$T_{cb}{}^a \equiv 0,$$

and $U_{cb}{}^a - V_{cb}{}^a \equiv 0$ as

$$U_{cb}{}^a \equiv V_{cb}{}^a.$$

It is easily seen that (2. 17) is valid if and only if we have

$$T_{cb}{}^a w^c v^b = 0$$

for any vector fields v^a and w^a such that $f_a v^a=0$ and $f_a w^a=0$.

Suppose now that $S_{cb}{}^a \equiv 0$, then we can put

(2. 18) $$S_{cb}{}^a = f_c P_b{}^a - f_b P_c{}^a.$$

Substituting this into (2. 6), (2. 9) and (2. 14), we find

$$S_c = -f_c P + P_c,$$

(2. 19) $$S_{cb} = f_c P_e f_b{}^e - f_b P_e f_c{}^e,$$

$$S_c{}^a = -f_c{}^e P_e{}^a$$

respectively, where $P = P_c f^c$, $P_c = P_c{}^a f_a$.

Thus we have

PROPOSITION 2. 1. *If* $S_{cb}{}^a \equiv 0$, *then* $S_{cb} \equiv 0$, *that is,*

(2. 20) $$L_{cb} \equiv L_{bc}.$$

A necessary and sufficient condition for the distribution M, i.e. $f_a d\eta^a = 0$, η^a being local coordinates, to be integrable is

(2. 21) $$\nabla_c f_b - \nabla_b f_c \equiv 0,$$

and consequently, from the definitions of $N_{cb}{}^a$ and $S_{cb}{}^a$ and the third equation of (2. 7), we have

PROPOSITION 2. 2. *A necessary and sufficient condition for the distribution* M, that is, $f_a d\eta^a=0$ *to be integrable is one of the following*:

$$S_{cb}{}^a-N_{cb}{}^a\equiv 0, \qquad (S_{cb}{}^a-N_{cb}{}^a)f_a\equiv 0, \qquad f_c{}^f L_{fb}\equiv 0.$$

It is well known [9, 10] that, in a manifold with an almost contact structure $(f_b{}^a, f^a, f_b)$, there exists a Riemannian metric g_{cb} such that

$$f_c{}^f f_b{}^e g_{fe}=g_{cb}-f_c f_b, \qquad f_b=f^c g_{cb}. \tag{2. 22}$$

We call the set $(f_b{}^a, f^a, f_b, g_{cb})$ of such tensor fields an *almost contact metric structure* [9, 10].

Let there be given, in a $(2m+1)$-dimensional differentiable manifold V, a differentiable 1-form $f=f_a d\eta^a$ such that

$$f\wedge\underbrace{df\wedge\cdots\wedge df}_{m}\neq 0$$

everywhere. Such a manifold is called a *contact manifold* and the covariant vector field f_b is called a *contact structure* [1, 2, 9]. It is well known [3] that in any contact manifold there exists always an almost contact metric structure $(f_b{}^a, f^a, f_b, g_{cb})$ such that

$$f_{cb}=f_c{}^e g_{eb}, \qquad f_{cb}=\nabla_c f_b-\nabla_b f_c.$$

Such an almost contact metric structure is usually called a *contact metric structure* [10]. However we shall, in the present paper, call an almost contact metric structure $(f_b{}^a, f^a, f_b, g_{cb})$ satisfying

$$\nabla_c f_b-\nabla_b f_c=2af_{cb}, \qquad f_{cb}=f_c{}^e g_{eb}, \tag{2. 23}$$

where a is a non-zero constant, a *contact metric structure.* It is easily seen that, if an almost contact metric structure is a contact structure, we have

$$L_{cb}=2a(-g_{cb}+f_c f_b) \tag{2. 24}$$

with non-zero constant a.

§ 3. Hypersurfaces in an almost complex space.

Let V be an $(n+1)$-dimensional differentiable manifold of class C^∞ and Γ_{ji}^h a symmetric linear connection of class C^∞ in V.[2] Let there be given an n-dimensional orientable submanifold M of class C^∞ differentiably immersed in V and a certain vector field C^h along M which does not belong to the tangent space of M everywhere. The set $\{M, C^h\}$ is called a *hypersurface* in V, where M and C^h are respectively called the *basic submanifold* and the *normal vector field* of the hyper-

2) The indices h, i, j, k, l run over the range $\{1, 2, \cdots, n, n+1\}$.

surface $\{M, C^h\}$.

Let the basic submanifold M be expressed by equations

$$\xi^h = \xi^h(\eta^a)$$

in local coordinates ξ^h in V, where η^a is a system of local coordinates in M. If we put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\eta^b, \tag{3.1}$$

then $B_b{}^h$ are n local vector fields along M, which are tangent to M and linearly independent at each point of M. Denoting by $(B^a{}_i, C_i)$ the inverse matrix of $\begin{pmatrix} B_b{}^h \\ C^h \end{pmatrix}$, we have

$$\begin{aligned} B^a{}_h B_b{}^h &= \delta_b^a, \qquad & B^a{}_h C^h &= 0, \\ C_h B_b{}^h &= 0, \qquad & C_h C^h &= 1, \end{aligned} \tag{3.2}$$

and

$$B_a{}^h B^a{}_i + C^h C_i = \delta_i^h. \tag{3.3}$$

We now define the *induced connection* Γ_{cb}^a induced on the hypersurface $\{M, C^h\}$ from the given connection Γ_{ji}^h by the equation

$$\Gamma_{cb}^a = (\partial_c B_b{}^h + B_c{}^j B_b{}^i \Gamma_{ji}^h) B^a{}_h \tag{3.4}$$

and a vector field l_c in M by

$$l_c = (\partial_c C^h + B_c{}^j C^i \Gamma_{ji}^h) C_h. \tag{3.5}$$

If we define the van der Waerden-Bortolotti covariant derivative $\nabla_c B_b{}^h$ of $B_b{}^h$ along the hypersurface by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + B_c{}^j B_b{}^i \Gamma_{ji}^h - B_a{}^h \Gamma_{cb}^a, \tag{3.6}$$

then $\nabla_c B_b{}^h$ is proportional to C^h. Therefore we can put

$$\nabla_c B_b{}^h = h_{cb} C^h, \tag{3.7}$$

where h_{cb} is the so-called second fundamental tensor of the hypersurface $\{M, C^h\}$. It is easily seen from (3.6) that

$$h_{cb} = h_{bc}. \tag{3.8}$$

If we define the covariant derivative $\nabla_c C^h$ of the normal vector field C^h along the hypersurface by

$$\nabla_c C^h = \partial_c C^h + B_c{}^j C^i \Gamma_{ji}^h, \tag{3.9}$$

then we have

$$\nabla_c C^h = -h_c{}^a B_a{}^h + l_c C^h, \tag{3.10}$$

where $h_c{}^a$ is a tensor field of type (1, 1) in the basic submanifold M defined by

$$h_c{}^a = -(\nabla_c C^h) B^a{}_h. \tag{3.11}$$

We assume now that the $(n+1)$-dimensional manifold V is an almost complex space, i.e. that V admits an almost complex structure F_i^h. The tensor field F_i^h satisfies

$$F_j^i F_i^h = -\delta_j^h. \tag{3.12}$$

Moreover we suppose that the normal vector field C^h of a hypersurface $\{M, C^h\}$ has the following property: the vector field $F_i^h C^i$ is tangent to M at each point. It is easily seen that there exists such a normal vector field C^h along any orientable submanifold M of V. In such a case, the hypersurface $\{M, C^h\}$ is called briefly an *almost contact hypersurface* in the almost complex space V.

