ON COMPLEX ANALYTIC MAPPINGS BETWEEN TWO ULTRAHYPERELLIPTIC SURFACES

By Mitsuru Ozawa

§1. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $u^2=g(w)$, respectively, where G and g are two entire functions each of which has no zero other than an infinite number of simple zeros. Let φ be an analytic mapping from R into S. Let $\mathfrak{P}_S$ be the projection map $(w, u)\to w$. Let Φ be the sifted mapping $\mathfrak{P}_S\circ\varphi$, then Φ is an entire function on R. Let $T(r, \Phi)$ be the Nevanlinna-Selberg characteristic function of Φ. Let $N(r\ R)$ be the quantity $N(r, \mathfrak{X})$ defined by Selberg [8], which is essentially one half of the integrated Euler characteristic of R defined by Sario [5].

Definition 1. *If $T(r, \Phi)$ satisfies the inequality*

$$\varlimsup_{r\to\infty}\frac{N(r, R)}{T(r, \Phi)}>2,$$

then we say φ a semi-degenerate analytic mapping from R into S.

Let $\mathfrak{P}_R$ be the projection map $(z, y)\to z$. If φ satisfies $\mathfrak{P}_S\circ\varphi(p)=\mathfrak{P}_S\circ\varphi(q)$ for $p\neq q$, $\mathfrak{P}_R p=\mathfrak{P}_R q$, then we say that φ satisfies the rigidity of projection map.

Definition 2. *If φ satisfies the rigidity of projection map, then we say φ a rigid analytic mapping from R into S.*

In the present paper we shall prove the following somewhat interesting

Theorem 1. *If φ exists and is a rigid analytic mapping from R into S, then there exists a suitable entire function $h(z)$ of z in such a manner that $f(z)^2G(z)=g\circ h(z)$ for a suitable entire function $f(z)$ of z.*

If φ is a semi-degenerate analytic mapping from R into S, then it is a rigid analytic mapping. If Φ is not single-valued with respect to z, we have $N(r, R)<2T(r, \Phi)+O(1)$ by Selberg's ramification theorem and hence

$$\varlimsup_{r\to\infty}\frac{N(r, R)}{T(r, \Phi)}\leqq 2$$

holds in our ultrahyperelliptic case. This contradicts the semi-degeneracy. Thus Φ must be single-valued for z, which is the desired rigidity of φ.

Received February 11, 1965.

§ 2. **Proof of the theorem.** Let E_1 be the closed set of z satisfying one of the following conditions:

1) $\varphi(p)=\varphi(q)$ if $p\neq q$ but $\mathfrak{P}_R(p)=\mathfrak{P}_R(q)=z$,

2) The projections of all the branch points of R.

If E_1 has a cluster point z_0, which is not any point under a branch point of R, then $(\varphi_1-\varphi_2)^2$ is a single-valued regular function having a countable number of zeros around z_0, where φ_j is a branch of φ. Let p_j be a point on R whose image $\varphi(p_j)$ is a branch point of S. Let t and T be the local parameters around p_j and $\varphi(p_j)$, respectively. Then T is a regular function of t. If p_j is not a branch point of R, then we have

