A NOTE ON ALGEBRAS OF REAL-ANALYTIC FUNCTIONS

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1. Let M and N be real-analytic manifolds with countable topology. We consider the totality of (real-valued) real-analytic functions on M and N; these form in a natural way function algebras $C(M)$ and $C(N)$, respectively. In this note we give a simple proof of the following theorem:

Theorem 1. *Assume that we have an isomorphism $\tilde{\phi}$ of $C(M)$ onto $C(N)$. Then there exists a real-analytic homeomorphism Φ of N onto M such that $\tilde{\phi}(f)(y) = f(\Phi(y))$ $(f \in C(M))$.*

Also we try to give a characterization of maximal ideal of $C(M)$.

It is well known that a result analogous to the above theorem holds in differentiable case [1]. Its proof, however, cannot be directly applicable to the above theorem since the method of localization through the partition of unity becomes infeasible in real-analytic case. Alternatively, use is made here of the following Cartan's result [2].

Proposition 1. *Let $f_i(x)$ be a finite number of real-analytic functions on M such that $f_i(x)$ have no common zero-points. Then there exist real-analytic functions $\varphi_i(x)$ on M satisfying $\sum f_i\varphi_i \equiv 1$.*

Now, let ζ be a homomorphism of $C(M)$ onto the real number field $\boldsymbol{R}$. We call such ζ a character of $C(M)$. Put $A_\zeta = \zeta^{-1}(0)$. It is clear that A_ζ is a maximal ideal of $C(M)$.

Lemma 1. *For a given character ζ, there exist a compact set K and a real-analytic function f of A_ζ such that f has no zero-point outside of K.*

Proof. There is a real-analytic function g on M with the property that for any real number α the set $S_\alpha = \{x \mid g(x) = \alpha\}$ is a compact set. Actually, the existence of such a g is an immediate consequence of the fact that M can be properly imbedded in a Euclidean space via real-analytic map such that its image becomes a closed set [3]. Then $f = g - \zeta(g)\cdot 1$ is in A_ζ, while the zero-points of f form a compact set. Hence f is a desired function.

Lemma 2. *Any A_ζ is given by a maximal ideal of $C(M)$ which consists of the functions vanishing at a single point $x_\zeta \in M$.*

Proof. Let f be a real analytic function of A given in Lemma 1. The zero-points of f are contained in a compact set K. We show that real-analytic functions

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of A_ζ have a common zero-point. To see this, assume the contrary. Then for any point x of K there is an $f_x \in A_\zeta$ with $f_x(x) \neq 0$. Since K is compact, we can find a finite number of functions f_{x_i} $(x_i \in K)$ such that $f_{x_i}(x)$ have no common zero-points in K. Therefore $f_{x_i}(x)$ and $f(x)$ have no common zero-points throughout M. Hence we have $1 \in A_\zeta$ by Proposition 1, which yields a contradiction. A_ζ being a maximal ideal, it follows readily that the common zero-points of A_ζ must be a single point, which leads to the conclusion of Lemma 2.

Once Lemma 2 is established, the proof of the theorem is straightforward by a well known reasoning, based on the following result also due to Cartan [2].

PROPOSITION 2. *Let $x_1, x_2, \cdots, x_l, \cdots$ be a sequence of points of M without accumulation points. Let r be any positive integer. Then there is a real-analytic function on M such that its partial derivatives up to order r take any assigned values at $x_1, x_2, \cdots, x_l, \cdots$.*

However, for the sake of completeness, we shall sketch the proof of the theorem. Lemma 2 and Proposition 2 show that the totality of characters of $C(M)$ can be canonically identified with the points of M. The same is true for characters of $C(N)$. Therefore, the given isomorphism $\tilde{\phi}$ of $C(M)$ onto $C(N)$ gives rise to a bijective map Φ of N onto M. Φ satisfies $f \circ \Phi(y) = \tilde{\phi}(f)(y)$. Now let $y_l \to y_0$. Then $\{\Phi(y_l)\}$ have at least one accumulation point. Otherwise, we have $\Phi(y_l) \to \infty$, hence by Proposition 2 we can find $f \in C(M)$ satisfying $f \circ \Phi(y_l) \to \infty$; this, however, contradicts the fact that $f \circ \Phi(y_l) = \tilde{\phi}(f)(y_l) \to \tilde{\phi}(f)(y_0)$. Since any distinct points of M can be separated by an $f \in C(M)$ by Proposition 2, it follows that $\{\Phi(y_l)\}$ contains only one accumulation point $\Phi(y_0)$. Thus Φ is continuous. The similar consideration of $\tilde{\phi}^{-1}$ yields that Φ gives a homeomorphism of N onto M. For any point $x_0 \in M$, consider admissible local coordinates $(\mu^1(x), \cdots, \mu^n(x))$ around x_0. From the real-analytic imbeddability of M in a Euclidean space, it follows that each $\mu^i(x)$ can be assumed to be in $C(M)$. Then $\mu^i \circ \Phi(y) = \tilde{\phi}(\mu^i)(y)$ implies that Φ is a real-analytic mapping of N into M. The same is true for Φ^{-1}. This completes the proof of the theorem.

2. We consider maximal ideals of $C(M)$. For any $f \in C(M)$, we denote the set of zero-points of f by Z_f. We say that a subset A of $C(M)$ satisfies the property (P) if the following is valid:

For any finite number of functions $f_i \in A$ and for any compact set K of M, we have

$$\cap Z_{f_i} \cap K^c \neq \phi.$$

The property (P) clearly is of finite character with respect to elements of A. Hence by Zorn's lemma, given any A with the property (P), there exists a maximal set $\tilde{A} \subset C(M)$ such that $\tilde{A}$ satisfies the property (P) and $\tilde{A} \supset A$.

THEOREM 2. *A maximal ideal I of $C(M)$ which is not induced by any character necessarily becomes a maximal set of $C(M)$ satisfying the property* (P).

Proof. Let I be a maximal ideal as stated in Theorem. It is easily seen that the functions of I have no common zero-points. We observe that if the conclusion of Lemma 1 is valid for I, then the proof of Lemma 2 shows that I is induced by a character. Therefore, I contains no real-analytic function f satisfying $Z_f \subset K$ for some compact set K of M. It follows that I satisfies the property (P); otherwise, we have a finite number of functions $f_i \in I$ and a compact set K such that $\cap Z_{f_i} \subset K$, so that $\sum f_i^2$ $(\in I)$ has its zero-points in K, which is a contradiction. Now set $\tilde{I}$ for a maximal set of $C(M)$ such that $\tilde{I}$ satisfies (P) and $\tilde{I} \supset I$. We show that $\tilde{I}$ is an ideal of $C(M)$. This is obvious from the maximality of $\tilde{I}$, combined with the fact that $Z_{f+g} \supset Z_f \cap Z_g$, $Z_{fg} = Z_f \cup Z_g \supset Z_f$. Also it is trivial to verify $\tilde{I} \neq C(M)$. Thus $\tilde{I} = I$ from the maximality of the ideal I. This completes the proof.

We do not know whether or not the converse of Theorem 2 is valid. We simply remark the following fact. Let $\tilde{I}$ be a maximal set of $C(M)$ satisfying (P). Assume furthermore that $\tilde{I}$ fulfills the supplementary condition (Q): any compact set $K \subset M$, there exists an $f \in \tilde{I}$ such that $Z_f \subset K^c$. Then $\tilde{I}$ becomes a maximal ideal of $C(M)$ which is not induced by any character. This is seen as follows: we have only to check that $\tilde{I}$ is a maximal ideal. Let $h \notin \tilde{I}$. Then for some finite number of functions f_i of $\tilde{I}$ and for a compact set K, we have $Z_h \cap (\cap Z_{f_i}) \subset K$ by virtue of the maximality of $\tilde{I}$. Therefore $h^2 + \sum f_i^2$ has the zero-points in K. Take a $g \in I$ with $Z_g \subset K^c$; such a g exists by (Q). Then $h^2 + \sum f_i^2$ and g are contained in the ideal generated by $\tilde{I}$ and h, which have no common zero-points. Hence Proposition 1 shows that the ideal $(\tilde{I}, h)$ becomes $C(M)$, which in turn implies the maximality of the ideal $\tilde{I}$.

Finally we note that the similar discussion to the above holds in differentiable case as well. In this case the condition (Q) however is always satisfied, which can be seen by making use of partition of unity. Denoting by $C^\infty(M)$ the algebras of differentiable functions on M, we thus obtain

THEOREM 3. *The family of maximal ideals of $C^\infty(M)$ consists of the following two families of subsets of $C^\infty(M)$:*

(i) *a set consisting of the zero-points of some character ζ: $I = \zeta^{-1}(0)$; this ideal corresponds to a point of M.*

(ii) *a maximal set satisfying the property* (P).

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