Let $\{M, C^h\}$ be an almost contact hypersurface in V. Then we can put

$$\begin{aligned} F_i^h B_b^i &= f_b^a B_a^h + f_b C^h, \\ F_i^h C^i &= -f^a B_a^h, \end{aligned} \tag{3.13}$$

because $F_i^h C^i$ is tangent to M. It follows easily from (3. 12) and (3. 13)

$$\begin{aligned} f_c^b f_b^a &= -\delta_c^a + f_c f^a, \qquad & f_b^a f^b &= 0, \\ f_b^a f_a &= 0, & f^a f_a &= 1. \end{aligned} \tag{3.14}$$

This means that the basic submanifold M admits an almost contact structure (f_b^a, f^a, f_b), which is called the *induced almost contact structure* of the hypersurface $\{M, C^h\}$.

If we differentiate covariantly both members of (3. 13), we obtain, taking account of (3. 7), (3. 10) and (3. 13),

$$\begin{aligned} (\nabla_j F_i^h) B_c^j B_b^i &= (\nabla_c f_b^a + h_{cb} f^a - h_c^a f_b) B_a^h + (\nabla_c f_b + h_{ce} f_b^e + l_c f_b) C^h, \\ (\nabla_j F_i^h) B_c^j C^i &= -(\nabla_c f^a - h_c^e f_e^a - l_c f^a) B_a^h + (h_c^e f_e - h_{ce} f^e) C^h. \end{aligned} \tag{3.15}$$

The Nijenhuis tensor N_{ji}^h of the almost complex structure F_i^h is by definition

$$N_{ji}^h = F_j^l \nabla_l F_i^h - F_i^l \nabla_l F_j^h - (\nabla_j F_i^l - \nabla_i F_j^l) F_l^h. \tag{3.16}$$

Taking account of (3. 13), (3. 15) and (3. 16), we find

$$\begin{aligned} N_{ji}^h B_c^j B_b^i = {} & [S_{cb}{}^a - f_c(h_b^e f_e^a - f_b^e h_e^a) + f_b(h_c^e f_e^a - f_c^e h_e^a) - (f_c l_b - f_b l_c) f^a] B_a^h \\ & + [S_{cb} + (h_c^e f_e f_b - h_b^e f_e f_c) - (f_c f_b^e - f_b f_c^e) l_e] C^h + f_c (\nabla_j F_i^h) C^j B_b^i - f_b (\nabla_j F_i^h) C^j B_c^i, \end{aligned} \tag{3.17}$$

$$\begin{aligned} N_{ji}^h B_c^j C^i = {} & [S_c{}^a + (h_c^a + h_f^d f_c^f f_d^a) - f_c f^e h_e^a + (h_c^e f_e - h_{ce} f^e) f^a + f_c^e l_e f^a] B_a^h \\ & + [S_c + h_{fe} f^f f_c^e + f^e l_e f_c - l_c] C^h + f_c (\nabla_j F_i^h) C^j C^i + (\nabla_j F_i^l) F_l^h C^j B_c^i, \end{aligned} \tag{3.18}$$

where S's are tensor fields in M defined by (2. 2). We have immediately from (3. 17)

THEOREM 3. 1. *For an almost contact hypersurface $\{M, C^h\}$ in an almost complex space V, the vector field $(N_{ji}^h B_c^j B_c^i) w^c v^b$ is tangent to M at each point of M,*

v^a and w^a being arbitrary vector fields in M satisfying conditions $f_a v^a=0, f_a w^a=0$, if and only if $S_{cb}\equiv 0$, that is, $L_{cb}\equiv L_{bc}$.

Taking account of Proposition 2. 1, we have also from (3. 17)

THEOREM 3. 2. *For an almost contact hypersurface $\{M, C^h\}$ in an almost complex space, we have $(N_{ji}{}^h B_c{}^j B_b{}^i) w^c v^b=0$, v^a and w^a being arbitrary vector fields in M satisfying $f_a v^a=0, f_a w^a=0$, if and only if $S_{cb}{}^a\equiv 0$.*

§4. Hypersurfaces in a complex space.

Let V be a complex space with complex structure $F_i{}^h$. Then the Nijenhuis tensor $N_{ji}{}^h$ of $F_i{}^h$ vanishes identically. As is well known (Cf. [17]), there exists in V a symmetric linear connection Γ_{ji}^h such that

$$\nabla_j F_i{}^h=0. \tag{4. 1}$$

In the present paper, by a *complex space* we mean a space admitting a complex structure $F_i{}^h$ and a symmetric linear connection satisfying (4. 1).

Let there be given an almost contact hypersurface $\{M, C^h\}$ in V. Then taking account of (4. 1), we have from (3. 15)

$$\begin{aligned} &\nabla_c f_b{}^a+h_{cb} f^a-h_c{}^a f_b=0,\\ &\nabla_c f_b+h_{ce} f_b{}^e+l_c f_b=0,\\ &\nabla_c f^a-h_c{}^e f_e{}^a-l_c f^a=0,\\ &h_c{}^e f_e-h_{ce} f^e=0. \end{aligned} \tag{4. 2}$$

On the other hand, since $N_{ji}{}^h=0$, it follows from (3. 17) and (3. 18)

$$\begin{aligned} &S_{cb}{}^a=f_c(h_b{}^e f_e{}^a-f_b{}^e h_e{}^a)-f_b(h_c{}^e f_e{}^a-f_c{}^e h_e{}^a)+(f_c l_b-f_b l_c) f^a,\\ &S_{cb}=f_c h_b-f_b h_c+(f_c f_b{}^e-f_b f_c{}^e) l_e,\\ &S_c{}^a=-(h_c{}^a+h_f{}^a f_c{}^f f_d{}^a)+f_c h_e{}^a f^e-f_c{}^e l_e f^a,\\ &S_c=-f_c{}^e h_e-f_c f^e l_e+l_c, \end{aligned} \tag{4. 3}$$

where we have put

$$h_c=h_{ce} f^e=h_c{}^e f_e \tag{4. 4}$$

because of the last equation of (4. 2).

From the first equation of (4. 3), we have

PROPOSITION 4. 1. *For an almost contact hypersurface in a complex space V, we have $S_{cb}{}^a\equiv 0$.*

From the expression (2. 3) for L_{cb} and the second equation of (4. 2), we find

$$L_{cb}=-(h_{cb}+h_{fe} f_c{}^f f_b{}^e)+f_c h_b-f_c{}^e l_e f_b. \tag{4. 5}$$

Thus we have

PROPOSITION 4. 2. *For an almost contact hypersurface in a complex space V, we have the expression* (4. 5) *for L_{cb} and*

$$L_{cb} \equiv L_{bc}. \tag{4. 6}$$

(Hermann [4]).

We have, from the second and the third equations of (4. 3),

$$\begin{aligned} h_{cb} &= (\nabla_c f_e) f_b{}^e + h_c f_b, \\ h_c{}^a &= -(\nabla_c f^e) f_e{}^a + h_c f^a, \end{aligned} \tag{4. 7}$$

the first of which, together with $h_{cb} = h_{bc}$, implies

$$(\nabla_c f_e) f_b{}^e - (\nabla_b f_e) f_c{}^e + h_c f_b - h_b f_c = 0. \tag{4. 8}$$

Substituting (4. 7) in the first equation of (4. 2), we get

$$\nabla_c f_b{}^a + (\nabla_c f_e) f_b{}^e f^a + (\nabla_c f^e) f_e{}^a f_b = 0. \tag{4. 9}$$

We have moreover from (4. 2)

$$(\nabla_c f_e) f^e = -l_c, \qquad (\nabla_c f^e) f_e = l_c. \tag{4. 10}$$

If we assume that the almost contact hypersurface is normal, that is, $S_{cb}{}^a = 0$, and consequently $S_{cb} = 0$, then from the second equation of (4. 3), we find

$$h_b + f_b{}^e l_e = \lambda f_b, \tag{4. 11}$$

where we have put

$$\lambda = h_e f^e = h_{fe} f^f f^e = h_e{}^d f^e f_d.$$

Thus (4. 5) becomes

$$L_{cb} = -(h_{cb} + h_{fe} f_c{}^f f_b{}^e) + \lambda f_c f_b. \tag{4. 12}$$

We have thus

PROPOSITION 4. 3. *If an almost contact hypersurface in a complex space V is normal, we have equation* (4. 11) *and the expression* (4. 12) *for L_{cb}.*

In the normal case, we have from the first and the third equation of (4. 3)

$$h_c{}^e f_e{}^a - f_c{}^e h_e{}^a + l_c f^a = f_c p^a, \tag{4. 13}$$

$$h_c{}^a + h_f{}^d f_c{}^f f_d{}^a = f_c h_e{}^a f^e - f_c{}^e l_e f^a, \tag{4. 14}$$

where

$$p^a = h_c{}^e f_e{}^a f^c + f^c l_c f^a.$$

From (4. 14) we find (4. 11) and consequently

$$h_c{}^e f_e{}^a - f_c{}^e h_e{}^a + l_c f^a = f_c (h_f{}^d f^f f_d{}^a + f^e l_e f^a), \tag{4. 15}$$

which is equivalent to (4. 13). Thus we see that (a) $S_{cb}{}^a=0$, (b) (4. 13), (c) (4. 14) are all equivalent to each other.