$$T(t)=\sum_{\nu=2}^{\infty} a_\nu t^\nu.$$

If p_j is a branch point of R, then we have by $\varphi_1\equiv\varphi_2$

$$T(t)=\sum_{\nu=2}^{\infty} a_\nu t^\nu,$$

which leads to a relation

$$w-w_j=\sum_{\nu=2}^{\infty} a_\nu(z-z_j)^\nu.$$

Then $\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}$ is a regular function of z having an infinite number of perfectly branched values, that is, every projection of the branch point of S is a perfectly branched value in Nevanlinna's sense. By the famous Nevanlinna ramification relation this is a contradiction. Thus E_1 must be a countable set in the z-plane. Let E_2 be the open set of z satisfying $\varphi(p)\neq\varphi(q)$ when $p\neq q$ but $\mathfrak{P}_R(p)=\mathfrak{P}_R(q)=z$. Then $\bar{E}_2$ is just the z-plane and $\varphi(\mathfrak{P}_R^{-1}(E_2))$ covers almost all parts of S excepting at most a countable set in S. By a slight discussion we can say that every branch point of R has its φ-image on a branch point of S and E_1 must coincide with the set of the projections of all the branch points of R. Let $h(z)$ be $\mathfrak{P}_S\circ\varphi\circ\mathfrak{P}_R^{-1}(z)$, then it is single-valued and analytic. Thus it is an entire function of z. Therefore $\varphi(R)$ covers S at most two exceptions having the same projection in the w-plane.

In the subsequent discussions it is necessary to consider the effects occurring from the choices of two analytic branches of $\mathfrak{P}_R^{-1}$ and $\mathfrak{P}_S^{-1}$ and to modify suitably as the case may be. In these cases we may adopt suitable sheet-exchanged analytic mappings. Then we can arrive at the same conclusion. Thus we do not list all the possible cases.

Let F be an analytic mapping from S into the F-plane, then $f\equiv F\circ\varphi(p)$ is an analytic mapping from R into the F-plane. Let $F^*(w)$ be the two-valued function corresponding to F, that is, $F^*(w)=F\circ\mathfrak{P}_S^{-1}(w)$. Let $f^*(z)$ be the corresponding function of f, that is, $f^*(z)=f\circ\mathfrak{P}_R^{-1}(z)$. Then we have the representations $F^*(w)=F_1^*(w)\pm F_2^*(w)\sqrt{g(w)}$, $f^*(z)=f_1^*(z)\pm f_2^*(z)\sqrt{G(z)}$. Therefore we have $\varphi=\mathfrak{P}_S^{-1}\circ h\circ\mathfrak{P}_R$ and

$$
\begin{aligned}
f_1^*(z) \pm f_2^*(z)\sqrt{G(z)} &= f\circ\mathfrak{P}_R^{-1}(z) = F\circ\varphi\circ\mathfrak{P}_R^{-1}(z) \\
&= F\circ\mathfrak{P}_S^{-1}\circ h\circ\mathfrak{P}_R\circ\mathfrak{P}_R^{-1}(z) = F\circ\mathfrak{P}_S^{-1}\circ h(z) \\
&= F^*\circ h(z) = F_1^*\circ h(z) \pm F_2^*\circ h(z)\sqrt{g\circ h(z)}.
\end{aligned}
$$

If ${}_1F$ is an analytic mapping from S into the ${}_1F$-plane, which preserves the projection map, that is, ${}_1F(p_1)={}_1F(q_1)$ for $p_1\neq q_1$, $\mathfrak{P}_S p_1=\mathfrak{P}_S q_1$, then ${}_1F\circ\varphi(p)={}_1F\circ\varphi(q)$ for $p\neq q$, $\mathfrak{P}_R p=\mathfrak{P}_R q$. Therefore, if we put ${}_1f={}_1F\circ\varphi$ and ${}_1f^*={}_1f\circ\mathfrak{P}_R^{-1}$, ${}_1F^*={}_1F\circ\mathfrak{P}_S^{-1}$, then ${}_1f^*$ and ${}_1F^*$ are single-valued and hence

$$ {}_1f_1^*(z)={}_1F_1^*\circ h(z). $$

Under these preparations we shall proceed to our original problem. If F is the analytic mapping from S into the F-plane which is defined by $F\circ\mathfrak{P}_S^{-1}(w)=\sqrt{g(w)}$, then we have

$$ f_1^*(z)\pm f_2^*(z)\sqrt{G(z)}=\sqrt{g\circ h(z)}. $$

Further if F is the analytic mapping from S into the F-plane which corresponds to $F\circ\mathfrak{P}_S^{-1}(w)=g(w)$, then we have

$$ f_1^{*2}+f_2^{*2}G\pm 2f_1^*f_2^*\sqrt{G}=g\circ h=f_1^{**} $$

and a single-valued function $g\circ h$ of z. Therefore $f_1^*f_2^*\equiv 0$. By the two-valuedness of $\sqrt{g\circ h(z)}$, $f_2^*\not\equiv 0$. Thus $f_1^*\equiv 0$. Hence we have the desired result: $g\circ h(z)=f_2^*(z)^2G(z)$.
This completes the proof of our theorem.

