VECTOR FIELDS IN RIEMANNIAN AND HERMITIAN MANIFOLDS WITH BOUNDARY

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The vector fields and tensor fields in a Riemannian manifold with boundary have been studied by Bochner [3], Duff and Spencer [4], Hsiung [5], Nakae [8], Takahashi [11] and one of the present authors [13].

The main purpose of the present paper is to study systematically vector fields in a Riemannian manifold with boundary and to study, applying the results in a Riemannian manifold, the contravariant and covariant almost analytic vector fields in an almost Hermitian manifold. We shall use the fact that the boundary of an almost Hermitian manifold admits the so-called almost contact structure studied by Sasaki [10] and others.

CONTENTS

I. Vector fields in a Riemannian manifold with boundary.

1. Hypersurfaces in a Riemannian manifold.

We consider an m-dimensional differentiable Riemannian manifold M of class C^∞ covered by a system of neighbourhoods with local coordinates (ξ^h), where and in the sequel the indices $h, i, j, k, \cdots, r, s, t$ run over the range $1, 2, \cdots, m$. We denote by g_{ji} the positive definite fundamental metric tensor, by ∇_j the covariant differentiation with respect to the Christoffel symbols $\{{h \atop ji}\}$ and by $K_{kji}{}^h$ the curvature tensor

$$(1.1)\qquad K_{kji}{}^h=\partial_k\{{h \atop ji}\}-\partial_j\{{h \atop ki}\}+\{{h \atop kt}\}\{{t \atop ji}\}-\{{h \atop jt}\}\{{t \atop ki}\},$$

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where ∂_k denotes partial differentiation with respect to the coordinate ξ^k. We denote by K_{ji} and K the Ricci tensor and curvature scalar:

$$K_{ji}=K_{tji}{}^t, \qquad K=K_{ji}g^{ji} \tag{1.2}$$

respectively.

We consider a hypersurface B in the Riemannian manifold M and represent it by parametric equations

$$\xi^h=\xi^h(\eta^a), \tag{1.3}$$

where and in the sequel the indices a, b, c, d, e, f run over the range $1, 2, \cdots, m-1$. We put

$$B_a{}^h=\partial_a\xi^h, \tag{1.4}$$

where ∂_a denotes partial differentiation with respect to η^a. The $B_a{}^h$ represent $m-1$ linearly independent contravariant vectors tangent to the hypersurface. The metric of the hypersurface is given by the metric tensor

$$'g_{cb}=g_{ji}B_c{}^jB_b{}^i. \tag{1.5}$$

Assuming that the Riemannian manifold and the hypersurface are both orientable, we choose the unit normal N^h to the hypersurface and coordinates η^a on the hypersurface in such a way that $N^h, B_1{}^h, \cdots, B_{m-1}{}^h$ form the positive sense of M, and $B_1{}^h, \cdots, B_{m-1}{}^h$ form the positive sense of B. We then have

$$g_{ji}N^jB_b{}^i=0, \qquad g_{ji}N^jN^i=1, \tag{1.6}$$

$$\sqrt{\mathfrak{g}}\,|N^h, B_a{}^h|=\sqrt{'\mathfrak{g}}, \tag{1.7}$$

where $|N^h, B_a{}^h|$ denotes the determinant formed by N^h and $B_1{}^h, \cdots, B_{m-1}{}^h$ and

$$\mathfrak{g}=|g_{ji}|, \qquad '\mathfrak{g}=|'g_{cb}| \tag{1.8}$$

are determinants formed by g_{ji} and $'g_{cb}$ respectively.

Denoting by $'\nabla_c$ the symbol of covariant differentiation along the hypersurface, we have the equations of Gauss

$$'\nabla_cB_b{}^h=\partial_cB_b{}^h+B_c{}^jB_b{}^i\left\{{h \atop ji}\right\}-B_a{}^h{}'\left\{{a \atop cb}\right\}=H_{cb}N^h, \tag{1.9}$$

where $'\left\{{a \atop cb}\right\}$ are Christoffel symbols formed with $'g_{cb}$ and H_{cb} are components of the second fundamental tensor of the hypersurface. We have also the equations of Weingarten

$$'\nabla_cN^h=\partial_cN^h+B_c{}^jN^i\left\{{h \atop ji}\right\}=-H_c{}^aB_a{}^h, \tag{1.10}$$

where $H_c{}^a=H_{cb}{}'g^{ba}$.

If we put

$$B^a{}_i=B_b{}^h{}'g^{ba}g_{ih}, \tag{1.11}$$

we have

$$B^a{}_iB_b{}^i=\delta_b^a, \qquad B^a{}_iN^i=0 \tag{1.12}$$

and

$$N_i N^h + B^a{}_i B_a{}^h = \delta_i^h, \tag{1.13}$$

and equations of Gauss are written as

$$'\nabla_c B^a{}_i = H_c{}^a N_i. \tag{1.14}$$

We now state Stokes' theorem in the following form:

STOKES' THEOREM. *We consider a compact orientable Riemannian manifold with compact orientable boundary B. Then, for an arbitrary vector field v^h, we have the integral formula*

$$\int_M \nabla_i v^i d\sigma = \int_B v_i N^i d'\sigma, \tag{1.15}$$

where

$$d\sigma = \sqrt{\mathfrak{g}}\, d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^m \tag{1.16}$$

is the volume element of M and

$$d'\sigma = \sqrt{'\mathfrak{g}}\, d\eta^1 \wedge d\eta^2 \wedge \cdots \wedge d\eta^{m-1} \tag{1.17}$$

is the surface element of B.

In the sequel we assume that the manifold M is compact orientable and the boundary B is also compact orientable and so we can always apply Stokes' theorem.

2. Killing vectors.

It is well known that an infinitesimal transformation v^h defines an infinitesimal motion when and only when it satisfies

$$\underset{v}{\mathcal{L}} g_{ji} = \nabla_j v_i + \nabla_i v_j = 0, \tag{2.1}$$

where $\underset{v}{\mathcal{L}}$ denotes Lie differentiation with respect to v^h (Yano [14]). A vector field satisfying this condition is called a Killing vector. A Killing vector satisfies

$$\nabla_i v^i = 0. \tag{2.2}$$

From (2.1) we get

$$\underset{v}{\mathcal{L}} \left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + K_{kji}{}^h v^k = 0,$$

from which

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i = 0. \tag{2.3}$$

Now by a straightforward computation we can prove

$$(g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i) v_h + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i)$$

(2. 4)

$$=\nabla^j[(\nabla_j v_i+\nabla_i v_j)v^i-v_j(\nabla_i v^i)],$$

which is valid for an arbitrary vector field v^h, where $\nabla^j=g^{ji}\nabla_i$. Integrating the both members of (2. 4) on the whole manifold M and applying Stokes' theorem to the right hand member, we get

$$\begin{aligned}&\int_M\left[(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i)v_h+\frac{1}{2}(\nabla^j v^i+\nabla^i v^j)(\nabla_j v_i+\nabla_i v_j)-(\nabla_j v^j)(\nabla_i v^i)\right]d\sigma\\ &=\int_B[(\nabla_j v_i+\nabla_i v_j)v^i-v_j(\nabla_i v^i)]N^j d'\sigma.\end{aligned}\tag{2. 5}$$

Suppose now that v^h is a Killing vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i=0 \quad\text{and}\quad \nabla_i v^i=0 \quad \text{in}\quad M$$

and

$$(\nabla_j v_i+\nabla_i v_j)N^j v^i=0 \qquad\qquad \text{on}\quad B.$$

Conversely, if a vector field v^h satisfies these conditions, then we have from (2. 5)

$$\frac{1}{2}\int_M(\nabla^j v^i+\nabla^i v^j)(\nabla_j v_i+\nabla_i v_j)d\sigma=0,$$

from which

$$\nabla_j v_i+\nabla_i v_j=0 \qquad\qquad \text{in}\quad M,$$

and consequently v^h is a Killing vector field. Thus we have

THEOREM 2. 1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a Killing vector field is that*

$$\left\{\begin{array}{lll} g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i=0, & \nabla_i v^i=0 & \textit{in}\quad M,\\ (\nabla_j v_i+\nabla_i v_j)N^j v^i=0 & & \textit{on}\quad B.\end{array}\right.\tag{2. 6}$$

This theorem has been obtained in [13] for a vector field tangential to B. But the theorem is true for any vector field v^h not necessarily tangential to B.

If the vector v^h vanishes on the boundary B, then the second condition in (2. 6) is automatically satisfied. Thus we have

PROPOSITION 2. 1. *A necessary and sufficient condition for an infinitesimal transformation v^h in M with boundary B leaving B invariant point by point to be a motion is that*

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i=0, \qquad \nabla_i v^i=0 \qquad\qquad \textit{in}\quad M.$$

Now we put, on the boundary B,

$$v^h=B_a{}^{h\prime}v^a+\alpha N^h,\tag{2. 7}$$

then we have

$$B_b{}^i v_i = 'v_b, \qquad N^i v_i = \alpha. \tag{2.8}$$

Differentiating the first equation of (2.8) covariantly along the boundary and taking account of (2.8), we find

$$\alpha H_{cb} + B_c{}^j B_b{}^i \nabla_j v_i = '\nabla_c 'v_b,$$

from which, transvecting with $'g^{cb}$ and taking account of $B_c{}^j B_b{}^i{}'g^{cb} = g^{ji} - N^j N^i$,

$$\alpha H_a{}^a + (\nabla_i v^i) - (\nabla_j v_i) N^j N^i = '\nabla_a 'v^a. \tag{2.9}$$

Differentiating next the second equation of (2.8) covariantly along the boundary and taking account of (2.8), we obtain

$$-H_c{}^b{}'v_b + B_c{}^j N^i(\nabla_j v_i) = '\nabla_c \alpha,$$

from which, transvecting with $'v^c$,

$$-H_{cb}{}'v^c{}'v^b + (\nabla_j v_i) v^j N^i - \alpha(\nabla_j v_i) N^j N^i = 'v^c{}'\nabla_c \alpha \tag{2.10}$$

by virtue of (2.7).