On the other hand, from the third equation of (4. 2), we have

$$h_c{}^a=-(\nabla_c f^e)f_e{}^a+h_c f^a. \tag{4.16}$$

Substituting (4. 16) into (4. 13), we find

$$\nabla_c f^a+(\nabla_f f^d)f_c{}^f f_d{}^a-f_c{}^e h_e f^a=f_c p^a. \tag{4.17}$$

Conversely, if (4. 17) is satisfied, we have (4. 13) by virtue of the third equation of (4. 2). Thus we see that (4. 13) and (4. 17) are equivalent to each other. Thus we have

THEOREM 4. 1. *For an almost contact hypersurface in a complex space, the following conditions are equivalent to each other*:

(a) $S_{cb}{}^a=0$,

(b) $h_c{}^e f_e{}^a-f_c{}^e h_e{}^a+l_c f^a=f_c(h_e{}^d f^e f_d{}^a+f^e l_e f^a)$,

(c) $h_c{}^a+h_e{}^d f_c{}^e f_d{}^a=f_c h_e{}^a f^e-f_c{}^e l_e f^a$,

(d) $\nabla_c f^a+(\nabla_e f^d)f_c{}^e f_d{}^a-l_c f^a=f_c(f^e\nabla_e f^a-f^e l_e f^a),\qquad h_b+f_b{}^e l_e=\lambda f_b$.

Since (b), (c) and (d) above can respectively written as (b′) $S_c{}^e f_e{}^a=0$, (c′) $S_c{}^b=0$ and (d′) $\qquad O_{cd}^{ea}(\nabla_e f^d)=(l_c-f_c f^e l_e)f^a,\qquad S_c=0,$

where

$$O_{cd}^{ea}=\delta_c^e\delta_d^a+f_c{}^e f_d{}^a-f_c f^e\delta_d^a, \tag{4.18}$$

we have

THEOREM 4. 2. *For an almost contact hypersurface in a complex space, the following conditions are equivalent to each other*:

(a) $S_{cb}{}^a=0$, (b′) $S_c{}^e f_e{}^a=0$, (c′) $S_c{}^a=0$,

(d′) $O_{cd}^{ea}(\nabla_e f^d)=(l_c-f_c f^e l_e)f^a,\qquad S_c=0$.

Coming back to the general case, if we take account of (4. 2), we have from (2. 2) and (4. 3)

$$N_{cb}{}^a=f_c(h_b{}^e f_e{}^a-f_b{}^e h_e{}^a)-f_b(h_c{}^e f_e{}^a-f_c{}^e h_e{}^a)+(h_{ce}f_b{}^e-h_{be}f_c{}^e)f^a. \tag{4.19}$$

Thus, if $N_{cb}{}^a=0$, then we have from (4. 19)

$$\begin{aligned} &h_{ce}f_b{}^e-h_{be}f_c{}^e=f_c(f_b{}^e h_e)-f_b(f_c{}^e h_e),\\ &h_c{}^e f_e{}^a-f_c{}^e h_e{}^a=f_c(h_e{}^d f^e f_d{}^a)-(f_c{}^e h_e)f^a. \end{aligned} \tag{4.20}$$

Conversely, if (4. 20) are satisfied, we have $N_{cb}{}^a=0$.

The second equation of (4. 20) is equivalent to

$$S_{cb}{}^a=(f_c S_b-f_b S_c)f^a.$$

On the other hand, the first equation of (4. 20) is equivalent to

$$L_{cb} \equiv 0,$$

if we take account of the expression (4. 5) for L_{cb}. Thus we have

THEOREM 4. 3. *For an almost contact hypersurface in a complex space, a necessary and sufficient condition for the induced f-structure $f_b{}^a$ to be integrable is that*

$$\text{(a)} \quad S_{cb}{}^a = (f_c S_b - f_b S_c) f^a, \qquad \text{(b)} \quad L_{cb} \equiv 0,$$
$$= (f_b f_c{}^e h_e - f_c f_b{}^e h_e) f^a.$$

We next suppose that the distribution M, i.e. $f_a d\eta^a = 0$ is integrable. A necessary and sufficient condition for that is $N_{cb}{}^a f_a = 0$. If this is the case, we have from (4. 19) that

$$h_{ce} f_b{}^e - h_b{}^e f_c{}^e \equiv 0 \tag{4. 21}$$

and consequently $N_{cb}{}^a \equiv 0$ which means that the f-structure $f_b{}^a$ is partially integrable. Conversely, if $N_{cb}{}^a \equiv 0$, then we have (4. 21) and consequently $N_{cb}{}^a f_a \equiv 0$. But, if we take account of (4. 12) and (4. 19), we see that the three conditions $N_{cb}{}^a \equiv 0$, $L_{cb} \equiv 0$ and $h_{cb} + h_{fe} f_c{}^f f_b{}^e \equiv 0$ are equivalent to each other. Thus, we have

THEOREM 4. 4. *For an almost contact hypersurface in a complex space, the following four conditions are equivalent to each other:*

(a) *The distribution M determined by the projection operator $l_c{}^a = -f_c{}^b f_b{}^a$ is integrable* ($N_{cb}{}^a f_a \equiv 0$).

(b) *The induced f-structure $f_b{}^a$ is partially integrable* ($N_{cb}{}^a \equiv 0$).

(c) $L_{cb} \equiv 0$.

(d) $h_{cb} + h_{fe} f_c{}^f f_b{}^e \equiv 0$.

Combining Theorems 4. 2, 4. 3 and 4. 4, we have

THEOREM 4. 5. *For a normal almost contact hypersurface in a complex space* ($S_{cb}{}^a = 0$), *the following three conditions are equivalent to each other:*

(a) *The induced f-structure $f_b{}^a$ is integrable* ($N_{cb}{}^a = 0$).

(b) *The induced f-structure is partially integrable* ($N_{cb}{}^a \equiv 0$).

(c) *The distribution M, i.e. $f_a d\eta^a = 0$ is integrable* ($N_{cb}{}^a f_a \equiv 0$).

We see from (4. 7) and (4. 9), taking account of (4. 10), that following three conditions are equivalent to each other:

(a) $\nabla_c f_b{}^a = 0$,

(b) $\nabla_c f_b = -l_c f_b, \qquad \nabla_c f^a = l_c f^a$,

(c) $h_{cb} = h_c f_b, \qquad h_c{}^a = h_c f^a$.

When the condition (c) above is satisfied, we have, from $h_{cb}=h_b f_c$, $h_b=\lambda f_b$. Therefore, we have

THEOREM 4.6. *For an almost contact hypersurface in a complex space, the following three conditions are equivalent to each other*:

(a) $\nabla_c f_b{}^a=0$.

(b) $\nabla_c f_b=-l_c f_b$ *and* $\nabla_c f^a=l_c f^a$.

(c) $h_{cb}=\lambda f_c f_b$ *and* $h_c{}^a=\lambda f_c f^a$,

λ *being a certain function. When one of these conditions is satisfied, the induced* f*-structure* $f_b{}^a$ *is integrable.*

§5. Hypersurfaces in a locally flat complex space.