In our theorem f would depend on the representations of G and g. We shall indicate this by $f_{G,g}$. If R is represented by $y^2=\alpha(z)^2G(z)$ and S by $u^2=\beta(w)^2g(w)$ for some meromorphic functions $\alpha(z)$ and $\beta(w)$. Then we have

$$ \beta(h(z))^2g(h(z))=f_{\alpha^2G,\,\beta^2g}(z)^2\cdot\alpha(z)^2G(z) $$

for a suitable meromorphic function $f_{\alpha^2G,\,\beta^2g}$. Thus we have

$$ f_{G,g}(z)^2\cdot\beta(h(z))^2=f_{\alpha^2G,\,\beta^2g}(z)^2\cdot\alpha(z)^2. $$

§ 3. **Examples.** We shall give here some examples.

1) Let R be the proper existence domain of the function $\sqrt{e^z-1}$ and S be R itself. Then we have

$$ f(z)^2(e^z-1)=e^{h(z)}-1 $$

for a rigid analytic mapping φ. If $h(z)$ is a transcendental entire function, then by the Lemma in [3] we have

$$ \varlimsup_{r\to\infty}\frac{N_2(r,1)}{T(r)}=1, $$

where N_2 denotes the N function with respect to the simple 1-points of e^h. By Pólya's theorem [4] the Nevanlinna characteristic function $T(r)$ of e^h is of infinite order and hence $N_2(r,1)$ has infinite order. However the left hand side function $f(z)^2(e^z-1)$ has its $N_2(r,0)$ function of order at most one. This is a contradiction, Thus $h(z)$ must be a polynomial. Its degree is denoted by ν. Then we have that the $N_2(r,1)$ function of e^h is of order ν. However that of $f^2(e^z-1)$ has order at most one. Therefore ν must be equal to one. Therefore we have the equation

$$f(z)^2(e^z-1)=e^{\alpha z+\beta}-1.$$

Let z be $2n\pi i$, then

$$e^\beta\cdot x^n=1, \qquad x=e^{2\alpha\pi i}.$$

Let n be zero, then we have $e^\beta=1$ and hence $\beta=2p\pi i$. Let n be one, then we have $x=1$ and hence $\alpha=q$, an integer $\neq 0$. Therefore we have

$$f(z)^2(e^z-1)=e^{qz+2p\pi i}-1=e^{qz}-1.$$

If $q\neq\pm 1$, then

$$\frac{e^{qz}-1}{e^z-1}$$

has some simple zeros, which is absurd. Thus $q=\pm 1$ must hold. If $q=1$, then we have

$$f(z)^2\equiv 1, \qquad h(z)=z+2p\pi i.$$

If $q=-1$, then we have

$$f(z)^2=-e^{-z}, \qquad h(z)=-z+2p\pi i.$$

Correspondingly we have several analytic mappings. Each solution in the above gives two analytic mappings φ_{p1} and φ_{p2}, which is the sheet exchange of φ_{p1}. There is no other rigid analytic mapping from R into itself.