Eliminating $(\nabla_j v_i) N^j N^i$ from (2.9) and (2.10), we obtain

$$(\nabla_j v_i) v^j N^i = H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a + \alpha(\nabla_i v^i) - 2\alpha('\nabla_a 'v^a) + '\nabla_a(\alpha' v^a), \tag{2.11}$$

from which

$$\begin{aligned}(\nabla_j v_i + \nabla_i v_j) N^j v^i \\ = (\nabla_j v_i) N^j v^i + H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a + \alpha(\nabla_i v^i) - 2\alpha('\nabla_a 'v^a) + '\nabla_a(\alpha' v^a).\end{aligned} \tag{2.12}$$

Thus we have

PROPOSITION 2.2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a Killing vector field is that*

$$\begin{cases} g^{ji}\nabla_j \nabla_i v^h + K_i{}^h v^i = 0, \quad \nabla^i v^i = 0 & \text{in } M, \\ (\nabla_j v_i) N^j v^i + H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a - 2\alpha('\nabla_a 'v^a) + '\nabla_a(\alpha' v^a) = 0 & \text{on } B. \end{cases} \tag{2.13}$$

Now if the vector v^h is tangential to B, then we have $\alpha = 0$ and consequently we have

PROPOSITION 2.3. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B to be a Killing vector field is that*

$$\begin{cases} g^{ji}\nabla_j \nabla_i v^h + K_i{}^h v^i = 0, \quad \nabla_i v^i = 0 & \text{in } M, \\ (\nabla_j v_i) N^j v^i + H_{cb}{}'v^c{}'v^b = 0 & \text{on } B. \end{cases} \tag{2.14}$$

If the vector field v^h is normal to the boundary B, then we have $'v^a = 0$ and $v^h = \alpha N^h$ and consequently

$$\begin{aligned}(\nabla_j v_i + \nabla_i v_j) N^j v^i &= \alpha(\nabla_j v_i + \nabla_i v_j) N^j N^i \\ &= 2[\alpha^2 H_a{}^a + \alpha(\nabla_i v^i)]\end{aligned}$$

by virtue of (2. 9). Thus we have

PROPOSITION 2. 4. *A necessary and sufficient condition for a vector field v^h in M normal to the boundary B to be a Killing vector field is that*

$$\begin{cases} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0, \qquad \nabla_i v^i = 0 & \text{in } M, \\ \alpha H_a{}^a = 0 & \text{on } B. \end{cases} \tag{2.15}$$

(Yano [13])

Now integrating the identity

$$(g^{ji}\nabla_j\nabla_i v^h)v_h + (\nabla^j v^i)(\nabla_j v_i) = \nabla^j[(\nabla_j v_i)v^i]$$

on M and applying Stokes' theorem, we find

$$\int_M [(g^{ji}\nabla_j\nabla_i v^h)v_h + (\nabla^j v^i)(\nabla_j v_i)]d\sigma = \int_B (\nabla_j v_i)N^j v^i d'\sigma. \tag{2.16}$$

From (2. 12) and (2. 16), we obtain

$$\begin{aligned} &\int_M [(g^{ji}\nabla_j\nabla_i v^h)v_h + (\nabla^j v^i)(\nabla_j v_i)]\, d\sigma \\ &= \int_B [(\nabla_j v_i + \nabla_i v_j)N^j v^i - H_{cb}{}'v^c{}'v^b - \alpha^2 H_a{}^a - \alpha(\nabla_i v^i) + 2\alpha('\nabla_a{}'v^a)]\, d'\sigma. \end{aligned} \tag{2.17}$$

Thus, forming (2. 5)$-$(2. 17), we obtain

$$\begin{aligned} &\int_M \Big[K_{ji}v^j v^i - (\nabla^j v^i)(\nabla_j v_i) \\ &\qquad + \frac{1}{2}(\nabla^j v^i + \nabla^i v^j)(\nabla_j v_i + \nabla_i v_j) - (\nabla_j v^j)(\nabla_i v^i) \Big] d\sigma \\ &= \int_B [H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a - 2\alpha('\nabla_a{}'v^a)]\, d'\sigma. \end{aligned}$$

Thus, if v^h is a Killing vector field, we have

$$\begin{aligned} &\int_M [K_{ji}v^j v^i - (\nabla^j v^i)(\nabla_j v_i)]\, d\sigma \\ &= \int_B [H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a - 2\alpha('\nabla_a{}'v^a)]\, d'\sigma. \end{aligned}$$

On the other hand, for a Killing vector field v^h, we have

$$B_c{}^j B_b{}^i(\nabla_j v_i + \nabla_i v_j) = '\nabla_c{}'v_b + '\nabla_b{}'v_c - 2\alpha H_{cb} = 0$$

from which

$$'\nabla_a{}'v^a = \alpha H_a{}^a.$$

Thus the above integral formula becomes

(2. 18) $$\int_M [K_{ji}v^jv^i-(\nabla^jv^i)(\nabla_jv_i)]\,d\sigma=\int_B [H_{cb}{}'v^c{}'v^b-\alpha({}'\nabla_a{}'v^a)]\,d'\sigma.$$

From this we have

PROPOSITION 2. 5. *If $K_{ji}v^jv^i\leqq 0$ and if a Killing vector field v^h satisfies one of the following alternate sets of conditions on B,*

(i) $H_{cb}{}'v^c{}'v^b\geqq 0,\quad \alpha=const.,$

(ii) $H_{cb}{}'v^c{}'v^b\geqq 0,\quad {}'\nabla_a{}'v^a=0,$

(iii) ${}'v^a=0,$

then we have

$$K_{ji}v^jv^i=0,\qquad \nabla_jv_i=0 \qquad \text{in } M$$

and in cases (i) and (ii),

$$H_{cb}{}'v^c{}'v^b=0 \qquad \text{on } B.$$

If $K_{ji}v^jv^i<0$ $(v^h\neq 0)$, then there is no such Killing vector field other than zero. (Bochner [3])

Thus if $K_{ji}v^jv^i<0$ $(v^h\neq 0)$ and $H_{cb}{}'v^c{}'v^b\geqq 0$, then there is no Killing vector tangent to B other than zero. If $K_{ji}v^jv^i<0$ $(v^h\neq 0)$, then there is no Killing vector normal to B other than zero.

3. Conformal Killing vectors.

It is well known that an infinitesimal transformation v^h defines an infinitesimal conformal motion when and only when it satisfies

(3. 1) $$\underset{v}{\mathcal{L}}g_{ji}=\nabla_jv_i+\nabla_iv_j=2\phi g_{ji}$$

for a certain scalar function ϕ. A vector field satisfying this condition is called a conformal Killing vector. The function ϕ above is found to be $(1/m)(\nabla_tv^t)$ and consequently (3. 1) can also be written as

(3. 2) $$\nabla_jv_i+\nabla_iv_j-\frac{2}{m}g_{ji}(\nabla_tv^t)=0.$$

From (3. 1) we get

$$\mathcal{L}\left\{{h \atop ji}\right\}=\nabla_j\nabla_iv^h+K_{kji}{}^hv^k=\delta^h_j\phi_i+\delta^h_i\phi_j-\phi^hg_{ji},$$

where $\phi_i=\nabla_i\phi$. From this we get, by transvection with g^{ji},

(3. 3) $$g^{ji}\nabla_j\nabla_iv^h+K_i{}^hv^i+\frac{m-2}{m}\nabla^h(\nabla_iv^i)=0.$$

Now by a straightforward computation we can prove

$$
\begin{aligned}
&\left[g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{m-2}{m}\nabla^h(\nabla_i v^i)\right]v_h \\
(3.4)\qquad &+\frac{1}{2}\left[\nabla^j v^i+\nabla^i v^j-\frac{2}{m}g^{ji}(\nabla_t v^t)\right]\left[\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}(\nabla_s v^s)\right] \\
&=\nabla^j\left[(\nabla_j v_i+\nabla_i v_j)v^i-\frac{2}{m}v_j(\nabla_i v^i)\right],
\end{aligned}
$$

which is valid for an arbitrary vector field v^h. Integrating the both members of (3.4) on M and applying Stokes' theorem on the right hand member, we get

$$
\begin{aligned}
&\int_M\left[\left(g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{m-2}{m}\nabla^h\nabla_i v^i\right)v_h\right. \\
(3.5)\qquad &\left.+\frac{1}{2}\left(\nabla^j v^i+\nabla^i v^j-\frac{2}{m}g^{ji}\nabla_t v^t\right)\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}\nabla_s v^s\right)\right]d\sigma \\
&=\int_B\left[(\nabla_j v_i+\nabla_i v_j)v^i-\frac{2}{m}v_j(\nabla_i v^i)\right]N^j d'\sigma.
\end{aligned}
$$

Suppose now that v^h is a conformal Killing vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{m-2}{m}\nabla^h\nabla_i v^i=0 \qquad \text{in } M$$

and

$$\left[(\nabla_j v_i+\nabla_i v_j)v^i-\frac{2}{m}v_j(\nabla_i v^i)\right]N^j=0 \qquad \text{on } B.$$