We consider in this section a complex space V which is locally flat, i.e. whose curvature tensor vanishes identically. Taking an almost contact hypersurface $\{M, C^h\}$ in such a complex space V, we obtain

$$\nabla_d\nabla_c B_b{}^h-\nabla_c\nabla_d B_b{}^h=-R_{dcb}{}^a B_a{}^h, \tag{5.1}$$

$$\nabla_d\nabla_c C^h-\nabla_c\nabla_d C^h=0, \tag{5.2}$$

as a consequence of the local flatness of V, where

$$R_{dcb}{}^a=\partial_d\Gamma^a_{cb}-\partial_c\Gamma^a_{db}+\Gamma^a_{de}\Gamma^e_{cb}-\Gamma^a_{ce}\Gamma^e_{db}$$

is the curvature tensor of the induced connection Γ^a_{cb}. Substituting (3.7) and (3.10) in the left hand sides of (5.1) and (5.2) we have

$$\begin{aligned} &R_{dcb}{}^a=h_d{}^a h_{cb}-h_c{}^a h_{db},\\ &\nabla_d h_{cb}-\nabla_c h_{db}+l_d h_{cb}-l_c h_{db}=0,\\ &\nabla_d h_c{}^a-\nabla_c h_d{}^a+h_d{}^a l_c-h_c{}^a l_d=0,\\ &h_{de}h_c{}^e-h_{ce}h_d{}^e=\nabla_d l_c-\nabla_c l_d. \end{aligned} \tag{5.3}$$

We suppose that an almost contact hypersurface $\{M, C^h\}$ of a locally flat complex space V satisfies

$$h_{cb}=\lambda f_c f_b, \qquad h_c{}^a=\lambda f_c f^a. \tag{5.4}$$

Substituting these equations in (5.3), we get

$$R_{dcb}{}^a=0, \qquad \nabla_d l_c-\nabla_c l_d=0, \qquad \nabla_b\lambda+\lambda l_b\equiv 0 \tag{5.5}$$

by means of Theorem 4.6. Summing up, we have

THEOREM 5.1. *If, in a locally flat complex space, an almost contact hypersurface satisfies one of three conditions* (a), (b), (c) *mentioned in Theorem* 4.6, *the induced connection is locally flat and the function* λ *appearing in Theorem* 4.6 *satisfy* $\nabla_b\lambda+\lambda l_b\equiv 0$.

Let there be given an almost contact hypersurface $\{M, C^h\}$ satisfying (5. 4) in a locally flat complex space V. If we suppose M to be simply connected, there exists in M, by virtur of the second equation of (5. 5), a function σ such that $\sigma_b=\nabla_b l$. On making use of this function σ, we put

$$\bar{C}^h=\rho^{-1}C^h, \qquad \rho=e^{\sigma}. \tag{5. 6}$$

We have thus a new hypersurface $\{M, \bar{C}^h\}$ having the submanifold M as its basic submanifold. The induced almost contact structure of the new hypersurface $\{M, \bar{C}^h\}$ is $(\bar{f}_b{}^a, \bar{f}^a, \bar{f}_b)$ as a consequence of (3. 13), where $\bar{f}_b{}^a$ is the same as the corresponding one of $\{M, C^h\}$ and

$$\bar{f}^a=\rho f^a, \qquad \bar{f}_b=\rho^{-1}f_b. \tag{5. 7}$$

The induced connection of $\{M, \bar{C}^h\}$ is the same as the induced connection Γ_{cb}^a of $\{M, C^h\}$ because of (3. 4). If we take account of (3. 10) and (5. 6), we have

$$\nabla_c\bar{C}^h=-\bar{h}_c{}^aB_a{}^h, \qquad \bar{h}_c{}^a=\rho^{-1}h_c{}^a, \tag{5. 8}$$

$\bar{l}_c$ being zero, and, if we take account of (3. 7) and (5. 6), we find

$$\nabla_cB_b{}^h=\bar{h}_{cb}\bar{C}^h, \qquad \bar{h}_{cb}=\rho h_{cb}, \tag{5. 9}$$

where $\bar{h}_{cb}$ and $\bar{h}_c{}^a$ are tensors determined, corresponding to h_{cb} and $h_c{}^a$, by the new hypersurface $\{M, \bar{C}^h\}$. We have from (5. 4), (5. 7), (5. 8) and (5. 9)

$$\bar{h}_{cb}=\bar{\lambda}\bar{f}_c\bar{f}_b, \qquad \bar{h}_c{}^a=\bar{\lambda}\bar{f}_c\bar{f}^a,$$

where we have put $\bar{\lambda}=\rho^{-1}\lambda$. Therefore we have from Theorem 4. 6

$$\nabla_c\bar{f}_b=0, \qquad \nabla_c\bar{f}^a=0.$$

Summing up, we have

THENREM 5. 2. *If, in a locally flat complex space V there exists an almost contact hypersurface $\{M, C^h\}$ with simply connected basic submanifold M, and if the hypersurface $\{M, C^h\}$ satisfies*

$$h_{cb}=\lambda f_cf_b, \qquad h_c{}^a=\lambda f_cf^a$$

with a certain function λ, then there exists in V an almost contact hypersurface $\{M, \bar{C}^h\}$ such that

$$\nabla_c\bar{f}_b{}^a=0, \qquad \nabla_c\bar{f}^a=0, \qquad \nabla_c\bar{f}_b=0,$$

where $(\bar{f}_b{}^a, \bar{f}^a, \bar{f}_b)$ is the induced almost contact structure of $\{M, \bar{C}^h\}$ and

$$\bar{f}_b{}^a=f_b{}^a, \qquad \bar{f}^a=\rho f^a, \qquad \bar{f}_b=\rho^{-1}f_b, \qquad \bar{C}^h=\rho^{-1}C^h,$$

ρ being a certain function in the common basic submanifold M.

Let CA^m be the $2m$-dimensional space of m complex numbers $(z^1, z^2, \cdots, z^m)$, $2m$ being equal to $n+1$. If we put

$$z^{\alpha}=x^{\alpha}+\sqrt{-1}\,x^{m+\alpha}, \qquad (\alpha=1, 2, \cdots, m),$$

then $(x^{\alpha}, x^{m+\alpha})$ are cartesian coordinates in CA^m.

Let there be given an almost contact hypersurface $\{M, C^h\}$ satisfying

$$\nabla_c f_b{}^a=0, \qquad \nabla_c f^a=0, \qquad \nabla_c f_b=0,$$
$$h_{cb}=\lambda f_a f_b, \qquad h_c{}^a=\lambda f_c f^a.$$

Then, $\{M, C^h\}$ is congruent to a portion of the almost contact hypersurface $\{\tilde{M}, \tilde{C}^h\}$ under the group of all affine transformations operating on CA^m and preserving the complex structure of CA^m, where the submanifold $\tilde{M}$ is defined by the equations

$$x^m=\varphi(t), \qquad x^{2m}=\psi(t), \tag{5.10}$$

$\varphi(t)$ and $\psi(t)$ being certain differentiable functions, and the normal vector field $\tilde{C}^h$ by the equations

$$\begin{gathered} \tilde{C}^1=\cdots=\tilde{C}^{m-1}=0, \qquad \tilde{C}^{m+1}=\cdots=\tilde{C}^{2m-1}=0, \\ \tilde{C}^m=-\frac{d\psi}{dt}, \qquad \tilde{C}^{2m}=\frac{d\varphi}{dt}. \end{gathered} \tag{5.11}$$

Thus we have

THEOREM 5.3. *If, in the 2m-dimensional space CA^m of m complex numbers $z^{\alpha}=x^{\alpha}+\sqrt{-1}\,x^{m+\alpha}$ $(\alpha=1, 2, \cdots, m)$, there is given an almost contact hypersurface $\{M, C^h\}$ with simply connected basic submanifold M, and if the hypersurface $\{M, C^h\}$ satisfies*

$$h_{cb}=\lambda f_c f_b, \qquad h_c{}^a=\lambda f_c f^a$$

with a certain function λ, then the basic submanifold M is conjugate to a submanifold defined by equations (5.10) *under the group of all affine transformations operating on CA^m and preserving the complex structure.*

§6. Hypersurfaces in an almost Hermitian space.

We consider an *almost Hermitian manifold* V of differentiability C^{∞} with almost complex structure $F_i{}^h$ and almost Hermitian structure G_{ji}. Then we have

$$F_j{}^i F_i{}^h=-\delta_j^h, \tag{6.1}$$

$$F_j{}^l F_i{}^k G_{lk}=G_{ji} \tag{6.2}$$

and the tensor

$$F_{ji}=F_j{}^l G_{li} \tag{6.3}$$

is skew-symmetric. If the Riemannian connection defined by G_{ji} satisfies

$$\nabla_j F_{ih}+\nabla_i F_{hj}+\nabla_h F_{ji}=0, \tag{6.4}$$

then the manifold is called an *almost Kählerian manifold*. If the Nijenhuis tensor $N_{ji}{}^h$ defined by (3.16) vanishes identically, the almost Hermitian manifold is called a *Hermitian manifold* and the almost Kählerian manifold a *Kählerian manifold*. A necessary and sufficient condition for an almost Hermitian manifold to be a Kählerian manifold is given by

(6. 5) $$\nabla_j F_i^{\ h}=0.$$

(Cf. Yano [17])

We now consider an n-dimensional orientable submanifold M in an almost Hermitian manifold V, V being $(n+1)$-dimensional. Denoting by C^h the unit normal to the submanifold M, we have a hypersurface $\{M, C^h\}$ and $F_i^{\ h}C^i$ is tangent to M. Then to any orientable submanifold M of n dimensions there corresponds uniquely a hypersurface $\{M, C^h\}$ with the unit normal vector field C^h and consequently the submanifold M can be identified with the hypersurface $\{M, C^h\}$. We call such a submanifold M briefly an *almost contact metric hypersurface*.

If we consider a hypersurface M, there exists in M the induced almost contact structure $(f_b^{\ a}, f^a, f_b)$ defined by (3. 13) and it satisfies the condition (3. 14). On the other hand the induced Riemannian metric on the hypersurface M is given by

(6. 6) $$g_{cb}=G_{ji}B_c^{\ j}B_b^{\ i}.$$

Transvecting (6. 2) with $B_c^{\ j}B_b^{\ i}$ and taking account of (3. 13), we find

(6. 7) $$f_c^{\ f}f_b^{\ e}g_{fe}+f_cf_b=g_{cb}.$$

Transvecting (6. 2) with $B_c^{\ j}C^i$ and taking account of (3. 13), we have

(6. 8) $$f_c^{\ f}f^ag_{fa}=0$$

or

(6. 9) $$f_{ca}f^a=0,$$

where

(6. 10) $$f_{ca}=f_c^{\ f}g_{fa}$$

is a skew-symmetric tensor.

Finally, transvecting (6. 2) with C^jC^i and taking account of (3. 13), we find

(6. 11) $$f^cf^bg_{cb}=1.$$

Thus, from $f^af_a=1$, (6. 8) and (6. 11), we see that

(6. 12) $$f^cg_{cb}=f_b.$$

Summing up, we see from (6. 7) and (6. 12) that in a hypersurface M of an almost Hermitian space V there exists an almost contact metric structure $(f_b^{\ a}, f^a, f_b, g_{cb})$ composed of the induced almost contact structure $(f_b^{\ a}, f^a, f_b)$ and the induced Riemannian metric g_{cb}. We call this structure the *induced almost contact metric structure* of the almost contact metric hypersurface.

Now the equations of Gauss (3. 7) and those of Weingarten (3. 10) in our metric case are respectively given by

(6. 13) $$\nabla_cB_b^{\ h}=h_{cb}C^h,$$

(6. 14) $$\nabla_c C^h = -h_c{}^a B_a{}^h,$$

where

(6. 15) $$h_c{}^a = h_{cb} g^{ba}$$

and the vector field l_b defined by (3. 5) vanishes identically.

Thus, differentiating (3. 13) covariantly along the hypersurface, we find the expressions (3. 15) for $(\nabla_j F_i{}^h) B_c{}^j B_b{}^i$ and $(\nabla_j F_i{}^h) B_c{}^j C^i$ with vanishing l_b, which imply the expressions (3. 17) for $N_{ji}{}^h B_c{}^j B_b{}^i$ and $N_{ji}{}^h B_c{}^j C^i$ with vanishing l_b.

§7. Hypersurfaces in a Kählerian space.

We assume now that the enveloping almost Hermitian manifold V is a Kählerian space. Then for an almost contact metric hypersurface of V we have from (3. 15) with vanishing l_b

(7. 1) $$\nabla_c f_b{}^a + h_{cb} f^a - h_c{}^a f_b = 0, \qquad \nabla_c f_b + h_{ce} f_b{}^e = 0,$$

because of $\nabla_j F_i{}^h = 0$, where the second equation is equivalent to

$$\nabla_c f^a - h_c{}^e f_e{}^a = 0.$$

When the second fundamental tensor h_{cb} has the form

(7. 2) $$h_{cb} = \alpha g_{cb} + \beta f_c f_b,$$

we say that the almost contact metric hypersurface is *contact umbilic* [13]. We have from (7. 1) and (2. 23)

PROPOSITION 7. 1. *When the almost contact metric hypersurface in a Kählerian manifold is contact umbilic, we have*

(7. 3) $$\nabla_c f_{ba} = -\alpha (g_{cb} f_a - g_{ca} f_b), \qquad \nabla_c f_b = \alpha f_{cb}$$

and the hypersurface is contact metric.

The enveloping almost Hermitian manifold being Kählerian, we have expressions (4. 3) for $S_{cb}{}^a$, S_{cb}, $S_c{}^a$, S_c with vanishing l_b and from (4. 5)

(7. 4) $$L_{cb} = -(h_{cb} + h_{fb} f_b{}^f f_q{}^e) + f_c h_b.$$

From the expression (4. 3) for $S_{cb}{}^a$ with vanishing l_b and (7. 2) we have

PROPOSITION 7. 2. *When the almost contact metric hypersurface in a Kählerian manifold is contact umbilic, then the almost contact metric hypersurface is normal.* (For the totally geodesic case, see Okumura [7, 8], Tashiro [11, 12])

Since the expression (4. 3) for $S_c{}^a$ can be written as

(7. 5) $$S_c{}^e g_{eh} = -(h_{cb} - h_{fe} f_c{}^f f_b{}^e) + f_c h_b,$$

we have from (4. 3)

PROPOSITION 7. 3. *For an almost contact metric hypersurface in a Kählerian manifold, the condition*

(7. 6) $$h_c = h_c{}^a f_a = \lambda f_c$$

with $\lambda = h_c f^c = h_{cb} f^c f^b$ *is equivalent to one of the following conditions*

(7. 7) $$S_{cb} = 0, \qquad S_c{}^e g_{eb} = S_b{}^e g_{ec}, \qquad S_c = 0.$$

(Cf. Okumura [7])

We shall now study the case in which the induced almost contact metric structure is normal, that is, $S_{cb}{}^a = 0$. We know that in this case all the S's vanish. Thus from (4. 3) with vanishing l_b, we have

(7. 8)
$$f_c(h_b{}^e f_e{}^a - f_b{}^e h_e{}^a) - f_b(h_c{}^e f_e{}^a - f_c{}^e h_e{}^a) = 0,$$
$$f_c h_b - h_c f_b = 0,$$
$$h_c{}^a + h_f{}^d f_c{}^f f_d{}^a = f_c h^a,$$
$$h_e f_c{}^e = 0,$$

from which

$$h_c{}^a + h_f{}^d f_c{}^f f_d{}^a = \lambda f_c f^a,$$

or equivalently

$$h_{cb} - h_{fe} f_c{}^f f_b{}^e = \lambda f_c f_b.$$

Conversely, if one of these is satisfied, we have $h_c = \lambda f_c$ and

$$h_c{}^e f_e{}^a - f_c{}^e h_e{}^a = 0,$$

and consequently all S's vanish. Thus we have

PROPOSITION 7. 4. *A necessary and sufficient condition for an almost contact metric hypersurface in a Kählerian manifold to be normal is*

(7. 9) $$h_c{}^a + h_f{}^d f_c{}^f f_d{}^a = \lambda f_c f^a$$

or equivalently

(7. 10) $$h_{cb} - h_{fe} f_c{}^f f_b{}^e = \lambda f_c f_b.$$

If (7. 9) is satisfied, then we have $h_c{}^a f_a = \lambda f_c$ and consequently we have, from the third equation of (4. 3) with vanishing l_b, $S_c{}^a = 0$. Conversely, if $S_c{}^a = 0$, then we see from (7. 5) that $h_b = \lambda f_b$ and consequently (7. 9) is satisfied because of (7. 8). Thus we have

PROPOSITION 7. 5. *A necessary and sufficient condition for an almost contact metric hypersurface in a Kählerian manifold to be normal is* $S_c{}^a = 0$. (Cf. Okumura [7])

Now (7. 8) is equivalent to

(7. 11) $$h_{ce}f_b{}^e+h_{be}f_c{}^e=0$$

which is, by virtue of the second equation of (7. 1), equivalent to

(7. 12) $$\nabla_c f_b+\nabla_b f_c=0.$$

This means that the vector field f^a is an infinitesimal translation, since f^a is a unit vector field. Thus we have

PROPOSITION 7. 6. *A necessary and sufficient condition for an almost contact metric hypersurface in a Kählerian manifold to be normal is that f^a defines an infinitesimal translation.* (Okumura [8])

Now if (7. 10) is satisfied, then we have $h_c=\lambda f_c$ and consequently (7. 4) gives

$$L_{cb}=-(h_{cb}+h_{fe}f_c{}^f f_b{}^e)+\lambda f_c f_b$$

which becomes, by virtue of (7. 10)

$$L_{cb}=-2h_{fe}f_c{}^f f_b{}^e.$$

Conversely, if L_{cb} has this form, then $S_{cb}=L_{cb}-L_{bc}=0$ and Proposition 7. 3 gives $h_c=\lambda f_c$, and consequently from (7. 4) and the equation above

$$h_{cb}-h_{fe}f_c{}^f f_b{}^e=\lambda f_c f_b.$$

Thus we have

PROPOSITION 7. 7. *A necessary and sufficient condition for an almost contact metric hypersurface in a Kählerian manifold to be normal is*

(7. 13) $$L_{cb}=-2h_{fe}f_c{}^f f_b{}^e.$$

From Propositions 7. 3, 7. 4, 7. 5, 7. 6 and 7. 7, we get

THEOREM 7. 1. *A necessary and sufficient condition for an almost contact metric hypersurface in a Kählerian manifold to be normal is that one of the following equivalent conditions is satisfied.*

(a) $$h_c{}^a+h_f{}^d f_c{}^f f_d{}^a=\lambda f_c f^a,$$

(b) $$h_{cb}-h_{fe}f_c{}^f f_b{}^e=\lambda f_c f_b,$$

(c) $$S_c{}^a=0,$$

(d) $$\nabla_c f_b+\nabla_b f_c=0,$$

(e) $$L_{cb}=-2h_{fe}f_c{}^f f_b{}^e.$$

If we assume now that an almost contact metric hypersurface in a Kählerian manifold is contact metric, then we have from (2. 24) and (7. 4)

(7. 14) $$h_{cb}+h_{fe}f_c{}^f f_b{}^e=2ag_{cb}-f_c\,(2af_b-h_b)$$

with a non-zero constant a. Transvecting (7. 14) with f^b, we find

$$h_c = h_{cb}f^b = \lambda f_c,$$

which implies together with (7. 14)

$$h_{cb} + h_{fe}f_c{}^f f_b{}^e = 2ag_{cb} + (\lambda - 2a)f_c f_b.$$

If we assume moreover for the hypersurface to be normal, then we have from Theorem 7. 1

$$h_{cb} - h_{fe}f_c{}^f f_b{}^e = \lambda f_c f_b.$$

Adding these two equations, we obtain

$$h_{cb} = ag_{cb} + (\lambda - a)f_c f_b.$$

Thus, taking account of Propositions 7. 1 and 7. 2, we have

THEOREM 7. 2. *A necessary and sufficient condition for an almost contact hypersurface in a Kählerian manifold to be normal and contact metric is that it is contact umbilic.* (Tashiro [11, 12], Tashiro and Tachibana [13])

We now assume that the tensor $f_b{}^a$ is a Killing tensor, that is, it satisfies (Yano [16])

$$\nabla_c f_b{}^a + \nabla_b f_c{}^a = 0.$$

Then, we have from the first equation of (7. 1)

$$2h_{cb}f^a - h_c{}^a f_b - h_b{}^a f_c = 0,$$

from which

$$h_{cb} = \lambda f_c f_b,$$

and consequently

$$\nabla_c f_b{}^a = 0, \qquad \nabla_c f_b = 0.$$

The converse being evident, we have

PROPOSITION 7. 8. *A necessary and sufficient condition for $f_b{}^a$ of an almost contact metric hypersurface in a Kählerian manifold to be a Killing tensor is*

(7. 15) $$h_{cb} = \lambda f_c f_b,$$

or equivalently

(7. 16) $$\nabla_c f_b{}^a = 0, \qquad \nabla_c f_b = 0.$$

We next assume that the tensor f_{ba} is harmonic. Since we have, from the first equation of (7. 1)

$$\nabla_c f_{ba} + \nabla_b f_{ac} + \nabla_a f_{cb} = 0,$$

the condition for f_{ba} to be harmonic is

$$g^{cb}\nabla_c f_b{}^a = -g^{cb}(h_{cb}f^a - h_c{}^a f_b) = 0,$$

from which

$$h_c = h_c{}^a f_a = \lambda f_c$$

with

$$\lambda = h_{cb} f^c f^b = h_{cb} g^{cb}.$$

The converse being evident, we have

PROPOSITION 7.9. *A necessary and sufficient condition for $f_b{}^a$ of an almost contact metric hypersurface in a Kählerian manifold to be harmonic is*

$$h_c = h_c{}^a f_a = \lambda f_c, \qquad \lambda = h_{cb} f^c f^b = h_{cb} g^{cb}. \tag{7.17}$$

We next study the case in which f_c is harmonic. Since the second equation of (7.1) gives $g^{cb} \nabla_c f_b = 0$, the condition for f_c to be harmonic is

$$\nabla_c f_b - \nabla_b f_c = 0. \tag{7.18}$$

In this case, we have $L_{cb} = 0$ from the definition of L_{cb}. Conversely, if

$$L_{cb} = f_c{}^e (\nabla_e f_b - \nabla_b f_e) = 0,$$

then we have

$$f_c{}^f L_{fb} = -(\nabla_c f_b - \nabla_b f_c) + f_c f^e \nabla_e f_b = 0,$$

from which, $f^b \nabla_b f_c = 0$, and consequently

$$\nabla_c f_b - \nabla_b f_c = 0.$$

Thus a necessary and sufficient condition for f_c to be harmonic is $L_{cb} = 0$. In this case, we have first of all

$$S_{cb} = 0, \qquad S_c = 0, \qquad h_c = \lambda f_c$$

and from the second equation of (7.1)

$$h_{ce} f_b{}^e - h_{be} f_c{}^e = 0,$$

from which

$$h_{cb} + h_{fe} f_c{}^f f_b{}^e = \lambda f_c f_b,$$

$$h_c{}^e f_e{}^a + f_c{}^e h_e{}^a = 0,$$

$$h_c{}^a - h_f{}^d f_c{}^f f_d{}^a = \lambda f_c f^a$$

and consequently $S_{cb}{}^a$ and $S_c{}^a$ take respectively the forms

$$S_{cb}{}^a = 2(f_c h_b{}^e - f_b h_c{}^e) f_e{}^a,$$

$$S_{cb} = 2 h_f{}^d f_c{}^f f_d{}^a.$$

Thus we have

THEOREM 7.3. *A necessary and sufficient condition for the vector field f_c of an almost contact metric hypersurface in a Kählerian manifold to be harmonic is $L_{cb} = 0$ and in this case the S's are given by*

$$S_{cb}{}^a = 2(f_c h_b{}^e - f_b h_c{}^e) f_e{}^a, \qquad S_{cb} = 0,$$
(7.19)
$$S_c{}^a = 2h_f{}^a f_c{}^f f_d{}^a, \qquad S_c = 0.$$