2) Let R be the surface defined by $y^2=(e^{z^2}-1)/z^2$ and S the surface defined by $y^2=e^z-1$. Then for an arbitrary rigid analytic mapping φ from R into S we have the equation

$$f(z)^2\frac{e^{z^2}-1}{z^2}=e^{h(z)}-1.$$

Quite similarly we can deduce that the function $h(z)$ is a polynomial of degree two. Further we have

$$\begin{cases} h=z^2+2n\pi i \\ f^2=z^2 \end{cases} \qquad \begin{cases} h=-z^2+2n\pi i \\ f^2=-z^2/e^{z^2}. \end{cases}$$

Correspondingly we have several rigid analytic mappings. Thus there is no other rigid analytic mapping not corresponding to the above solutions.

To solve the equation

$$f^2G=g\circ h$$

is very difficult in almost all cases. It would be necessary to investigate more precisely the right hand side term by Nevanlinna's theory. If it is known that there is no analytic mapping from R into S, then this equation has no solution. For example, if the order ρ_G of G is of a finite non-integral value and g is $(e^p-\gamma)(e^p-\delta)$, $\gamma\neq\delta$, $\gamma\delta\neq 0$ with an entire function p, then $P(R)=2$ and $P(S)=4$ [2], [3]. Hence there is no analytic mapping from R into S. Thus we can say that

$$f^2 G=(e^{p\circ h}-\gamma)(e^{p\circ h}-\delta)$$

has no solution. Similarly we have the impossibility of the equations

$$f^2\frac{e^{2z}-1}{z}=e^{2h}-1, \qquad f\not\equiv 0$$

and

$$f^2\frac{(e^{2z}-1)(z-a)}{z}=e^{2h}-1, \qquad a\neq n\pi i, \qquad f\not\equiv 0,$$

respectively. However it had not yet been known the impossibility of the equations

$$f^2(e^{2z}-1)=\frac{e^{2h}-1}{h}, \qquad f\not\equiv 0$$

and

$$f^2(e^{2z}-1)=\frac{(e^{2h}-1)(h-a)}{h}, \qquad a\neq n\pi i, \qquad f\not\equiv 0,$$

respectively.

3) We shall consider the first case:

$$f^2(e^{2z}-1)=\frac{e^{2h}-1}{h}, \qquad f\not\equiv 0.$$

By the same Lemma $h(z)$ must be a polynomial. Further the degree of $h(z)$ must be one. Thus we have $h(z)=\alpha z+\beta$. Thus the equation is reduced to the following one:

$$f^2(e^{2z}-1)(\alpha z+\beta)=e^{2\alpha z+2\beta}-1.$$

Let z be $n\pi i$, then we have

$$Ce^{2\alpha n\pi i}=1, \qquad C=e^{2\beta}.$$

Therefore we have that α is an integer and $C=1$, that is, $\beta=m\pi i$. Then the equation is reduced to the form

$$f^2(e^{2z}-1)(pz-m\pi i)=e^{2pz}-1.$$

If $p\neq\pm 1$, then it is a contradiction. If $p=1$, then $f^2(z-m\pi i)=1$, which is absurd. If $p=-1$, then $f^2(z+m\pi i)=e^{-2z}$, which is also absurd. Thus we have the desired impossibility. Hence we have the following fact: There is no rigid analytic mapping from R into S, where R is the surface defined by $y^2=e^{2x}-1$ and S the surface defined by $y^2=(e^{2x}-1)/x$.

4) Quite similarly we have the impossiblity of the second equation

$$f^2(e^{2z}-1)=\frac{e^{2h}-1}{h}(h-a), \qquad a\neq n\pi i, \qquad f\not\equiv 0.$$

Then there is no rigid analytic mapping from R: $y^2=e^{2x}-1$ into S: $y^2=(e^{2x}-1)(x-a)/x$, $a\neq n\pi i$.