Conversely if a vector field v^h satisfies these conditions, then we have from (3.5)

$$\frac{1}{2}\int_M\left(\nabla^j v^i+\nabla^i v^j-\frac{2}{m}g^{ji}\nabla_t v^t\right)\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}\nabla_s v^s\right)d\sigma=0,$$

from which

$$\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}(\nabla_t v^t)=0 \qquad \text{in } M,$$

and consequently v^h is a conformal Killing vector field. Thus we have

THEOREM 3.1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a conformal Killing vector field is that*

$$
(3.6)\qquad \begin{cases}
g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\dfrac{m-2}{m}\nabla^h(\nabla_i v^i)=0 & \textit{in } M, \\[2ex]
\left(\nabla_j v_i+\nabla_i v_j-\dfrac{2}{m}g_{ji}\nabla_t v^t\right)N^j v^i=0 & \textit{on } B.
\end{cases}
$$

If the vector v^h vanishes on the boundary B, then the second condition of (3.6) is automatically satisfied. Thus we have

PROPOSITION 3. 1. *A necessary and sufficient condition for an infinitesimal transformation v^h in M with boundary B leaving B invariant point by point to be a conformal motion is that*

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{m-2}{m}\nabla^h\nabla_i v^i=0 \qquad in\ M.$$

Now from (2. 12), we find

$$\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}\nabla_t v^t\right)N^j v^i$$

$$=(\nabla_j v_i)N^j v^i+H_{cb}'v^c{}'v^b+\alpha^2 H_a{}^a+\frac{m-2}{m}\alpha(\nabla_i v^i)-2\alpha('\nabla_a{}'v^a)+'\nabla_a(\alpha' v^a).$$

Thus we have

PROPOSITION 3. 2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a conformal Killing vector is that*

$$(3.7)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\dfrac{m-2}{m}\nabla^h\nabla_i v^i=0 & in\ M, \\ (\nabla_j v_i)N^j v^i+H_{cb}'v^c{}'v^b+\alpha^2 H_a{}^a+\dfrac{m-2}{m}\alpha(\nabla_i v^i) & \\ \qquad\qquad -2\alpha('\nabla_a{}'v^a)+'\nabla_a(\alpha' v^a)=0 & on\ B. \end{cases}$$

If the vector v^h is tangential to B, then we have $\alpha=0$ and consequently we have

PROPOSITION 3. 3. *A necessary and sufficient condition for a vector field v^h in M tangential to B to be a conformal Killing vector field is that*

$$(3.8)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\dfrac{m-2}{m}\nabla^h\nabla_i v^i=0 & in\ M, \\ (\nabla_j v_i)N^j v^i+H_{cb}'v^c{}'v^b=0 & on\ B. \end{cases}$$

If the vector v^h is normal to B, then $'v^a=0$ and $v^h=\alpha N^h$ and consequently

$$\left(\nabla_j v_i+\nabla_i v_j-\frac{2}{m}g_{ji}\nabla_t v^t\right)N^j v^i$$

$$=\alpha(\nabla_j v_i+\nabla_i v_j)N^j N^i-\frac{2}{m}\alpha(\nabla_i v^i)=2\left[\alpha^2 H_a{}^a+\frac{m-1}{m}\alpha(\nabla_i v^i)\right]$$

by virtue of (2. 9). Thus we have

PROPOSITION 3. 4. *A necessary and sufficient condition for a vector field v^h in M normal to B to be a conformal Killing vector field is that*

$$
\left\{
\begin{aligned}
&g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{m-2}{m}\nabla^h\nabla_i v^i = 0 &&\text{in } M\\
&\alpha^2 H_a{}^a + \frac{m-1}{m}\alpha(\nabla_i v^i) = 0 &&\text{on } B.
\end{aligned}
\right. \tag{3.9}
$$

(Yano [13])

Now forming (3. 5)$-$(2. 17), we obtain

$$
\begin{aligned}
&\int_M \Big[K_{ji}v^j v^i + \frac{m-2}{m}(\nabla^j\nabla_i v^i)v_j - (\nabla^j v^i)(\nabla_j v_i)\\
&\qquad + \frac{1}{2}\Big(\nabla^j v^i + \nabla^i v^j - \frac{2}{m}g^{ji}\nabla_t v^t\Big)\Big(\nabla_j v_i + \nabla_i v_j - g_{ji}\nabla_s v^s\Big)\Big] d\sigma\\
&= \int_B \Big[H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a + \frac{m-2}{m}\alpha(\nabla_i v^i) - 2\alpha('\nabla_a{}'v^a)\Big] d'\sigma
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_M \Big[K_{ji}v^j v^i - \frac{2(m-1)}{m}(\nabla_j v^j)(\nabla_i v^i)\\
&\qquad + \frac{1}{2}\Big(\nabla^j v^i + \nabla^i v^j - \frac{2}{m}g^{ji}\nabla_t v^t\Big)\Big(\nabla_j v_i + \nabla_i v_j - \frac{2}{m}g_{ji}\nabla_s v^s\Big)\Big] d\sigma\\
&= \int_B [H_{cb}{}'v^c{}'v^b + \alpha^2 H_a{}^a - 2\alpha('\nabla_a{}'v^a)] d'\sigma.
\end{aligned}
$$

On the other hand, for a conformal Killing vector field v^h, we have

$$
\begin{aligned}
&B_c{}^j B_b{}^i\Big(\nabla_j v_i + \nabla_i v_j - \frac{2}{m}g_{ji}\nabla_t v^t\Big)\\
&= '\nabla_c{}'v_b + '\nabla_b{}'v_c - 2\alpha H_{cb} - \frac{2}{m}{}'g_{cb}(\nabla_i v^i) = 0,
\end{aligned}
$$

from which

$$
'\nabla_a{}'v^a = \alpha H_a{}^a + \frac{m-1}{m}(\nabla_i v^i).
$$

Thus for a conformal Killing vector v^h, we have, from the above equation,

$$
\begin{aligned}
&\int_M \Big[K_{ji}v^j v^i - \frac{2(m-1)}{m}(\nabla_j v^j)(\nabla_i v^i)\Big] d\sigma\\
&= \int_B \Big[H_{cb}{}'v^c{}'v^b - \alpha('\nabla_a{}'v^a) - \frac{m-1}{m}\alpha(\nabla_i v^i)\Big] d'\sigma,
\end{aligned} \tag{3.10}
$$

from which we have

PROPOSITION 3. 5. *If* $K_{ji}v^j v^i \leqq 0$ *and if a conformal Killing vector field* v^h *satisfies one of the following alternate sets of conditions on* B

(i) $$H_{cb}{}'v^c{}'v^b \geqq 0, \quad \alpha = 0,$$

(ii) $$H_{cb}{}'v^{c}{}'v^{b}\geqq 0,\quad ({}'\nabla_{a}{}'v^{a})+\frac{m-1}{m}(\nabla_{i}v^{i})=0,$$

(iii) $$'v^{a}=0,\qquad \nabla_{\iota}v^{\iota}=0,$$

then we have

$$K_{ji}v^{j}v^{i}=0,\qquad \nabla_{j}v^{j}=0 \qquad \textit{in } M$$

and in cases (i) *and* (ii)

$$H_{cb}{}'v^{c}{}'v^{b}=0.$$

Thus if $K_{ji}v^{j}v^{i}<0\ (v^{h}\neq 0)$, *then there exists no such conformal Killing vector field other than zero.*

Thus if $K_{ji}v^{j}v^{i}<0\ (v^{h}\neq 0)$ and $H_{cb}{}'v^{c}{}'v^{b}\geqq 0$ then there is no conformal Killing vector tangent to B other than zero. If $K_{ji}v^{j}v^{i}<0\ (v^{h}\neq 0)$, then there is no conformal Killing vector normal to B such that $\nabla_{i}v^{i}=0$ on B other than zero.

4. Harmonic vectors.

A harmonic vector is defined as a vector satisfying

(4.1) $$\nabla_{j}v_{i}-\nabla_{i}v_{j}=0,\qquad \nabla_{i}v^{i}=0.$$

For a harmonic vector v_{i}, we have

$$g^{ji}\nabla_{j}(\nabla_{i}v_{h}-\nabla_{h}v_{i})+\nabla_{h}(g^{ji}\nabla_{j}v_{i})=0,$$

from which

$$g^{ji}\nabla_{j}\nabla_{i}v_{h}-K_{h}{}^{i}v_{i}=0$$

or

(4.2) $$g^{ji}\nabla_{j}\nabla_{i}v^{h}-K_{i}{}^{h}v^{i}=0.$$

By a straightforward computation, we can prove

(4.3) $$(g^{ji}\nabla_{j}\nabla_{i}v^{h}-K_{i}{}^{h}v^{i})v_{h}+\frac{1}{2}(\nabla^{j}v^{i}-\nabla^{i}v^{j})(\nabla_{j}v_{i}-\nabla_{i}v_{j})+(\nabla_{j}v^{j})(\nabla_{i}v^{i})$$
$$=\nabla^{j}[(\nabla_{j}v_{i}-\nabla_{i}v_{j})v^{i}+v_{j}(\nabla_{i}v^{i})],$$

which is valid for an arbitrary vector field v^{h}. So integrating the both members of (4.3) on the whole M and applying Stokes' theorem to the right hand member, we get

(4.4) $$\int_{M}\left[(g^{ji}\nabla_{j}\nabla_{i}v^{h}-K_{i}{}^{h}v^{i})v_{h}+\frac{1}{2}(\nabla^{j}v^{i}-\nabla^{i}v^{j})(\nabla_{j}v_{i}-\nabla_{i}v_{j})+(\nabla_{j}v^{j})(\nabla_{i}v^{i})\right]d\sigma$$
$$=\int_{B}[(\nabla_{j}v_{i}-\nabla_{i}v_{j})v^{i}+v_{j}(\nabla_{i}v^{i})]N^{j}d'\sigma.$$

Suppose that v^{h} is a harmonic vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i = 0 \qquad \text{in } M$$

and

$$[(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j = 0 \qquad \text{on } B.$$

Conversely if a vector field v^h satisfies these conditions, then we have from (4. 4)

$$\int_M \left[\frac{1}{2}(\nabla^j v^i - \nabla^i v^j)(\nabla_j v_i - \nabla_i v_j) + (\nabla_j v^j)(\nabla_i v^i)\right] d\sigma = 0,$$

from which

$$\nabla_j v_i - \nabla_i v_j = 0, \qquad \nabla_i v^i = 0 \qquad \text{in } M.$$

Thus we have

THEOREM 4. 1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a harmonic vector field is that*

$$\text{(4. 5)} \qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i = 0 & \text{in } M, \\ [(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j = 0 & \text{on } B. \end{cases}$$

If the vector field v^h vanishes on the boundary B, then the second condition of (4. 5) is automatically satisfied. Thus we have

PROPOSITION 4. 1. *A necessary and sufficient condition for a vector field v^h in M with boundary B vanishing on B to be a harmonic vector field is that*

$$g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i = 0 \qquad \text{in } M.$$

From (2. 11), we find

$$\begin{aligned} &[(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j \\ &= (\nabla_j v_i)N^j v^i - H_{cb}{}'v^{c\prime}v^b - \alpha^2 H_a{}^a + 2\alpha('\nabla_a{}'v^a) - '\nabla_a(\alpha' v^a). \end{aligned}$$

Thus we have

PROPOSITION 4. 2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a harmonic vector field is that*

$$\text{(4. 6)} \qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i = 0 & \text{in } M, \\ (\nabla_j v_i)N^j v^i - H_{cb}{}'v^{c\prime}v^b - \alpha^2 H_a{}^a + 2\alpha('\nabla_a{}'v^a) - '\nabla_a(\alpha' v^a) = 0 & \text{on } B. \end{cases}$$

If the vector v^h is tangential to B, then we have $\alpha=0$ and consequently we have

PROPOSITION 4. 3. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B to be a harmonic vector field is that*

$$\text{(4. 7)} \qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h - K_i{}^h v^i = 0 & \text{in } M, \\ (\nabla_j v_i)N^j v^i - H_{cb}{}'v^{c\prime}v^b = 0 & \text{on } B. \end{cases}$$

(Yano [13])

If the vector field v^h is normal to the boundary B, then $'v^a=0$ and $v^h=\alpha N^h$ and consequently

$$[(\nabla_j v_i-\nabla_i v_j)v^i+v_j(\nabla_i v^i)]N^j=\alpha(\nabla_i v^i).$$

Thus we have

PROPOSITION 4. 4. *A necessary and sufficient condition for a vector field v^h in M normal to the boundary B to be a harmonic vector is that*

$$\left\{\begin{array}{lr} g^{ji}\nabla_j\nabla_i v^h-K_i{}^h v^i=0 & \quad \textit{in}\ \ M, \\ \alpha(\nabla_i v^i)=0 & \quad \textit{on}\ \ B. \end{array}\right. \tag{4. 8}$$

(Yano [13])

From (2. 11) and (2. 16), we find

$$\begin{aligned} &\int_M[(g^{ji}\nabla_j\nabla_i v^h)v_h+(\nabla^j v^i)(\nabla_j v_i)]\,d\sigma \\ &=\int_B[(\nabla_j v_i-\nabla_i v_j)v^i+v_j(\nabla_i v^i)]N^j d'\sigma \\ &+\int_B[H_{cb}{}'v^c{}'v^b+\alpha^2 H_a{}^a-2\alpha('\nabla_a{}'v^a)]\,d'\sigma. \end{aligned} \tag{4. 9}$$

Forming (4. 9)$-$(4. 4), we obtain

$$\int_M\left[K_{ji}v^j v^i+(\nabla^j v^i)(\nabla_j v_i)-\frac{1}{2}(\nabla^j v^i-\nabla^i v^j)(\nabla_j v_i-\nabla_i v_j)-(\nabla_j v^j)(\nabla_i v^i)\right]d\sigma$$

$$=\int_B[H_{cb}{}'v^c{}'v^b+\alpha^2 H_a{}^a-2\alpha('\nabla_a{}'v^a)]d'\sigma.$$

Thus for a harmonic vector v^h, we have

$$\begin{aligned} &\int_M[K_{ji}v^j v^i+(\nabla^j v^i)(\nabla_j v_i)]\,d\sigma \\ &=\int_B[H_{cb}{}'v^c{}'v^b+\alpha^2 H_a{}^a-2\alpha('\nabla_a{}'v^a)]\,d'\sigma, \end{aligned}$$

from which we have

PROPOSITION 4. 5. *If $K_{ji}v^j v^i\geqq 0$ and if a harmonic vector field v^h satisfies one of the following alternate sets of conditions on B,*

(i) $\quad H_{cb}{}'v^c{}'v^b\leqq 0,\quad \alpha=0,$

(ii) $\quad H_{cb}{}'v^c{}'v^b\leqq 0,\quad H_a{}^a\leqq 0,\quad '\nabla_a{}'v^a=0,$

(iii) $\quad 'v^a=0,\quad H_a{}^a\leqq 0,$

then we have

$$K_{ji}v^jv^i=0, \qquad \nabla_j v_i=0 \qquad\qquad in \quad M,$$

and in cases (i) *and* (ii)

$$H_{cb}{}'v^{c'}v^b=0 \qquad\qquad on \quad B.$$

Thus if $K_{ji}v^jv^i>0$ $(v^h\neq 0)$, *then there is no such vector field other than zero.* (Bochner [3])

Thus, if $K_{ji}v^jv^i>0$ $(v^h\neq 0)$ and $H_{cb}{}'v^{c'}v^b\leqq 0$, then there is no harmonic vector tangential to the boundary B other than zero. If $K_{ji}v^jv^i>0$ $(v^h\neq 0)$ and $H_a{}^a\leqq 0$, then there is no harmonic vector normal to the boundary B other than zero.

II. Vector fields in an almost Hermitian manifold.

5. Hermitian manifolds.

We consider a differentiable manifold of even dimension $m=2n$ and of class C^∞ and suppose that the manifold admits a tensor field $F_i{}^h$ of type (1, 1) and of class C^∞ which satisfies

$$F_j{}^iF_i{}^h=-A_j^h, \tag{5. 1}$$

where A_j^h is the unit tensor. A tensor field F satisfying (5. 1) is said to define an almost complex structure and a manifold admitting an almost complex structure is called an almost complex manifold.

It is now well-known (Newlander and Nirenberg [9]) that an almost complex structure F is induced from a complex structure if and only if the Nijenhuis tensor

$$N_{ji}{}^h=(F_j{}^t\partial_tF_i{}^h-F_i{}^t\partial_tF_j{}^h)-(\partial_jF_i{}^t-\partial_iF_j{}^t)F_t{}^h \tag{5. 2}$$

vanishes identically. The Nijenhuis tensor $N_{ji}{}^h$, skew-symmetric in j and i, satisfies

$$N_{ji}{}^h-F_i{}^sF_r{}^hN_{js}{}^r=0, \tag{5. 3}$$

$$N_{ji}{}^h+F_j{}^tF_i{}^sN_{ts}{}^h=0. \tag{5. 4}$$

If we introduce tensors

$$O_{ir}^{sh}=\frac{1}{2}(A_i^sA_r^h-F_i{}^sF_r{}^h), \tag{5. 5}$$

$$^*O_{ir}^{sh}=\frac{1}{2}(A_i^sA_r^h+F_i{}^sF_r{}^h), \tag{5. 6}$$

equations (5. 3) and (5. 4) can be written as

$$O_{ir}^{sh}N_{js}{}^r=0 \quad\text{and}\quad {}^*O_{ji}^{ts}N_{ts}{}^h=0 \tag{5. 7}$$

respectively.

In general if a tensor $T_{\cdots i \cdots}^{\cdots h \cdots}$ satisfies

$$O_{ir}^{sh}T_{\cdots s\cdots}^{\cdots r\cdots}=0 \quad\text{or}\quad {}^*O_{ir}^{sh}T_{\cdots s\cdots}^{\cdots r\cdots}=0,$$

the tensor is said to be hybrid or pure in i and h respectively.

Equation (5. 1) can be written as

$$F_i^h + F_i^s F_r^h F_s^r = 0 \quad \text{or} \quad {}^*O_{ir}^{sh} F_s^r = 0$$

and consequently the tensor F_i^h is pure in i and h. Equations (5. 3) and (5. 4) show that N_{ji}^h is hybrid in i and h and pure in j and i.

The tensors O and $^*O$ satisfy

$$(5.8)\qquad \begin{gathered} O + {}^*O = A, \\ O\cdot O = O, \qquad O\cdot {}^*O = 0, \qquad {}^*O\cdot O = 0, \qquad {}^*O\cdot {}^*O = {}^*O, \end{gathered}$$

where A represents the tensor $A_i^s A_r^h$. Thus the conditions

$$O\cdot T = 0 \qquad \text{and} \qquad {}^*OT = T$$

are equivalent and

$${}^*O\cdot T = 0 \qquad \text{and} \qquad O\cdot T = T$$

are also equivalent.

Suppose that P^{ji} is hybrid in j and i, then we have

$$P^{ji} = {}^*O_{ts}^{ji} P^{ts}.$$

If Q_{ji} is pure in j and i, then we have

$$Q_{ji} = O_{ji}^{vu} Q_{vu}.$$

Using (5. 8) and these equations we can easily prove that if P^{ji} is hybrid in j and i and Q_{ji} pure in j and i, then the contracted product $P^{ji}Q_{ji}$ vanishes identically.

From an arbitrary positive definite Riemannian metric a_{ji} in M, we can construct another Riemannian metric

$$g_{ji} = \frac{1}{2}(a_{ji} + a_{ts} F_j^t F_i^s),$$

which is also positive definite and satisfies

$$(5.9)\qquad g_{ts} F_j^t F_i^s = g_{ji}.$$

This equation is also written as

$$(5.10)\qquad O_{ji}^{ts} g_{ts} = 0$$

and shows that g_{ji} is hybrid.

A Riemannian metric g_{ji} on an almost complex manifold satisfying (5. 10) is called a Hermitian metric. An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold and a complex manifold with a Hermitian metric is called a Hermitian manifold. In an almost Hermitian manifold the tensor

$$(5.11)\qquad F_{ji} = F_j^t g_{ti}$$

is skew-symmetric and of rank $2n$.

If we denote by ∇_i the covariant differentiation with respect to a Hermitian metric g_{ji}, then the Nijenhuis tensor $N_{ji}{}^h$ can be written as follows:

$$N_{ji}{}^h=(F_j{}^t\nabla_tF_i{}^h-F_i{}^t\nabla_tF_j{}^h)-(\nabla_jF_i{}^t-\nabla_iF_j{}^t)F_t{}^h. \tag{5.12}$$

We now define the tensors

$$F_{jih}=\nabla_jF_{ih}+\nabla_iF_{hj}+\nabla_hF_{ji}, \tag{5.13}$$

$$F_h=g^{ji}\nabla_jF_{ih}=-\nabla_iF_h{}^i, \tag{5.14}$$

$$G_{ji}{}^h=\nabla_jF_i{}^h+\nabla_iF_j{}^h. \tag{5.15}$$

We call an almost Kähler manifold an almost Hermitian manifold in which $F_{jih}=0$ and a Kähler manifold a Hermitian manifold in which $F_{jih}=0$.

The covariant components $N_{jih}=N_{ji}{}^tg_{th}$ of the Nijenhuis tensor can be written in the form

$$N_{jih}=F_j{}^tF_{tih}-F_i{}^tF_{tjh}+2F_j{}^t(\nabla_hF_{it}), \tag{5.16}$$

from which, transvecting with F^{ih},

$$F_j{}^tF_{tih}F^{ih}+2F_j=0 \tag{5.17}$$

by virtue of

$$N_{jih}F^{ih}=0 \quad \text{and} \quad F_{tj}{}^t=0.$$

We see from (5.17) that the vector F_j vanishes in an almost Kähler manifold and consequently the tensor F_{ji}, satisfying $F_{jih}=0$ and $F_j=0$, is a harmonic tensor.

We also see from (5.16) that an almost Kähler manifold is a Kähler manifold if and only if ∇_jF_{ih} vanishes identically.

We call an almost Tachibana manifold an almost Hermitian manifold in which $G_{ji}{}^h=0$ and a Tachibana manifold a Hermitian manifold in which $G_{ji}{}^h=0$.

The Nijenhuis tensor can also be written as

$$N_{ji}{}^h=-4(\nabla_jF_i{}^t)F_t{}^h+2G_{ji}{}^tF_t{}^h+F_j{}^tG_{ti}{}^h-F_i{}^tG_{tj}{}^h, \tag{5.18}$$

from which we see that, for a Tachibana manifold we have $\nabla_jF_i{}^t=0$. Thus a Tachibana manifold is a Kähler manifold.

Coming back to a general almost Hermitian manifold, we denote covariant components of the curvature tensor by

$$K_{kjih}=K_{kji}{}^tg_{th} \tag{5.19}$$

and put

$$H_{kj}=\frac{1}{2}K_{kjih}F^{ih} \tag{5.20}$$

and

$$K^*_{ji}=-H_{jr}F_i{}^r. \tag{5.21}$$

From the Ricci identity

$$\nabla_k\nabla_jF_i{}^h-\nabla_j\nabla_kF_i{}^h=K_{kjt}{}^hF_i{}^t-K_{kji}{}^tF_t{}^h,$$

we get

(5. 22) $$\nabla_r \nabla_j F_i{}^r = (K_{jr} - K^*_{jr}) F_i{}^r - \nabla_j F_i,$$

which gives the expression for the difference $K_{ji} - K^*_{ji}$. In a Kähler manifold, K^*_{ji} coincides with K_{ji}.

6. Hypersurfaces in an almost Hermitian manifold.

We consider a hypersurface $\xi^h = \xi^h(\eta^a)$ in an almost complex manifold. The transform $F_i{}^h B_a{}^i$ of $B_a{}^i$ by $F_i{}^h$ can be expressed as a linear combination of $B_a{}^h$ and N^h:

(6. 1) $$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h + f_b N^h,$$

where the coefficients $f_b{}^a$ and f_b are defined by

(6. 2) $$f_b{}^a = F_i{}^h B_b{}^i B^a{}_h$$

and

(6. 3) $$f_b = F_i{}^h B_b{}^i N_h$$

respectively.

The transform $F_i{}^h N^i$ of N^i by $F_i{}^h$ is perpendicular to N^i and consequently tangent to the hypersurface, and hence we have the equation of the form

(6. 4) $$F_i{}^h N^i = -h^a B_a{}^h,$$

where the coefficient h^a is defined by

(6. 5) $$h^a = -F_i{}^h N^i B^a{}_h.$$

Transforming again the both members of (6. 1) by F and taking account of (6. 1) and (6. 4), we find

$$-B_b{}^h = f_b{}^c (f_c{}^a B_a{}^h + f_c N^h) - f_b h^a B_a{}^h,$$

from which

(6. 6) $$f_b{}^c f_c{}^a = -\delta_b^a + f_b h^a,$$

(6. 7) $$f_b{}^c f_c = 0.$$

Transforming again the both members of (6. 4) by F and taking account of (6. 1) and (6. 4), we get

$$-N^h = -h^c (f_c{}^a B_a{}^h + f_c N^h),$$

from which

(6. 8) $$f_c{}^a h^c = 0,$$

(6. 9) $$f_c h^c = 1.$$

The equations (6. 6), (6. 7), (6. 8) and (6. 9) show that the tensor $f_b{}^a$ and vectors f_c, h^c define the so-called almost contact structure. (Sasaki [10], Tashiro [12])

We next suppose that the almost complex manifold is Hermitian, then we have

(6. 10) $$g_{ji}=g_{ts}F_j{}^tF_i{}^s.$$

Substituting this into

$$g_{ji}B_c{}^jB_b{}^i={}'g_{cb},$$

we get

$$g_{ts}F_j{}^tF_i{}^sB_c{}^jB_b{}^i={}'g_{cb}$$
$$g_{ts}(f_c{}^eB_e{}^t+f_cN^t)(f_b{}^dB_d{}^s+f_bN^s)={}'g_{cb},$$

from which

(6. 11) $$f_c{}^ef_b{}^d{}'g_{ed}+f_cf_b={}'g_{cb}.$$

Transvecting equation (6. 11) with h^b and taking account of (6. 8) and (6. 9), we find

(6. 12) $$f_c={}'g_{cd}h^d$$

and consequently we shall write f^a in place of h^a. Thus equations (6. 6), (6. 7), (6. 8) and (6. 9) become

(6. 13) $$f_c{}^bf_b{}^a=-\delta_c^a+f_cf^a, \qquad f_b{}^af_a=0,$$
$$f_b{}^af^b=0, \qquad f_af^a=1.$$

If we put

(6. 14) $$f_{ca}=f_c{}^b{}'g_{ba},$$

the equations (6. 2), (6. 3) and (6. 5) are written as

(6. 15) $$F_{ih}B_b{}^iB_a{}^h=f_{ba}$$

(6. 16) $$F_{ih}B_b{}^iN^h=f_b$$

(6. 17) $$F_{ih}N^iB_a{}^h=-f_a$$

respectively. Equation (6. 15) shows that f_{ba} is a skew symmetric tensor.

We differentiate (6. 1) covariantly along the hypersurface and obtain

(6. 18) $$B_c{}^jB_b{}^i(\nabla_jF_i{}^h)=({}'\nabla_cf_b{}^a+H_{cb}f^a-H_c{}^af_b)B_a{}^h+({}'\nabla_cf_b+H_{ca}f_b{}^a)N^h,$$

from which, transvecting with B_{ah},

(6. 19) $$B_c{}^jB_b{}^iB_a{}^h(\nabla_jF_{ih})={}'\nabla_cf_{ba}+H_{cb}f_a-H_{ca}f_b,$$

and, transvecting with N_h,

(6. 20) $$B_c{}^jB_b{}^iN^h(\nabla_jF_{ih})={}'\nabla_cf_b+H_{ca}f_b{}^a.$$

We next differentiate (6. 4) covariantly along the hypersurface and obtain

(6. 21) $$B_c{}^jN^i(\nabla_jF_i{}^h)=-({}'\nabla_cf^a-H_c{}^bf_b{}^a)B_a{}^h,$$

from which, transvecting with B_{ah},

(6. 22) $$B_c{}^jN^iB_a{}^h(\nabla_jF_{ih})=-({}'\nabla_cf_a-H_c{}^bf_{ba}).$$

7. Contravariant almost analytic vectors.

When an infinitesimal transformation v^h leaves the almost complex structure invariant, that is, when it satisfies

$$\underset{v}{\mathcal{L}} F_i{}^h = v^t \partial_t F_i{}^h - F_i{}^t \partial_t v^h + F_t{}^h \partial_i v^t = 0, \tag{7. 1}$$

the vector v^h is called a contravariant almost analytic vector field. If the almost complex manifold is almost Hermitian, equation (7. 1) can be written as

$$\underset{v}{\mathcal{L}} F_i{}^h = v^t \nabla_t F_i{}^h - F_i{}^t \nabla_t v^h + F_t{}^h \nabla_i v^t = 0, \tag{7. 2}$$

from which

$$S_{ih} = (\underset{v}{\mathcal{L}} F_i{}^r) g_{rh} = v^t \nabla_t F_{ih} - F_i{}^t \nabla_t v_h - F_h{}^t \nabla_i v_t = 0 \tag{7. 3}$$

and

$$S^{ih} = (\underset{v}{\mathcal{L}} F_s{}^h) g^{si} = v^t \nabla_t F^{ih} + F_t{}^i \nabla^t v^h + F_t{}^h \nabla^i v^t = 0. \tag{7. 4}$$

From (7. 3) and (7. 4), we find

$$O_{ih}^{sr}(\underset{v}{\mathcal{L}} g_{sr}) = 0 \qquad \text{and} \qquad O_{sr}^{ih}(\underset{v}{\mathcal{L}} g^{sr}) = 0 \tag{7. 5}$$

respectively.

Now, by a straightforward computation, we can show that the tensor

$$\frac{1}{2}(F_j{}^t F_{ti}{}^h + F_i{}^t F_{tj}{}^h) - G_{ji}{}^t F_t{}^h$$

is pure in j and i. Since $\underset{v}{\mathcal{L}} F_i{}^h = 0$ and $\underset{v}{\mathcal{L}} g^{ji}$ is hybrid in j and i for a contravariant almost analytic vector field v^h, we have, by the above remark,

$$\left[\frac{1}{2}(F_j{}^t F_{ti}{}^h + F_i{}^t F_{tj}{}^h) - G_{ji}{}^t F_t{}^h\right](\underset{v}{\mathcal{L}} g^{ji}) = 0$$

or

$$\frac{1}{2} F_{ji}{}^h (\underset{v}{\mathcal{L}} F^{ji}) = G_{ji}{}^t F_t{}^h (\nabla^j v^i). \tag{7. 6}$$

On the other hand, applying the operator $g^{ji}\nabla_j$ to the both sides of (7. 2), we find

$$F_t{}^h [g^{ji} \nabla_j \nabla_i v^t + K_i{}^t v^i - F_i{}^t \underset{v}{\mathcal{L}} F^i - G_{ji}{}^s F_s{}^t (\nabla^j v^i)] = 0, \tag{7. 7}$$

from which

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i - F_i{}^h (\underset{v}{\mathcal{L}} F^i) - G_{ji}{}^t F_t{}^h (\nabla^j v^i) = 0 \tag{7. 8}$$

or,

$$g^{ji} \nabla_j \nabla_i v^h + K_i{}^h v^i - F_i{}^h (\underset{v}{\mathcal{L}} F^i) - \frac{1}{2} F_{ji}{}^h (\underset{v}{\mathcal{L}} F^{ji}) = 0 \tag{7. 9}$$

by virtue of (7. 6). This equation gives a necessary condition for a vector field v^h to be contravariant almost analytic.

Now by a straightforward computation we can prove

$$\left\{g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i-F_i{}^h(\underset{v}{\mathcal{L}}F^i)-\frac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji})\right\}v_h+\frac{1}{2}S^{ji}S_{ji}$$
$$=\nabla^j\{F_j{}^t(\underset{v}{\mathcal{L}}F_t{}^h)v_h\}, \tag{7.10}$$

which is valid for an arbitrary vector field v^h, where

$$S_{ji}=(\underset{v}{\mathcal{L}}F_j{}^t)g_{ti}. \tag{7.11}$$

Integrating the both members of (7.10) on the whole manifold M and applying Stokes' theorem to the right hand member, we get

$$\int_M\left[\left\{g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i-F_i{}^h(\underset{v}{\mathcal{L}}F^i)-\frac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji})\right\}v_h+\frac{1}{2}S^{ji}S_{ji}\right]d\sigma$$
$$=\int_B\{F_j{}^t(\underset{v}{\mathcal{L}}F_t{}^h)\}N^j v_h d'\sigma. \tag{7.12}$$

Suppose now that v^h is a contravariant almost analytic vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i-F_i{}^h(\underset{v}{\mathcal{L}}F^i)-\frac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji})=0 \qquad \text{in } M$$

and

$$\{F_j{}^t(\underset{v}{\mathcal{L}}F_t{}^h)\}N^j v_h=0 \qquad \text{on } B.$$

Conversely if a vector field v^h satisfies these conditions, then we have, from (7.12),

$$\frac{1}{2}\int_M S^{ji}S_{ji}\,d\sigma=0,$$

from which

$$S_{ji}=0 \quad \text{or} \quad \underset{v}{\mathcal{L}}F_i{}^h=0 \qquad \text{in } M,$$

and consequently v^h is a contravariant almost analytic vector. Thus we have

THEOREM 7.1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a contravariant almost analytic vector is that*

$$\begin{cases} g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i-F_i{}^h(\underset{v}{\mathcal{L}}F^i)-\frac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji})=0 & \textit{in } M, \\ \{F_j{}^t(\underset{v}{\mathcal{L}}F_t{}^h)\}N^j v_h=0 & \textit{on } B. \end{cases} \tag{7.13}$$

If the vector v^h vanishes on the boundary B, then the second condition in (7.13) is automatically satisfied. Thus we have

PROPOSITION 7.1. *A necessary and sufficient condition for an infinitesimal transformation v^h in M leaving the boundary B invariant point by point to be an almost analytic transformation is that*

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i-F_i{}^h(\underset{v}{\mathcal{L}}F^i)-\frac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji})=0 \qquad \textit{in} \quad M.$$

In the case of almost Kähler manifold, this condition reduces to

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i=0 \qquad \text{in } M,$$

and consequently combining Propositions 2. 1 and 7. 1, we get

PROPOSITION 7. 2. *An infinitesimal motion in an almost Kähler manifold leaving the boundary invariant point by point is an automorphism.*

PROPOSITION 7. 3. *An infinitesimal almost analytic transformation in an almost Kähler manifold leaving the volume invariant and the boundary B invariant point by point is an automorphism.* (Ba [2])

Suppose that an infinitesimal conformal transformation v^h in M leaves the boundary B invariant point by point, then we have, by Proposition 3. 1,

$$g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i+\frac{m-2}{m}\nabla^h\nabla_i v^i=0 \qquad \text{in} \quad M.$$

On the other hand, in the case of an almost Kähler manifold, (7. 12) reduces to

$$\int_M\left[\{g^{ji}\nabla_j\nabla_i v^h+K_i{}^h v^i\}v_h+\frac{1}{2}S^{ji}S_{ji}\right]d\sigma=0$$

for a vector field v^h vanishing on B. Thus substituting the above equation into this integral formula, we get

$$\int_M\left[-\frac{m-2}{m}\{\nabla^h(\nabla_i v^i)\}v_h+\frac{1}{2}S^{ji}S_{ji}\right]d\sigma=0,$$

$$\int_M\left[-\frac{m-2}{m}\nabla^h\{(\nabla_i v^i)v_h\}+\frac{m-2}{m}(\nabla_i v^i)(\nabla_h v^h)+\frac{1}{2}S^{ji}S_{ji}\right]d\sigma=0,$$

$$\int_M\left[\frac{m-2}{m}(\nabla_j v^j)(\nabla_i v^i)+\frac{1}{2}S^{ji}S_{ji}\right]d\sigma=\frac{m-2}{m}\int_B\{(\nabla_i v^i)v_h\}N^h d'\sigma=0,$$

from which

$$S_{ji}=0 \qquad \text{for} \qquad m=2$$

and

$$\nabla_i v^i=0, \qquad S_{ji}=0 \qquad \text{for} \quad m>2.$$

Thus we have

PROPOSITION 7. 4. *An infinitesimal conformal transformation in M leaving the boundary invariant point by point is almost analytic when $m=2$ and an automorphism when $m>2$.* (Ba [2])

Now putting

$$v^h=B_a{}^{h\prime}v^a+\alpha N^h$$

on the boundary, we have

$$\begin{aligned}\{F_j{}^t(\underset{v}{\mathcal{L}}F_t{}^h)\}N^j v_h &= -\{(\underset{v}{\mathcal{L}}F_j{}^s)g_{si}\}B_c{}^j f^c v^i \\ &= -\{\underset{v}{\mathcal{L}}F_{ji} - F_j{}^s(\underset{v}{\mathcal{L}}g_{si})\}B_c{}^j f^c v^i \\ &= -[(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j f^c v^i - (\underset{v}{\mathcal{L}}g_{si})(f_c{}^b B_b{}^s + f_c N^s) f^c v^i] \\ &= -[(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j f^c - (\underset{v}{\mathcal{L}}g_{ji})N^j]v^i.\end{aligned}$$

Thus we have

PROPOSITION 7. 5. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a contravariant almost analytic vector is that*

$$(7.14)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i - F_i{}^h(\underset{v}{\mathcal{L}}F^i) - \dfrac{1}{2}F_{ji}{}^h(\underset{v}{\mathcal{L}}F^{ji}) = 0 & \textit{in } M, \\ [(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j f^c - (\underset{v}{\mathcal{L}}g_{ji})N^j]v^i = 0 & \textit{on } B. \end{cases}$$

In the case of almost Kähler manifold, these conditions reduce to

$$(7.15)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0 & \text{in } M, \\ [(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j f^c - (\nabla_j v_i + \nabla_i v_j)N^j]v^i = 0 & \text{on } B, \end{cases}$$

and consequently combining Theorem 2. 1 and Proposition 7. 5, we get

PROPOSITION 7. 6. *An infinitesimal motion in an almost Kähler manifold leaving the fundamental form F_{ji} on the boundary invariant is an automorphism.*

PROPOSITION 7. 7. *An infinitesimal almost analytic transformation in an almost Kähler manifold leaving the volume invariant in M and the fundamental tensors F_{ji} and g_{ji} invariant on B is an automorphism.* (Ba [1])

Now, if the vector v^h is tangential to the boundary B, then we have

$$(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j B_b{}^i = (v^t\nabla_t F_{ji} + F_{ti}\nabla_j v^t + F_{jt}\nabla_i v^t)B_c{}^j B_b{}^i = \underset{'v}{\mathcal{L}} f_{cb}$$

and consequently

$$\begin{aligned}&[(\underset{v}{\mathcal{L}}F_{ji})B_c{}^j f^c - (\underset{v}{\mathcal{L}}g_{ji})N^j]v^i \\ &= (\underset{'v}{\mathcal{L}} f_{cb})f^c{}'v^b - (\underset{v}{\mathcal{L}}g_{ji})N^j v^i.\end{aligned}$$

Hence we have

PROPOSITION 7. 8. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B to be a contravariant almost analytic vector is that*

$$(7.16)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i - F_i{}^h(\underset{v}{\mathcal{L}} F^i) - \dfrac{1}{2} F_{ji}{}^h(\underset{v}{\mathcal{L}} F^{ji}) = 0 & \textit{in } M, \\[2ex] (\underset{'v}{\mathcal{L}} f_{cb}) f^{c'} v^b - (\underset{v}{\mathcal{L}} g_{ji}) N^j v^i = 0 & \textit{on } B. \end{cases}$$

In the case of almost Kähler manifold, these conditions reduce to

$$(7.17)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0 & \text{in } M, \\ (\underset{'v}{\mathcal{L}} f_{cb}) f^{c'} v^b - (\underset{v}{\mathcal{L}} g_{ji}) N^j v^i = 0 & \text{on } B. \end{cases}$$

On the other hand, for a conformal Killing vector tangential to B, we have

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{m-2}{m}\nabla^h(\nabla_i v^i) = 0 \qquad \text{in } M,$$

$$(\underset{v}{\mathcal{L}} g_{ji}) N^j v^i = 0 \qquad \text{on } B.$$

Substituting these into integral formula (7. 12), we find

$$\int_M \left[-\frac{m-2}{m}\{\nabla^h(\nabla_i v^i)\} v_h + \frac{1}{2} S^{ji} S_{ji} \right] d\sigma = -\int_B (\underset{'v}{\mathcal{L}} f_{cb}) f^{c'} v^b d'\sigma$$

or

$$(7.18)\qquad \int_M \left[\frac{m-2}{m}(\nabla_j v^j)(\nabla_i v^i) + \frac{1}{2} S^{ji} S_{ji} \right] d\sigma = -\int_B (\underset{'v}{\mathcal{L}} f_{cb}) f^{c'} v^b d'\sigma$$

by virtue of

$$\begin{aligned} -\int_M \{\nabla^h(\nabla_i v^i)\} v_h d\sigma &= -\int_M [\nabla^h\{(\nabla_i v^i) v_h\}] d\sigma + \int_M (\nabla_j v^j)^2 d\sigma \\ &= -\int_B (\nabla_i v^i) v_j N^j d'\sigma + \int_M (\nabla_j v^j)^2 d\sigma \\ &= \int_M (\nabla_j v^j)^2 d\sigma. \end{aligned}$$

From (7. 18) we can see that $\underset{'v}{\mathcal{L}} f_{cb} = 0$ implies

$$\nabla_i v^i = 0 \quad \text{and} \quad S_{ji} = 0, \quad \text{for} \quad m > 2, \quad \text{and} \quad S_{ji} = 0 \quad \text{for} \quad m = 2.$$

Thus we have

PROPOSITION 7. 9. *An infinitesimal conformal transformation in an almost Kähler manifold tangential to the boundary B and leaving f_{cb} invariant along B is an automorphism for $m>2$, and analytic for $m=2$.* (Ba [2])

Suppose next that an infinitesimal transformation is conformal and at the same time almost analytic, then we have, from (3. 6) and (7. 17) $\nabla_i v^i$=constant. But if v^h is tangential to B, we have

$$\int_M \nabla_i v^i d\sigma = \int_B v^i N_i d'\sigma = 0$$

and consequently $\nabla_i v^i = 0$. Thus the transformation is an automorphism. Hence we have

PROPOSITION 7.10. *If an infinitesimal transformation in an almost Kähler manifold leaving invariant the boundary B is conformal and almost analytic, then it is an automorphism.* (Ba [2])

We now consider a very special infinitesimal transformation v^h which is tangent to the boundary B and whose transform by $F_i{}^h$ is normal to the boundary. If we represent the transformation by

$$v^h = B_a{}^h{}'v^a,$$

then, by assumption

$$F_i{}^h v^i = F_i{}^h B_a{}^i{}'v^a = (f_a{}^c B_c{}^h + f_a N^h)'v^a$$

must be in the direction of N^h, from which we have

$$'v^a = \lambda f^a. \tag{7.19}$$

Thus, from Proposition 7.8, we have

PROPOSITION 7.11. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B and whose transform by F is normal to the boundary to be contravariant almost analytic is that*

$$\left\{\begin{array}{ll} g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i - F_i{}^h(\underset{v}{\mathcal{L}} F^i) - \dfrac{1}{2} F_{ji}{}^h(\underset{v}{\mathcal{L}} F^{ji}) = 0 & \textit{in}\ M, \\ (\underset{v}{\mathcal{L}} g_{ji}) N^j v^i = 0 & \textit{on}\ B. \end{array}\right. \tag{7.20}$$

From this we have

PROPOSITION 7.12. *An infinitesimal almost analytic transformation in an almost Kähler manifold M tangential to the boundary B, whose transform by F is normal to the boundary B and which preserves the volume is an automorphism.* (Ba [2])

We also have, from Proposition 7.9,

PROPOSITION 7.13. *An infinitesimal conformal transformation in an almost Kähler manifold tangential to the boundary B and whose transform by F is normal to the boundary B is an automorphism.* (Ba [2])

8. Covariant almost analytic vectors.

When a covariant vector field w_i satisfies

$$(\partial_j F_i{}^t - \partial_i F_j{}^t) w_t - F_j{}^t \partial_t w_i + F_i{}^t \partial_j w_t = 0, \tag{8.1}$$

the vector w_i is called a covariant almost analytic vector field. If the almost complex manifold is almost Hermitian, the equation (8.1) can be written as

(8. 2) $$(\nabla_j F_i{}^t - \nabla_i F_j{}^t) w_t - F_j{}^t \nabla_t w_i + F_i{}^t \nabla_j w_t = 0,$$

from which, by taking the symetric part with respect to j and i, we get

(8. 3) $$^*O_{ji}^{ts}(\nabla_t w_s - \nabla_s w_t) = 0.$$

We also have, transvecting $\nabla_k F^{ji}$ to (8. 2),

(8. 4) $$(\nabla_k F^{ji})(\nabla_j F_i{}^t) w_t = 0.$$

If we put

$$P_{ji} = (\nabla_j F_i{}^t - \nabla_i F_j{}^t) w_t, \qquad Q_{ji} = F_j{}^t \nabla_t w_i - F_i{}^t \nabla_j w_t,$$

then we have, for a covariant almost analytic vector w_i,

$$P_{ji} = Q_{ji},$$

$$P_{ji} P^{ji} = 2F_{ji}{}^t (\nabla^j F^{is}) w_t w_s,$$

$$P_{ji} Q^{ji} = F_j{}^t (\nabla_t w_i + \nabla_i w_t)(G^{jis} - 2\nabla^i F^{js}) w_s.$$

Suppose now that the manifold is an almost Kähler manifold, then we have $P_{ji}P^{ji}=0$ and consequently $P_{ji}=0$, $Q_{ji}=0$ for a covariant almost analytic vector w_i. But in an almost Kähler manifold, $P_{ji}=0$ is equivalent to $w^t \nabla_t F_{ji} = 0$.

Suppose next that the manifold is an almost Tachibana manifold, then, $\nabla_j F_{ih}$ being skew symmetric in all indices, we have from (8. 4)

$$(w^t \nabla_t F_{ji})(w^s \nabla_s F^{ji}) = 0$$

and consequently

$$w^t \nabla_t F_{ji} = 0,$$

from which $P_{ji}=0$ and $Q_{ji}=0$, for a covariant almost analytic vector field w_i. Thus we see that a necessary and sufficient condition for a vector field w_i in an almost Kähler or Tachibana manifold to be covariant almost analytic is

(8. 5) $$w^a \nabla_a F_{ji} = 0 \quad \text{and} \quad F_j{}^t \nabla_t w_i - F_i{}^t \nabla_j w_t = 0.$$

Coming back to a general almost Hermitian manifold, we can show, by a straightforward calculation that

$$N_{ji}{}^h w_h = -2{}^*O_{ji}^{ts}(\nabla_t w_s - \nabla_s w_t)$$

or

(8. 6) $$N_{ji}{}^h w_h = 0$$

and

$$(\nabla_j F_i{}^t - \nabla_i F_j{}^t)(\nabla^j w^i) F_t{}^h = -\frac{1}{2} N_{ji}{}^h (\nabla^j w^i),$$

from which

(8. 7) $$(\nabla_j F_i{}^t - \nabla_i F_j{}^t)(\nabla^j w^i) F_t{}^h w_h = 0$$

for a covariant almost analytic vector field w_i.

We next apply the operator $F_h{}^j \nabla^i$ to (8. 2) and change the indices, then we get

$$\begin{aligned}(8.8)\qquad & g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^j+F_i{}^t\nabla^s(F_{tsr}w^r)\\ & +(\nabla^t w^s)G_{tsr}F_i{}^r+F_i{}^s(w^t\nabla_t F_s+F_t\nabla^t w_s)=0.\end{aligned}$$

On the other hand, by a straightforward computation, we can get

$$\begin{aligned}(8.9)\qquad & [g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^j+F_i{}^t\nabla^s(F_{tsr}w^r)\\ & +(\nabla^t w^s)G_{tsr}F_i{}^r+F_i{}^s(w^t\nabla_t F_s+F_t\nabla^t w_s)\\ & -(\nabla_t F_{sr}-\nabla_s F_{tr})(\nabla^t w^s)F_i{}^r]w^i+\frac{1}{2}T^{ji}T_{ji}=-\nabla^j(T_{ji}F_r{}^i w^r),\end{aligned}$$

where

$$(8.10)\qquad T_{ji}=(\nabla_j F_i{}^t-\nabla_i F_j{}^t)w_t-F_j{}^t\nabla_t w_i+F_i{}^t\nabla_j w_t.$$

Integrating the both members of (8. 9) on the whole manifold M and applying Stokes' theorem to the right hand member, we get

$$\begin{aligned}(8.11)\qquad & \int_M\Big[\{g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^i+F_i{}^t\nabla^s(F_{tsr}w^r)\\ & +(\nabla^t w^s)G_{tsr}F_i{}^r+F_i{}^s(w^t\nabla_t F_s+F_t\nabla^t w_s)\\ & -(\nabla_t F_{sr}-\nabla_s F_{tr})(\nabla^t w^s)F_i{}^r\}w^i+\frac{1}{2}T^{ji}T_{ji}\Big]d\sigma\\ & =-\int_B T_{ji}F_r{}^i N^j w^r d'\sigma.\end{aligned}$$

Suppose now that w_i is a covariant almost analytic vector field. Then it satisfies (8. 7) and (8. 8) on M and

$$T_{ji}F_r{}^i N^j w^r=0 \qquad \text{on } B.$$

Conversely, if (8. 7) and (8. 8) are satisfied on M and the condition above is satisfied on B, then, we have from (8. 11)

$$\frac{1}{2}\int_M T^{ji}T_{ji}\,d\sigma=0$$

from which

$$T_{ji}=0 \qquad \text{on } M,$$

and consequently w_i is a covariant almost analytic vector field. Thus we have

THEOREM 8. 1. *A necessary and sufficient condition for a vector field w_i in M with boundary B to be a covariant almost analytic vector is that*

$$(8.12)\qquad \begin{cases}(\nabla_j F_i{}^t-\nabla_i F_j{}^t)(\nabla^j w^i)F_t{}^h w_h=0 & \text{on } M,\\ g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^j+F_i{}^t\nabla^s(F_{tsr}w^r) & \end{cases}$$

$$\left.\begin{array}{ll} +(\nabla^t w^s)G_{tsr}F_i{}^r+F_i{}^s(w^t\nabla_t F_s+F_t\nabla^t w_s)=0 & \text{on } M, \\ T_{ji}F_r{}^i N^j w^r=0 & \text{on } B. \end{array}\right.$$

If the vector w_i vanishes on the boundary B, then the last condition in (8. 12) is automatically satisfied. Thus we have

PROPOSITION 8. 1. *A necessary and sufficient condition for a vector field w_i in M vanishing on the boundary B to be covariant almost analytic is that*

$$\left\{\begin{array}{ll} (\nabla_j F_i{}^t-\nabla_i F_j{}^t)(\nabla^j w^i)F_t{}^h w_h=0 & \text{on } M, \\ g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^i+F_i{}^t\nabla^s(F_{tsr}w^r) & \\ \quad +(\nabla^t w^s)G_{tsr}F_i{}^r+F_i{}^s(w^t\nabla_t F_s+F_t\nabla^t w_s)=0 & \text{on } M. \end{array}\right.$$

In the case of almost Kähler manifold, these conditions reduce to

$$\left\{\begin{array}{ll} (\nabla_t F_{ji})(\nabla^j w^i)F_h{}^t w^h=0 & \text{on } M \\ g^{kj}\nabla_k\nabla_j w_i-(2K^*_{ji}-K_{ji})w^j+(\nabla^t w^s)G_{tsr}F_i{}^r=0 & \text{on } M. \end{array}\right. \tag{8. 13}$$

On the other hand, taking account of

$$\nabla_r\nabla_j F_i{}^r=(K_{jr}-K^*_{jr})F_i{}^r$$

derived from (5. 22) and of the first equation of (8. 5), we have

$$\begin{aligned} &-2(K^*_{ji}-K_{ji})w^j+(\nabla^t w^s)G_{tsr}F_i{}^r \\ &=-2F_i{}^t(\nabla_r\nabla_j F_t{}^r)w^j+F_i{}^r(\nabla^t w^j)G_{tjr} \\ &=2F_i{}^t(\nabla_j F_{ts})(\nabla^s w^j)+F_i{}^r(\nabla^t w^j)(\nabla_t F_{jr}+\nabla_j F_{tr}) \\ &=F_i{}^r(\nabla^t w^j)(\nabla_t F_{jr}-\nabla_j F_{tr}) \end{aligned}$$

and consequently, from the second equation of (8. 13)

$$(g^{kj}\nabla_k\nabla_j w_i-K_{ji}w^j)w^i=-F_i{}^r(\nabla^t w^j)(\nabla_t F_{jr}-\nabla_j F_{tr})w^i=0$$

by virtue of (8. 7). Thus, from (4. 4) in which $w_i=0$ on B, we have

$$\int_M\left[\frac{1}{2}(\nabla^j w^i-\nabla^i w^j)(\nabla_j w_i-\nabla_i w_j)+(\nabla_j w^j)(\nabla_i w^i)\right]d\sigma=0,$$

from which

$$\nabla_j w_i-\nabla_i w_j=0, \quad \nabla_i w^i=0 \qquad \text{on } M.$$

Thus we have

PROPOSITION 8. 2. *A covariant almost analytic vector field on an almost Kähler manifold vanishing on the boundary B is harmonic.*

In the case of almost Tachibana manifold, we have

$$F_{tsr}w^r=3(\nabla_r F_{ts})w^r=0$$

and

$$-(K^*_{ji}-K_{ji})w^j=(\nabla_j F^{ts})(\nabla_i F_{ts})w^j=0,$$

and consequently, the condition of Proposition 8. 1 reduces to

$$(\nabla_j F_i{}^t)(\nabla^j w^i) F_t{}^h w_h = 0 \qquad \text{on} \quad M,$$

$$g^{kj} \nabla_k \nabla_j w_i - K_{ji} w^j = 0 \qquad \text{on} \quad M.$$

Thus we have

PROPOSITION 8. 3. *A covariant almost analytic vector field in an almost Tachibana manifold vanishing on the boundary B is harmonic.*

Now, putting

$$w^h = B_a{}^h{}' w^a + \alpha N^h$$

and supposing

$$f_b{}^a{}' w_a = 0$$

that is,

$$'w_a = (f^c{}' w_c) f_a$$

on the boundary B, we have

$$T_{ji} F_r{}^i N^j w^r = -N^j w^i (\nabla_j w_i) + H_{cb}{}' w^c{}' w^b + \alpha^2 H_a{}^a$$
$$-2\alpha('\nabla_a{}' w^a) + '\nabla_a(\alpha' w^a) + \alpha^2 f_c{}^a f_a{}^b H_b{}^c + \alpha f^b{}' w_b('\nabla_a f^a).$$

Thus, following Proposition 4. 2, we have

PROPOSITION 8. 4. *A covariant almost analytic vector field w^h in an almost Hermitian manifold M which is tangent to the boundary B and has the direction of f^a on B is harmonic.*

If we suppose that the manifold M is Kählerian, then we have

$$\alpha^2 f_c{}^a f_a{}^b H_b{}^c + \alpha f^b{}' w_b('\nabla_a f^a) = \frac{1}{2} \alpha^2 L_a{}^a$$

where L_{cb} is the so-called Levi tensor defined by

$$L_{cb} = f_c{}^a('\nabla_b f_a - '\nabla_a f_b)$$

and consequently we have

PROPOSITION 8. 5. *A covariant almost analytic vector field w_h whose projection on the boundary B has the direction of f^a is harmonic if the contracted Levi tensor of the boundary B vanishes identically.*

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ADDED IN PROOF. After we had submitted the paper to the journal, we found that the following paper dealing with the similar topics as ours appeared.

HILT, A. L., AND C. C. HSIUNG, Vector fields and infinitesimal transformations on almost-hermitian manifolds with boundary. Canadian J. Math. **17** (1965), 213–238.

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