Thus, if f_c is harmonic, we have $h_c = \lambda f_c$ and

$$L_{cb} = -(h_{cb} + h_{fe} f_c{}^f f_b{}^e) + \lambda f_c f_b = 0,$$

from which, transvecting with g^{cb},

$$\lambda = h_{cb} f^c f^b = h_{cb} g^{cb}.$$

Thus, from Proposition 7.9 and Theorem 7.3, we have

PROPOSITION 7.10. *If, for an almost contact metric hypersurface in a Kählerian manifold, the vector field f_c is harmonic, then the tensor field f_{cb} is also harmonic.*

If we assume finally that the induced f-structure $f_b{}^a$ of an almost contact metric hypersurface in a Kählerian manifold is integrable, i.e. $N_{cb}{}^a = 0$, then taking account of $l_b = 0$, we find from (4.20)

$$h_{cb} = f_c h_b + f_b h_c - \lambda f_c f_b, \qquad \lambda = h_{cb} f^c f^b \tag{7.20}$$

which is equivalent to

$$\nabla_c f_b{}^a = f_c (f_b h^a - h_b f^a), \tag{7.21}$$

or

$$\nabla_c f_b = -f_c f_b{}^e h_e \tag{7.22}$$

by virtue of (4.2) with vanishing l_b. Thus we have

PROPOSITION 7.11. *A necessary and sufficient condition for the induced f-structure $f_b{}^a$ of an almost contact metric hypersurface in a Kählerian manifold to be integrable is that one of the conditions* (7.20), (7.21) *and* (7.22) *be satisfied.*

We assume that $S_{cb}{}^a = 0$ and $N_{cb}{}^a = 0$. Then $S_{cb}{}^a = 0$ implies $S_c = 0$, from which we find

$$h_b = \lambda f_b.$$

Therefore, we have

$$h_{cb} = \lambda f_c f_b, \qquad \nabla_c f_b{}^a = 0, \qquad \nabla_c f_b = 0$$

respectively from (7.20), (7.21), (7.22). We have thus

THEOREM 7.4. *A necessary and sufficient condition that the induced f-structure be integrable for a normal almost contact metric hypersurface of an Kählerian manifold is that one of the following conditions be satisfied:*

(a) $\nabla_c f_b{}^a = 0$, (b) $\nabla_c f_b = 0$, (c) $h_{cb} = \lambda f_c f_b$.

§ 8. Hypersurfaces in a locally flat Kählerian space.

We suppose that the enveloping manifold V is a Fubini space. Then the curvature tensor of V is given by (See for example Yano [17])

$$K_{kjih}=k(G_{kh}G_{ji}-G_{jh}G_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})$$

with a constant k, where $F_{ji}=F_j{}^lG_{li}$. We consider now an almost contact metric hypersurface in a Fubini space V. Substituting the above expression of K_{kjih} in the equations of Gauss and Codazzi

$$K_{kjih}B_d{}^kB_c{}^jB_b{}^iB_a{}^h=K_{dcba}-(h_{da}h_{cb}-h_{ca}h_{db}),$$

$$K_{jjih}B_d{}^kB_c{}^jB_b{}^iC^h=\nabla_dh_{cb}-\nabla_ch_{db},$$

we find

$$k(g_{da}g_{cb}-g_{ca}g_{db}+f_{da}f_{cb}-f_{ca}f_{db}-2f_{dc}f_{ba})=K_{dcba}-(h_{da}h_{cb}-h_{ca}h_{db}),$$

(8. 1)

$$k(f_df_{cb}-f_cf_{db}-2f_{dc}f_b)=\nabla_dh_{cb}-\nabla_ch_{db},$$

where K_{dcba} is the curvature tensor of the hypersurface. If we now assume that the hypersurface is normal and the induced f-structure $f_b{}^a$ is integrable, we have from Theorem 7. 4

$$\nabla_cf_b=0, \qquad h_{cb}=\lambda f_cf_b. \tag{8. 2}$$

If we take account of the well known formula

$$\nabla_d\nabla_cf_b-\nabla_c\nabla_df_b=-K_{dcb}{}^af_a,$$

we have by means of the first equation of (8. 2) $K_{dcba}f^a=0$, which implies $k=0$ and consequently $K_{kjih}=0$, $K_{dcba}=0$ as a consequence of (8. 1) and the second equation of (8. 2). Thus we have

THEOREM 8. 1. *In a Fubini space, which is not locally flat, there exists no normal almost contact hypersurface whose induced f-structure $f_b{}^a$ is integrable.* (Cf. Theorem 5. 2)

We have, taking account of (8. 2), from Theorem 5. 1

THEOREM 8. 2. *If, in the Euclidean space E^{n+1} of even dimensions with the natural Kählerian structure, there is given a normal almost contact hypersurface M such that its induced f-structure is integrable, then the basic submanifold M is conjugate to a portion of a submanifold defined by* (5. 10) *under the group of all motions of E^{n+1} preserving the complex structure.*

We now suppose that the enveloping Kählerian manifold is Euclidean, then we have equations of Gauss and Codazzi

$$K_{dcba}=h_{da}h_{cb}-h_{ca}h_{db}, \tag{8. 3}$$

$$\nabla_dh_{cb}-\nabla_ch_{db}=0. \tag{8. 4}$$

If the almost contact metric hypersurface is *normal*, the vector field f^a is a Killing vector field and consequently we have (Cf. [16])

$$\nabla_c \nabla_b f_a + K_{dcba} f^d = 0. \tag{8.5}$$

Substituting (7. 1) and (8. 3) into (8. 5), we find

$$-\nabla_c (h_{be} f_a{}^e) + (h_{da} h_{cb} - h_{ca} h_{db}) f^d = 0,$$

$$-(\nabla_c h_{be}) f_a{}^e - h_{be}(h_{ca} f^e - h_c{}^e f_a) + \lambda (f_a h_{cb} - f_b h_{ca}) = 0,$$

or

$$(\nabla_c h_{be}) f_a{}^e - h_{be} h_c{}^e f_a + \lambda h_{cb} f_a = 0,$$

from which, by transvecting f^a, we obtain

$$h_c{}^e h_e{}^a = \lambda h_c{}^a, \tag{8.6}$$

and consequently

$$(\nabla_c h_b{}^c) f_c{}^a = 0.$$

The last equation implies together with (8. 4)

$$\nabla_c h_{ba} = A f_c f_b f_a, \tag{8.7}$$

A being a certain function, since $h_{ba} = h_{ab}$.

Differentiating covariantly the both sides of the equation $h_{be} f^e = \lambda f_b$ and taking account of (8. 7), we have

$$A f_c f_b + h_{be} \nabla_c f^e = (\nabla_c \lambda) f_b + \nabla_c f_b,$$

from which, transvecting f^b and taking account of $(\nabla_c f_b) f^b = 0$, we find

$$\nabla_c \lambda = A f_c. \tag{8.8}$$

Differentiating covariantly the both sides of (8. 8), we have

$$\nabla_d \nabla_c \lambda = (\nabla_d A) f_c + A \nabla_d f_c,$$

from which, transvecting $\nabla^d f^c$, we obtain

$$A(\nabla_d f_c \nabla^d f^c) = 0, \tag{8.9}$$

where $\nabla^d f^c = g^{de} \nabla_e f^c$.

Now, we consider the case $A = 0$ everywhere. Then the hypersurface being supposed to be connected, (8. 8) implies that the function λ is constant. The equation (8. 7) implies $\nabla_c h_{ba} = 0$, which means that all proper values of $h_c{}^a$ are constant. Taking a proper value C of $h_c{}^a$, we have from (8. 6)

$$C^2 = \lambda C,$$

which implies $C = \lambda$ or $C = 0$. We denote by D_1 the distribution spanned by all proper vectors with proper value λ and by D_0 the distribution spanned by all proper vectors with proper value 0. Then, D_1 and D_0 are both integrable. In fact, we take two vector fields v^a and w^a belonging to D_1. From $h_c{}^a v^c = \lambda v^a$ we have

$$h_c{}^a (w^d \nabla_d v^c) = \lambda w^d \nabla_d v^a$$

by virtue of $\nabla_c h_{ba}=0$. This equation shows that $w^d \nabla_d v^a$ belongs to D_1. Therefore, D_1 is integrable. Similarily, we can show that the distribution D_0 is also integrable.

Now, we have

$$w^d \nabla_d (v^c B_c{}^h) = (w^d \nabla_d v^c) B_c{}^h,$$

v^a and w^a belonging respectively to D_1 and D_0, which shows that D_1 is parallel along the integral manifold of D_0. We have further

$$w^d \nabla_d C^h = 0$$

for any vector field w^a belonging to D_0, which shows that the normal vector field C^h is parallel along the integral manifold of D_0. Therefore, the integral manifold of D_0 is a portion of a plane.

If v^a belongs to D_1 and w^a belongs to D_0, supposing that the constant λ is non-zero, we have

$$v^c \nabla_c \left(\xi^h + \frac{1}{\lambda} C^h \right) = v^c B_c{}^h - \frac{1}{\lambda} h_c{}^a v^c B_a{}^h = 0,$$

$$w^c \nabla_c \left(\xi^h + \frac{1}{\lambda} C^h \right) = w^c B_c{}^h,$$

that is, if we move along the integral manifold of D_1, the point $\xi^h + (1/\lambda) C^h$ does not move and if we move along the integral manifold of D_0, the locus of the point $\xi^h + (1/\lambda) C^h$ is parallel to the integral manifold of D_0.

Summing up, we see that in case $\lambda \neq 0$, the almost contact metric hypersurface is a portion of

$$S^n \quad \text{or} \quad S^r \times E^{n-r} \quad (1 \leqq r < n),$$

where S^r denotes an r-dimensional sphere and E^s an s-plane. The dimension number r is necessarily odd. In fact, the vector field f^a belongs to D_1. If we take a vector v^a belonging to D_1 and being orthogonal to f^a, then we see by making use of (7. 11) that $f_c{}^a v^c$ belongs to D_1. Thus we see that the distribution D_1 is odd dimensional.

Suppose next that $\lambda = 0$. Then, we have

$$h_c{}^e h_e{}^a = 0. \tag{8. 10}$$

Let v^a be a proper vector of $h_c{}^a$ with proper value α. Then, we have, transvecting (8. 10) with v^a, $\alpha = 0$. Thus, all proper values of $h_c{}^a$ being zero, we have $h_c{}^a = 0$ and consequently the almost contact metric hypersurface is a portion of a hyperplane. Summing up, we have

PROPOSITION 8. 1. *If, for a normal almost contact metric hypersurface, which is connected, of the Euclidean space E^{n+1} of even dimensions with the natural flat Kählerian structure, the function λ is constant, i.e. if the function A vanishes identically, then the hypersurface is a portion of the following*

$$S^n, \qquad S^r \times E^{n-r}, \qquad E^n,$$

r being an odd number such that $1\leqq r<n$, where S^r is an r-dimensional sphere immbeded naturally in E^{n+1} and E_p is a p-plane.

In the next step, we consider the case in which the function A appearing in (8. 7) does not vanish somewhere. If this is the case, we find from (8. 9)

$$\nabla_c f_b=0$$

in the open set U consisting of all points where the function A does not vanish. Then, taking account of Theorems 7. 4 and 8. 2, we have

PROPOSITION 8. 2. *If, for a normal almost contact metric hypersurface of the Euclidean space E^{n+1} of even dimensions with the natural flat Kählerian structure, the function A does not vanish somewhere, then the open set U consisting of all points where A does not vanish is conjugate to a portion of a hypersurface defined by* (5. 10) *under the group of all motions of E^{n+1} preserving the complex structure and the curvature tensor of the hypersurface vanishes identically in U.*

Let W be the closed set complementary to the open set U. Then, the function A vanishes identically in W and consequently the function λ is constant in W because of (8. 8). Therefore, the set W^0 of all interior points of W is a portion of S^n, $S^r\times E^{n-r}$ or E^n by virtue of Proposition 8. 1. When W^0 is a portion of S^n or $S^r\times E^{n-r}$, W^0 is a symmetric space with non-vanishing curvature tensor, and consequently the curvature tensor of the hypersurface does not vanish in the boundary of W. On the other hand, by means of Proposition 8. 2, the curvature tensor vanishes identically in U, and hence vanishes in the boundary of U, that is, in the boundary of W. This contradicts the fact that the curvature tensor does not vanish in the boundary of W. Consequently, W^0 is not a portion of S^n or $S^r\times E^{n-r}$ and then it is necessarily a portion of a hyperplane E^n. Thus, we have

PPOPOSITION 8. 3. *If the function A does not vanish somewhere, the hypersurface is conjugate to a portion of a hypersurface defined by* (5. 10).

Combining Propositions 8. 1 and 8. 3, we have

THEOREM 8. 3. *If an almost contact metric hypersurface, which is connected, in the Euclidean space E^{n+1} of even dimensions with the natural flat Kählerian structure is normal, and, if it is complete, then the hypersurface is one of the following*

$$S^n, \qquad S^r\times E^{n-r}, \qquad E^n,$$

r being an odd number such that $1\leqq r<n$, or, the hypersurface is conjugate to a hypersurface defined by (5. 10). *When the hypersurface is S^n or $S^r\times E^{n-r}$, the rank of the Levi tensor L_{cb} is equal to $n-1$ or $r-1$, respectively. In other cases, L_{cb} is of rank* 0. (For analytic case, cf. Okumura [7])

Let there be given, in the Euclidean space E^{n+1}, an almost contact metric hypersurface and assume that its induced f-structure is integrable. Then, substituting (7. 20) in the expression (8. 3) of the curvature tensor, we find

$$K_{dcba}=(f_d h_c-f_c h_d)(f_b h_a-f_a h_b).$$

Thus, K_{dcba} vanishes if and only if $h_c=h_{cb}f^b=\lambda f_c$. Therefore, by virtue of Theorem 4. 3, $S_{cb}{}^a=0$ if $K_{dcba}=0$. Conversely, if we suppose $S_{cb}{}^a=0$, we have $h_c=\lambda f_c$ from Theorem 4. 3, and consequently $K_{dcba}=0$. Summing up, we have

THEOREM 8. 4. *In the Euclidean space of even dimension with the natural Kählerian structure, a necessary and sufficient condition for an almost contact metric hypersurface to be normal is that its curvature tensor vanish identically, its induced f-structure being assumed to be integrable.*

We consider now an almost contact umbilic hypersurface. It is normal and contact by virtue of Theorem 7. 2. The hypersurface being contact, the rank of the Levi tensor L_{cb} is necessarily equal to $n-1$. Then, we have from Theorem 8. 3

THEOREM 8. 5. *If, in the Euclidean space E^{n+1} of even dimensions with the natural Kahlerian structure, an almost contact metric hypersurface is contact umbilic, then the hypersurface is a portion of a hypersphere S^n.* (Tashiro [11, 12], Tashiro and Tachibana [13], Kurita [5])

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