§ **4.** We shall here discuss the case where S is closed and hyperelliptic of genus $\geqq 2$. The torus case should be excluded, since it does not determine the projection map uniquely. Thus we can define the rigidity of φ similarly. Let φ be rigid. Every branch point of R corresponds to a branch point of S. Thus there is at least one branch point of S which is covered infinite times by $\varphi(R)$. Thus $h(z)$ should be a transcendental meromorphic function (not entire in general). We finally have an equation

$$f^2G=g\circ h,$$

for a suitable meromorphic function f. Here g has the following form

$$\prod_{\nu=1}^{2p}(z-a_\nu) \quad \text{or} \quad \prod_{\nu=1}^{2p-1}(z-a_\nu), \qquad p\geqq 3.$$

Thus we have the equations

$$f(z)^2G(z)=\prod_{\nu=1}^{2p}(h(z)-a_\nu) \quad \text{or} \quad \prod_{\nu=1}^{2p-1}(h(z)-a_\nu).$$

We can easily construct an example. Let R be the surface of $y^2=e^{8x}-1$ and S the surface of $y^2=\prod_{\nu=1}^{8}(x-\varepsilon^\nu)$, $\varepsilon=\exp(\pi i/4)$. Then we may take $h(z)=e^z$ and $f(z)\equiv 1$. However our example does not belong to the semi-degenerate class, though it satisfies the rigidity of projection map. In fact we have the following theorem:

THEOREM 2. *If there is an analytic mapping from an ultrahyperellptic surface R into a hyperelliptic surface S of genus greater than* 1, *then it satisfies*

$$\overline{\lim}_{r\to\infty} N(r, R)/T(r, \Phi)\leqq p.$$

Proof. If an analytic mapping φ is not rigid, then Φ is two-valued for z. Then by Selberg's ramification theorem $N(r, R)<2T(r, \Phi)+O(1)$, which implies the non-semi-degeneracy of φ. If φ is rigid, then

$$f^2G=\prod_{\nu=1}^{2p}(h-a_\nu),$$

where we may assume that S is defined by $y^2=\prod_{\nu=1}^{2p}(x-a_\nu)$. Then $\prod(h-a_\nu)$ has no pole of odd multiplicity. Therefore

$$2N(r, R)+O(\log r)=N(r; 0, G)\leqq N(r; 0, f^2G)$$
$$\leqq T(r, f^2G)=2pT(r, h)+O(1),$$

which leads again to the desired result, since h is transcendental in our case.

§ 5. In our discussions $h(z)$ has played the central role. In general $h(z)$ is a transcendental meromorphic function. Thus there is at most two exceptional values. Here we mean α the exceptional value if α is not taken infinitely many times. Thus there are at most four Picard's exceptional points on S. This case may happen when S is a closed hyperelliptic surface. Such an example has already listed in § 4.

If S is an ultrahyperelliptic surface, then $h(z)$ is an entire function. If it is transcendental, then $h(z)$ has at most one finite Picard's exceptional value. Thus there are at most two Picard's points on S. Let R be the surface defined by $y^2=\exp e^x-\gamma$ and S the surface defined by $y^2=e^x-\gamma$. If $\gamma\neq 1$, then we may take $h(z)=e^z$ and $f^2\equiv 1$. Then there are two Picard's points on S, which lie over $z=0$.

If $\gamma=1$, then $h(z)=e^z$ and $f^2\equiv 1$. Then there is only one Picard's point on S, which is the branch point over $z=0$. If $h(z)$ is a polynomial, then $\lim_{z\to\infty} h(z)=\infty$ and hence $\varphi(p)$ has the limiting point, which is the ideal boundary point of S, when p tends to the ideal boundary point of R. The function $h(z)$ has the limiting value when z tends to ∞ if and only if $\varphi(p)$ has constantly finite valence on S, which is equivalent to the growth condition

$$\varlimsup_{r\to\infty} \frac{T(r, \Phi)}{\log r} < \infty.$$

Its valence ν is equal to the degree of $h(z)$. In this case we say φ *degenerate.* If φ is degenerate, then every end of S corresponds to some end of R and vice versa. Thus we can establish a sufficient condition for the non-existence of a degenerate analytic mapping using the Picard great theorem. Its formulation is quite similar as in [2].

§ **6.** We shall offer some unsolved problems.

1) Is there any non-rigid analytic mapping from an ultrahyperelliptic surface R into another such surface S?

It would be negative. Sario's results in [5], [6] would play a role for this problem and the subsequent problem.

2) Is there any rigid but non-semi-degenerate analytic mapping from R into S? There would be another way to attack this problem. It reduces to the following problem.

3) To investigate the behavior of $g\circ h(z)$ especially a quantitative estimation of the counting function of its simple zeros, when g and h are transcendental.

This is really necessary to solve the equation

$$f^2 G = g\circ h$$

in some cases and to investigate the analytic mapping problems correspondingly.

4) To seek for some relations between the order ρ_G of G and that ρ_g of g when there exists an analytic mapping from R into S. It is very plausible to conjecture that ρ_G is a multiple of ρ_g. However it is necessary to fix G and g suitably, since the representation of an ultrahyperelliptic surface is not unique. It should be remarked that $\rho_G<\infty$ and $\rho_g>0$ must hold in our problem. Let $G(z)$ be the famous θ_1-function, that is,

$$\theta_1(z) = -i \sum_{n=-\infty}^{+\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)iz}, \qquad |q|<1.$$

Then it is known that $\rho_G=\rho_{\theta_1}=2$ and θ_1 has simple zeros at $m\pi+n\pi\tau$, $\tau=\omega_3/\omega_1$, where m and n run over all the integers and $2\omega_1$ and $2\omega_3$ are two primitive periods of Weierstrass' $\wp$-function. Further θ_1 is an entire function of $\sin z$ of order 0 [9]. This entire function is denoted by $g(z)$. Then $g(z)$ has the following form

$$g(z) = g'(0) z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{4\sin^2 n\pi\tau}\right)$$

and hence it has no zero other than an infinite number of simple zeros at 0, $\pm 2 \sin n\pi\tau$. Thus it is concluded that the equation

$$f^2\theta_1 = g \circ h$$

has at least one pair of solutions $h(z) = \sin z$, $f(z) \equiv 1$. Let R be the surface defined by $y^2 = \theta_1(x)$ and S the surface defined by $y^2 = g(x)$. Then there is an analytic mapping φ, induced by $\sin z$, from R into S. However $\rho_G = 2$ and $\rho_g = 0$. Thus we do not persist that our conjecture remains true in this case. Hence we assume that $\rho_g > 0$. If $\rho_G = \infty$, $0 < \rho_g \leqq \infty$, then evidently ρ_G is a multiple of ρ_g. This case is trivial and may be omitted in our problem. It is not yet known any counter examples so far as we concern. If it is positively answered, then it gives very effective criterion in our case. This problem would reduce to the above problem 3).

References

[1] NEVANLINNA, R., Eindeutige analytische Funktionen. Berlin 1936.

[2] OZAWA, M., On complex analytic mappings. Kōdai Math. Sem. Rep. **17** (1965), 93–102.

[3] OZAWA, M., On ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **17** (1965), 103–108.

[4] POLYA, G., On an integral function of an integral function. Journ. London Math. Soc. 1 (1926), 12–15.

[5] SARIO, L., Meromorphic functions and conformal metrics on Riemann surfaces. Pacific Journ. Math. **12** (1962), 1079–1097.

[6] SARIO, L., Value distribution under analytic mappings of arbitrary Riemann surfaces. Acta Math. **109** (1963), 1–10.

[7] SARIO, L., General value distribution theory. Nagoya Math. Journ. **23** (1963), 213–229.

[8] SELBERG, H., Algebroide Funktionen und Umkehrfunktionen Abelscher Integrale. Avh. Norske Vid. Akad. Oslo Nr. 8 (1934), 1–72.

[9] TITCHMARCH, E. C., The theory of functions. 2nd edition, Oxford (1939).

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY.