RICCI'S FORMULA FOR NORMAL GENERAL CONNECTIONS AND ITS APPLICATIONS

BY TOMINOSUKE ŌTSUKI AND CHORNG-SHI HOUH

In [15], Ōtsuki gave a kind of Ricci's formulas for a space with an integrable normal general connection, that is the distribution of the tangent subspaces of the space associated with the connection is completely integrable. In the present paper, the authors will give a generalized Ricci's formula without the condition of integrability and its applications for induced general connections on subspaces.

§1. Preliminaries.

Let $\mathfrak{X}$ be an n-dimensional differentiable manifold with a general connection[1] which is written in terms of local coordinates u^i as

(1.1) $$\gamma=\partial u_j\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h),$$

where $\partial u_j=\partial/\partial u^j$ and d^2u^i denotes the differential of order 2 of u^i.

The components of the curvature tensor of γ are given by

(1.2) $$R_i{}^j{}_{hk}=\left\{P_l^j\left(\frac{\partial\Gamma_{mk}^l}{\partial u^h}-\frac{\partial\Gamma_{mh}^l}{\partial u^k}\right)+\Gamma_l{}^j{}_h\Gamma_m{}^l{}_k-\Gamma_l{}^j{}_k\Gamma_{mh}^l\right\}P_i^m$$
$$-\delta_{m,h}^j\Lambda_{ik}^m+\delta_{m,k}^i\Lambda_{ih}^m,\ ^{1)}$$

where

$$\Lambda_{ih}^j=\Gamma_{ih}^j-\frac{\partial P_i^j}{\partial u^h}$$

and $\delta_{m,h}^j$ denote the covariant derivarives of the Kronecker's δ_m^j with respect to γ.

γ is called *normal* when the tensor $P=P_i^j\partial u_j\otimes du^i$ of type (1, 1) is normal.[2] Let Q be the tensor such that $Q=P^{-1}$ on the image of P and $Q=P$ on the kernel of P at each point of $\mathfrak{X}$ regarding P as a homomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$. The tensor field $A=PQ=QP$ with local components A_i^j is called the *canonical projection* of γ. The components $'R_i{}^j{}_{hk}$ and $''R_i{}^j{}_{hk}$ of the curvature tensors of the contravariant part $'\gamma=Q\gamma$ and the covariant part $''\gamma=\gamma Q$ of the normal general connection γ can be written respectively as

Received October 31, 1964.

1) See [8], §6.

2) See [11], §1.

(1. 3) $$'R_i{}^j{}_{hk}=A_l^j\left(\frac{\partial A_{mk}^l}{\partial u^h}-\frac{\partial' A_{mh}^l}{\partial u^k}+'A_{th}^l{}'A_{mk}^t-'A_{tk}^l{}'A_{mh}^t\right)A_i^m$$

and

(1. 4) $$''R_i{}^j{}_{hk}=A_l^j\left(\frac{\partial''\Gamma_{mk}^l}{\partial u^h}-\frac{\partial''\Gamma_{mh}^l}{\partial u^k}+''\Gamma_{th}^l{}''\Gamma_{mk}^t-''\Gamma_{tk}^l{}''\Gamma_{mh}^t\right)A_i^m.$$ [3]

A tensor V of type (β, α) with local components $V_{i_1\cdots i_\alpha}^{j_1\cdots j_\beta}$ is called *A-invariant if*

$$V_{i_1\cdots i_\alpha}^{j_1\cdots j_\beta}=A_k^{j_\tau}V_{i_1i_2\cdots i_\alpha}^{j_1\cdots k\cdots j_\beta}=V_{i_1\cdots k\cdots i_\alpha}^{j_1j_2\cdots j_\beta}A_{i_\sigma}^k,$$

$$\tau=1, 2, \cdots, \beta;\ \sigma=1, 2, \cdots, \alpha$$

The basic covariant derivatives of an A-invariant tensor V of type (β, α) with respect to γ can be writtten as[4]

(1. 5) $$V_{i_1\cdots i_\alpha|h}^{j_1\cdots j_\beta}=\frac{\partial V_{i_1\cdots i_\alpha}^{j_1\cdots j_\beta}}{\partial u^h}+\sum_{\tau=1}^{\beta}{}'A_{kh}^{j_\tau}V_{i_1i_2\cdots i_\alpha}^{j_1\cdots k\cdots j_\beta}$$
$$-\sum_{\sigma=1}^{\alpha}{}''\Gamma_{i_\sigma h}^k V_{i_1\cdots k\cdots i_\alpha}^{j_1j_2\cdots j_\beta},$$

The tensor of type $(\beta, \alpha+1)$ with local components

(1. 6) $$V_{i_1\cdots i_\alpha||h}^{j_1\cdots j_\beta}=V_{i_1\cdots i_\alpha|l}^{j_1\cdots j_\beta}A_h^l$$

is also A-invariant.[5]

§2. Ricci's formula for spaces with normal general connections.

Making use of the notations in §1, let γ be a normal general connection on an n-dimensional differentiable manifold $\mathfrak{X}$. Let V be an A-invariant tensor of type (β, α), then we have (1. 5). Since the tensor with the components $V_{i_1\cdots i_\alpha||h}^{j_1\cdots j_\beta}$ is A-invariant, we have

$$\begin{aligned}
V_{i_1\cdots i_\alpha||h||k}^{j_1\cdots j_\beta}&=V_{i_1\cdots i_\alpha||h|q}^{j_1\cdots j_\beta}A_k^q\\
&=\left(\frac{\partial V_{i_1\cdots i_\alpha||h}^{j_1\cdots j_\beta}}{\partial u^q}+\sum_{\tau=1}^{\beta}{}'A_{lq}^{j_\tau}V_{i_1i_2\cdots i_\alpha||h}^{j_1\cdots l\cdots i_\beta}\right.\\
&\quad\left.-\sum_{\sigma=1}^{\alpha}{}''\Gamma_{i_\sigma q}^l V_{i_1\cdots l\cdots i_\alpha||h}^{j_1j_2\cdots j_\beta}-''\Gamma_{hq}^l V_{i_1\cdots i_\alpha||l}^{j_1\cdots j_\beta}\right)A_k^q\\
&=\left(\frac{\partial V_{i_1\cdots i_\alpha|p}^{j_1\cdots j_\beta}}{\partial u^q}+\sum_{\tau=1}^{\beta}{}'A_{lq}^{j_\tau}V_{i_1i_2\cdots i_\alpha|p}^{j_1\cdots l\cdots j_\beta}-\sum_{\sigma=1}^{\alpha}{}''\Gamma_{i_\sigma q}^l V_{i_1\cdots l\cdots i_\alpha|p}^{j_1j_2\cdots j_\beta}\right)A_h^pA_k^q\\
&\quad+V_{i_1\cdots i_\alpha|p}^{j_1\cdots j_\beta}\left(\frac{\partial A_h^p}{\partial u^q}-A_l^p{}''\Gamma_{hq}^l\right)A_k^q.
\end{aligned}$$

3) See [15], §2.
4) See [11], §4.
5) See [11], Theorem 4. 1.

Making use of (1. 5), the equation can be written as

$$=\Big\{\frac{\partial^2 V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha}}{\partial u^q\partial u^p}+\sum_{\tau=1}^{\beta}\frac{\partial' A^{j_\tau}_{lp}}{\partial u^q}V^{j_1\cdots l\cdots j_\beta}_{i_1 i_2\cdots i_\alpha}+\sum_{\tau=1}^{\beta}{}'A^{j_\tau}_{lp}\frac{\partial^{j_1\cdots l\cdots j_\beta}_{i_1 i_2\cdots\cdots i_\alpha}}{\partial u^q}$$

$$-\sum_{\sigma=1}^{\alpha}\frac{\partial'' \Gamma^l_{i_\sigma p}}{\partial u^q}V^{j_1 j_2\cdots j_\beta}_{i_1\cdots l\cdots i_\alpha}-\sum_{\sigma=1}^{\alpha}{}''\Gamma^l_{i_\sigma p}\frac{\partial V^{j_1 j_2\cdots j_\beta}_{i_1\cdots l\cdots i_\alpha}}{\partial u^q}$$

$$+\sum_{\tau=1}^{\beta}{}'A^{j_\tau}_{lq}V^{j_1\cdots l\cdots j_\beta}_{i_1 i_2\cdots i_\alpha|p}-\sum_{\sigma=1}^{\alpha}\Gamma^l_{i_\sigma q}V^{j_1 j_2\cdots j_\beta}_{i_1\cdots l\cdots i_\alpha|q}\Big\}A^p_h A^q_k$$

$$+V^{j_1 j_2\cdots j_\beta}_{i_1\cdots i_\alpha|p}\Big(\frac{\partial A^p_h}{\partial u^q}-A^{p}_{l}{}''\Gamma^l_{hq}\Big)A^q_k.$$

Hence, by virtue of (1. 5), we have

$$V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||h||k}-V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||k||h}$$

$$=\Big\{-\sum_{\tau=1}^{\beta}\Big(\frac{\partial' A^{j_\tau}_{lq}}{\partial u^p}-\frac{\partial' A^{j_\tau}_{lp}}{\partial u^q}+{}'A^{j_\tau}_{mp}{}'A^m_{lq}-{}'A^{j_\tau}_{mq}{}'A^m_{lp}\Big)V^{j_1\cdots l\cdots j_\beta}_{i_1 i_2\cdots i_\alpha}$$

$$+\Big(\frac{\partial''\Gamma^l_{i_\sigma q}}{\partial u^p}-\frac{\partial''\Gamma^l_{i_\sigma p}}{\partial u^q}+{}''\Gamma^l_{mp}{}''\Gamma^m_{i_\sigma q}-{}''\Gamma^l_{mq}{}''\Gamma^m_{i_\sigma p}\Big)V^{j_1 j_2\cdots j_\beta}_{i_1\cdots l\cdots i_\alpha}\Big\}A^p_h A^q_k$$

$$+V^{j\cdots j}_{i\cdots i/p}\Big\{\Big(\frac{\partial A^p_h}{\partial u^q}-A^{p}_{l}{}''\Gamma^l_{pq}\Big)A^p_k-\Big(\frac{\partial A^p_k}{\partial u^q}-A^{p}_{l}{}''\Gamma^l_{kq}\Big)A^q_h\Big\}$$

Making use of $A^2=A$, $\lambda(\gamma Q)=A$ and $(\gamma Q)A=\gamma Q$.[6] We have

$$\Big(\frac{\partial A^p_h}{\partial u^q}-A^{p}_{l}{}''\Gamma^l_{hq}\Big)A^q_k$$

$$=\Big(\frac{\partial A^p_h}{\partial u^q}-A^{p}_{l}{}''A^l_{hq}-A^p_l\frac{\partial A^l_h}{\partial u^q}\Big)A^q_k$$

$$=\Big(\frac{\partial A^p_l}{\partial u^q}A^l_h-A^{p}_{l}{}''A^l_{hq}\Big)A^q_k=\Big(\frac{\partial A^p_l}{\partial u^q}-A^{p}_{m}{}''A^m_{lq}\Big)A^l_h A^q_k$$

Hence the last terms of the equation can be written as

$$V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|l}\Big\{\Big(\frac{\partial A^l_p}{\partial u^q}-\frac{\partial A^l_q}{\partial u^p}\Big)-A^l_m({}''A^m_{pq}-{}''A^m_{qp})\Big\}A^p_h A^q_k.$$

Putting $N^j_i=\delta^j_i-A^j_i$, we have

$$V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||m}\frac{\partial A^m_p}{\partial u^q}A^p_h=-V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||m}\frac{\partial N^m_p}{\partial u^q}A^p_h=V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||m}N^m_p\frac{\partial A^p_h}{\partial u^q}=0.$$

Hence, denoting the torsion tensor of $''\gamma$ by

(2. 1) $$''S^j_{ih}=\frac{1}{2}({}''\Gamma^j_{ih}-{}''\Gamma^j_{hi}),$$

6) $''\gamma A={}''\gamma$ follows $''A^j_{lh}A^l_i={}''A^j_{ih}$.

the above quantity can be written as

$$-\left\{V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|l}\left(\frac{\partial N^l_p}{\partial u^q}-\frac{\partial N^l_q}{\partial u^p}\right)+2V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||l}{}''S^l_{pq}\right\}A^p_hA^q_k.$$

Since the tensors $V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha}$ and $V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||h||k}$ are A-invariant and the tensor $V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|h}$ is so with respect to $i_1, \cdots, i_\alpha, j_1, \cdots, j_\beta$, by (1. 3) and (1. 4) we have the following generalized Ricci's formula

$$\begin{aligned} &V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||h||k}-V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||k||h} \\ &=\left\{-\sum_{\tau=1}^{\beta}{}'R^{j_\tau}_{lpq}V^{j_1\cdots l\cdots j_\beta}_{i_1\cdots i_\alpha}+\sum_{\sigma=1}^{\alpha}V^{j_1\cdots j_\beta}_{i_1\cdots l\cdots i_\alpha}{}''R^l_{i_\sigma pq}\right. \\ &\left.\quad -2V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha||l}{}''S^l_{pq}-V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|l}\left(\frac{\partial N^l_p}{\partial u^q}-\frac{\partial N^l_q}{\partial u^p}\right)\right\}A^p_hA^q_k. \end{aligned} \tag{2. 2}$$

Applying this formula for A and a scalar φ, we get

$$\begin{aligned} A^j_{i||h||k}-A^j_{i||k||h}=&\left\{-{}'R_i{}^j{}_{pq}+{}''R_i{}^j{}_{pq}-2A^j_{i||l}{}''S^l_{pq}\right. \\ &\left.-A^j_{i|l}\left(\frac{\partial N^l_p}{\partial u^q}-\frac{\partial N^l_p}{\partial u^p}\right)\right\}A^p_hA^q_k \end{aligned} \tag{2. 3}$$

and

$$\varphi_{||h||k}-\varphi_{||k||h}=-\left\{2\varphi_{||l}{}''S^l_{pq}+\varphi_{|l}\left(\frac{\partial N^l_p}{\partial u^q}-\frac{\partial N^l_q}{\partial u^p}\right)\right\}A^p_hA^q_k, \tag{2. 4}$$

because ${}'R_i{}^j{}_{hk}=A^j_l{}'R_i{}^l{}_{hk}={}'R^j_{lhk}A^l_i$ and ${}''R^j_{ihk}=A^j_l{}''R_i{}^l{}_{hk}={}''R_l{}^j{}_{hk}A^l_i$ by (1. 3) and (1. 4).

§ 3. Application of the Ricci's formula.

Let V and W be tensors of type (β, α) and (τ, σ) respectively. Then, by means of the formula of the basic covariant differentiation of a normal general connection,[7] we have

$$\begin{aligned} (V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha}W^{q_1\cdots q_\tau}_{p_1\cdots p_\sigma})_{|h}=&V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|h}A^{q_1}_{m_1}\cdots A^{q_\tau}_{m_\tau}W^{m_1\cdots m_\tau}_{l_1\cdots l_\sigma}A^{l_1}_{p_1}\cdots A^{l_\sigma}_{p_\sigma} \\ &+A^{j_1}_{m_1}\cdots A^{j_\beta}_{m_\beta}V^{m_1\cdots m_\beta}_{l_1\cdots l_\alpha}A^{l_1}_{i_1}\cdots A^{l_\alpha}_{i_\alpha}W^{q_1\cdots q_\tau}_{p_1\cdots p_\sigma|h}. \end{aligned} \tag{3. 1}$$

Especially, when V and W are A-invariant, (3. 1) can be written as

$$(V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha}W^{q_1\cdots q_\tau}_{p_1\cdots p_\sigma})_{|h}=V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha|h}W^{q_1\cdots q_\tau}_{p_1\cdots p_\sigma}+V^{j_1\cdots j_\beta}_{i_1\cdots i_\alpha}W^{q_1\cdots q_\tau}_{p_1\cdots p_\sigma|h}. \tag{3. 2}$$

Let V be an A-invariant tensor of type (β, α) $(\beta, \alpha\geqq 1)$ and W be the tensor of type $(\beta-1, \alpha-1)$ with components

$$W^{j_2\cdots j_\beta}_{i_2\cdots i_\alpha}=V^{lj_2\cdots j_\beta}_{li_2\cdots i_\alpha}$$

7) See [11], § 4, (4. 2′).

by the contraction. By means of (1. 5), we get

$$\delta^i_j V^{jj_2\cdots j_\beta}_{ii_2\cdots i_\alpha|h}=W^{j_2\cdots j_\beta}_{i_2\cdots i_\alpha|h}+('\Lambda^l_{kh}-''\Gamma^l_{kh})V^{kj_2\cdots j_\beta}_{li_2\cdots i_\alpha}.$$

Since we have

$$\delta^j_{i|h}=\frac{\partial A^j_i}{\partial u^h}+'\Lambda^j_{lh}A^l_i-A^j_l{}''\Gamma^l_{ih} \quad (3.3)$$ [8]

and

$$A^j_l\frac{\partial A^l_m}{\partial u^h}A^m_i=0,$$

we have

$$\delta^l_{k|h}V^{kj_2\cdots j_\beta}_{li_2\cdots i_\alpha}=('\Lambda^l_{ph}A^p_k-A^l_p{}''\Gamma^p_{kh})V^{kj_2\cdots j_\beta}_{li_2\cdots i_\alpha}$$
$$=('\Lambda^l_{kh}-''\Gamma^l_{kh})V^{kj_2\cdots j_\beta}_{li_2\cdots i_\alpha}.$$

Hence the above equation can be written as

$$\delta^i_j V^{jj_2\cdots j_\beta}_{ii_2\cdots i_\alpha|h}=W^{j_2\cdots j_\beta}_{i_2\cdots i_\alpha|h}+\delta^i_{j|h}V^{jj_2\cdots j_\beta}_{ii_2\cdots i_\alpha}, \quad (3.4)$$ [9]

where $W^{j_2\cdots j_\beta}_{i_2\cdots i_\alpha}=V^{lj_2\cdots j_\beta}_{li_2\cdots i_\alpha}$.

LEMMA 1. *Let γ be a normal general connection, then we have*

$$A^j_{i|h}=\delta^j_{i|h}, \quad (3.5)$$

$$A^j_{i,h}=\delta^j_{i,h} \quad (3.6)$$

and

$$Q^j_{i|h}=0. \quad (3.7)$$

Proof. Since $A^j_lN^l_i=N^j_lA^l_i=0$, we have

$$N^j_{i|h}='\Lambda^j_{lh}N^l_kA^k_i-A^j_lN^l_k{}''\Gamma^k_{ih}$$ [8]

from which we get (3. 5).

Since $P^j_lN^l_i=N^j_lP^l_i=0$, we have analogously $N^j_{i,h}=0$,[10] hence (3. 6). Lastly, since Q^j_i is A-invariant, we have

$$Q^j_{i|h}=\frac{\partial Q^j_i}{\partial u^h}+'\Lambda^j_{mh}Q^m_i-Q^j_m{}''\Gamma^m_{ih}$$
$$=\frac{\partial Q^j_i}{\partial u^h}+\left(Q^j_l\Gamma^l_{mh}-\frac{\partial A^j_m}{\partial u^h}\right)Q^m_i$$
$$-Q^j_m\left(\Gamma^m_{lh}Q^l_i+P^m_l\frac{\partial Q^l_i}{\partial u^h}\right)=0,$$

by means of

---

8) See [11], § 4, (4. 2′).

9) This formula is analogous to [8], (3. 11) regarding to a regular general connections.

10) See [8], § 2, (2. 14) or § 3, Theorem 3. 5.

$$\text{(3.8)} \qquad \begin{cases} {}'\Lambda^j_{ih} = Q^j_k \Gamma^k_{ih} - \dfrac{\partial A^j_i}{\partial u^h}, \\ {}''\Gamma^j_{ih} = \Gamma^j_{kh} Q^k_i + P^j_k \dfrac{\partial Q^k_i}{\partial u^h}. \end{cases}$$ [11]

LEMMA 2. *Let γ be a normal general connection satisfying one of the following conditions*

(i) $\bar{D}A = A\omega$, (ii) $DA = A\omega$ *and* (iii) $DA = P\omega$

where $\omega = \rho_h du^h$ is a differential form and D and $\bar{D}$ are the covariant and the basic covariant differential operators[12] *of γ respectively. Then ω is exact.*

Proof. Let m be the rank of the matrix (P^j_i).

Case (i): $\bar{D}A = A\omega$. This condition is written as

$$\text{(3.9)} \qquad A^j_{i|h} = A^j_i \rho_h,$$

from which by means of (3.3), (3.4) and (3.8) we get

$$m\rho_h = \delta^i_j A^j_{i|h} = \delta^i_{j|h} A^j_i$$
$$= \left(\frac{\partial A^i_j}{\partial u^h} + {}'\Lambda^i_{jh} - {}''\Gamma^i_{jh} \right) A^j_i = -P^i_k \frac{\partial Q^k_i}{\partial u^h}.$$

On the other hand, Since $A^2 = A$, $N^2 = N$, $PN = NP = QN = NQ = 0$ and $PQ = QP = A$, we have

$$\text{(3.10)} \qquad A^j_i \frac{\partial A^i_j}{\partial u^h} = N^j_i \frac{\partial N^i_j}{\partial u^h} = 0,$$

$$\text{(3.11)} \qquad P^j_i \frac{\partial N^i_j}{\partial u^h} = Q^j_i \frac{\partial N^i_j}{\partial u^h} = 0$$

and

$$\text{(3.12)} \qquad (P+N)(Q+N) = 1.$$

Making use of these relations, we get

$$m\rho_h = -P^i_k \frac{\partial Q^k_i}{\partial u^h} = -(P^i_k + N^i_k) \frac{\partial (Q^k_i + N^k_i)}{\partial u^h},$$

hence

$$\text{(3.13)} \qquad m\rho_h = -\frac{\partial}{\partial u^h} \log \det (Q+N).$$

This shows that ρ_h is a gradient vector, if $m \neq 0$. But the case $m = 0$ is trivial.

Case (ii): $DA = A\omega$. Making use of the fundamental relation between D and $\bar{D}$

11) See [11], §4, (4.9).

12) See [11], §§3, 4.

(3.14) $$\iota_A \cdot D = \iota_P \cdot \bar{D}$$ [13]

and the condition (ii), we get

(3.15) $$A^j_i \rho_h = P^j_i A^l_{m|h} P^m_i,$$

from which by (3.3), (3.5) and (3.8) we get

$$m\rho_h = P^i_k P^k_j A^j_{i|h} = P^i_k P^k_j \delta^j_{i|h} = P^i_k P^k_j \left(\frac{\partial A^j_i}{\partial u^h} + 'A^j_{ih} - ''\Gamma^j_{ih} \right)$$
$$= P^i_k P^k_j \left(Q^j_l \Gamma^l_{ih} - \Gamma^j_{lh} Q^l_i - P^j_l \frac{\partial Q^l_i}{\partial u^h} \right) = -P^i_k P^k_j P^j_l \frac{\partial Q^l_i}{\partial u^h}$$
$$= \frac{1}{2} \frac{\partial \operatorname{tr} P^2}{\partial u^h} + P^i_k P^k_j \frac{\partial N^j_i}{\partial u^h},$$

that is

(3.16) $$m\rho_h = \frac{1}{2} \frac{\partial \operatorname{tr} P^2}{\partial u^h}.$$

Case (iii): $DA = P\omega$. Making use of (3.11), we get from this condition (iii)

(3.17) $$P^j_i \rho_h = P^j_l A^l_{m|h} P^m_i,$$

which is equivalent to

$$A^j_i \rho_h = Q^j_k P^k_l A^l_{m|h} P^m_i = A^j_{m|h} P^m_i$$

since $A^j_{i|h}$ is A-invariant with respect to i and j. Hence, by (3.5), (3.3), (3.8) and (3.11) we have

$$m\rho_h = P^i_j A^j_{i|h} = -P^i_j P^j_l \frac{\partial Q^l_i}{\partial u^h} = \frac{\partial P^i_j}{\partial u^h} A^j_i + P^i_j \frac{\partial N^j_i}{\partial u^h} = \frac{\partial \operatorname{tr} P}{\partial u^h},$$

that is

(3.18) $$m\rho_h = \frac{\partial \operatorname{tr} P}{\partial u^h}.$$

Thus, the lemma is proved.

LEMMA 3. *Let γ be a normal general connection such as in lemma 2, then the curvature forms of $'\gamma = Q\gamma$ and $''\gamma = \gamma Q$ are commuatative with $P = \lambda(\gamma)$ on the image of P, that is*

(3.19) $$P^j_l {}'R_i{}^l{}_{pq} A^p_h A^q_k = {}'R_l{}^j{}_{pq} P^l_i A^p_h A^q_k,$$

(3.20) $$P^j_l {}''R_i{}^l{}_{pq} A^p_h A^q_k = {}''R_l{}^j{}_{pq} P^l_i A^p_h A^q_k.$$

Proof. In the following, we may assume $m > 0$.

Case (i): $\bar{D}A = A\omega$. By Lemma 2 and (3.13), putting $\varphi = -(1/m) \log\det(Q+N)$, we have $\rho_h = \varphi_{|h}$ and so from (3.9) $A^j_{i||h} = A^j_i \varphi_{||h}$. Using (3.2), we get

13) See [11], §3, (3.14).

$$A^j_{i||h||k}=A^j_{i||k}\varphi_{||h}+A^j_i\varphi_{||h||k}$$
$$=A^j_i\varphi_{||k}\varphi_{||h}+A^j_i\varphi_{||h||k}$$

and so

$$A^j_{i||h||k}-A^j_{i||k||h}=A^j_i(\varphi_{||h||k}-\varphi_{||k||h}).$$

On the other hand, the right of (2. 3) can be written in this case as

$$(-'R_i{}^j{}_{pq}+''R_i{}^j{}_{pq})A^p_hA^q_k$$
$$-A^j_i\left\{2\varphi_{||l}''S^l_{pq}+\varphi_{|l}\left(\frac{\partial N^l_p}{\partial u^q}-\frac{\partial N^l_q}{\partial u^p}\right)\right\}A^p_hA^q_k$$
$$=(-'R_i{}^j{}_{pq}+''R_i{}^j{}_{pq})A^p_hA^q_k+A^j_i(\varphi_{||h||k}-\varphi_{||k||h}).$$

hence we obtain

$$(-'R_i{}^j{}_{pq}+''R_i{}^j{}_{pq})A^p_hA^q_k=0. \tag{3. 21}$$

Case (ii): $DA=A\omega$. By Lemma 1, (3. 16), putting $\varphi=(1/2m)\operatorname{tr}P^2$, we have $\rho_h=\varphi_{|h}$. Since $A^j_{i|h}$ is A-invariant with respect to j and i,[14] (3. 15) is equivalent to

$$A^j_{i|h}=Q^j_lQ^l_i\varphi_{|h}, \tag{3. 15}$$

from which we get

$$A^j_{i||h}=Q^j_lQ^l_i\varphi_{||h},$$
$$A^j_{i||h||k}=Q^j_lQ^l_i\varphi_{||h||k}+(Q^j_lQ^l_i)_{||k}\varphi_{||h}=Q^j_lQ^l_i\varphi_{||h||k}-\delta^l_{m||k}Q^j_lQ^m_i\varphi_{||h}$$
$$=Q^j_lQ^l_i\varphi_{||h||k}-A^l_{m||k}Q^j_lQ^m_i\varphi_{||h}=Q^j_lQ^l_i\varphi_{||h||k}-Q^l_tQ^t_mQ^j_lQ^m_i\varphi_{||h}\varphi_{||k}$$

by means of Lemma 1, (3. 4) and (3. 15′). In this case, the right of (2. 3) can be written as

$$(-'R_i{}^j{}_{pq}+''R_i{}^j{}_{pq})A^p_hA^q_k+Q^j_lQ^l_i(\varphi_{||h||k}-\varphi_{||k||h}),$$

hence we obtain also (3. 21).

Case (iii): $DA=P\omega$. By Lemma 1, (3. 18), putting $\varphi=(1/m)\operatorname{tr}P$, we have $\rho_h=\varphi_{|h}$. (3. 17) is equivalent to

$$A^j_{i|h}=Q^j_i\varphi_{|h}, \tag{3. 17′}$$

from which we get

$$A^j_{i||h||k}=Q^j_i\varphi_{||h||k}.$$

Analogously, we get also (3. 21).

Lastly, according to [15], Theorem 2, we have

$$P^j_l{}'R_i{}^l{}_{hk}={}''R_l{}^j{}_{hk}P^l_i.$$

from this relation and (3. 21), we get easily (3. 19) and (3. 20). The lemma is completely proved.

---

14) See [11], §4, Theorem 4. 1.

§4. Some relations between the Ricci's formula and induced general connections.

Let γ be a general connection on $\mathfrak{X}$ given by (1.1). Let $\mathfrak{Y}$ be an m-dimensional submanifold given by

$$u^j = u^j(v^\alpha), \tag{4.1}$$

in terms of local coordinates u^j of $\mathfrak{X}$ and v^α of $\mathfrak{Y}$. Let Z be a field of $(n-m)$-dimensional tangent subspaces of $\mathfrak{X}$ given on $\mathfrak{Y}$ which is complementary with the tengent space of $\mathfrak{Y}$ at each point of $\mathfrak{Y}$. Let $\{X_\alpha, X_\lambda\}$, $\alpha=1, \cdots, m$; $\lambda=m+1, \cdots, n$, be a local field of n-frames of $\mathfrak{X}$ on $\mathfrak{Y}$ such that

$$X_\alpha = X^j_\alpha \partial/\partial u^j,\ X^j_\alpha = \frac{\partial u^j}{\partial v^\alpha} \text{ and } X_\lambda = X^j_\lambda \partial/\partial u^j \in Z$$

and $\{Y^\beta, Y^\mu\}$ with local components Y^β_i, Y^μ_i be its dual. Then, we say the general connection on $\mathfrak{Y}$

$$\gamma^* = \partial v_\beta \otimes Y^\beta_j \iota^* (P^j_i d^2 u^i + \Gamma^j_{ih} du^i \otimes du^h)\ [15] \tag{4.2}$$

the *induced general connection on* $\mathfrak{Y}$ *from* γ *by means of* Z.

A normal general connection γ is called *contravariantly proper* or *covariantly proper* if

$$N^j_k \Gamma^k_{lm} A^l_i A^m_h = 0 \tag{4.3}$$

or

$$A^j_k A^k_{lm} N^l_i A^m_h = 0. \tag{4.4}$$

LEMMA 4. *Let γ be a normal general connection such that $DA = A\omega$ or $DA = P\omega$ as in Lemma 2, then γ is contravariantly proper.*

Proof. By the assumption and Lemma 1, we have

$$0 = N^j_k A^k_{i,h} = N^j_k \delta^k_{i,h} = N^j_k (\Gamma^k_{ih} P^l_i - P^j_l A^l_{ih}) = N^j_k \Gamma^k_{lh} P^l_i,$$

hence

$$N^j_k \Gamma^k_{lm} A^l_i A^m_h = 0.$$

A normal general connection γ is called *integrable* if the distribution of the tangent subspaces $P_x = P(T_x(\mathfrak{X}))$, $x \in \mathfrak{X}$, is completely integrable.

THEOREM 4.1. *Let γ be an integrable normal general connection on $\mathfrak{X}$ and γ^* be the induced general connection from γ on a maximal integral submanifold $\mathfrak{Y}$ of the distribution of the image tangent subspaces of $P = \lambda(\gamma)$ by means of N. If γ is contravariantly proper, then $(Q\gamma)^* = Q^* \gamma^*$ and the curvature tensor of γ^* is induced from that of γ.*

Proof. By the assumption, we can take local coordinates v^i of $\mathfrak{X}$ such that the maximal integral submanifolds of the distribution of P_x, $x \in \mathfrak{X}$, are given by

$$v^\mu = \text{constant}, \quad \mu = m+1, \cdots, n.$$

15) See [15], §3. $\iota: \mathfrak{Y} \to \mathfrak{X}$ denotes the imbedding mapping.

Then we have

$$\begin{cases} X^j_\alpha=\delta^j_\alpha=Y^j_\alpha, \\ A^j_\alpha=\delta^j_\alpha,\ A^\mu_i=P^\mu_i=Q^\mu_i=0,\ N^j_\alpha=0,\ N^\mu_\lambda=\delta^\mu_\lambda \end{cases}$$

$$\alpha=1,2,\cdots,m;\ \lambda,\mu=m+1,\cdots,n$$

and (4. 3) can be written as

$$\Gamma^\mu_{\alpha\beta}=0$$

from which we get

$$\Lambda^\mu_{\alpha\beta}=0.$$

On the other hand, putting

$$\gamma^*=\partial v_\beta\otimes(P^{*\beta}_\alpha d^2v^\alpha+\Gamma^{*\beta}_{\alpha\gamma}dv^\alpha\otimes dv^\gamma),$$

we get by the above relations

$$P^{*\beta}_\alpha=Y^\beta_j P^j_\alpha=Y^\beta_\gamma P^\gamma_\alpha=P^\beta_\alpha,$$

$$\Gamma^{*\beta}_{\alpha\gamma}=Y^\beta_j\Gamma^j_{\alpha\gamma}=\Gamma^\beta_{\alpha\gamma}+Y^\beta_\mu\Gamma^\mu_{\alpha\gamma}=\Gamma^\beta_{\alpha\gamma}$$

from which we have $Q^*\gamma^*=(Q\gamma)^*$. Furthermore we have

$$\delta^\beta_{\alpha,\gamma}=\Gamma^\beta_{i\gamma}P^i_\alpha-P^\beta_k\Lambda^k_{\alpha\gamma}=\Gamma^\beta_{\delta\gamma}P^\delta_\alpha-P^\beta_\delta\Lambda^\delta_{\alpha\gamma}=\delta^{*\beta}_{\alpha,\gamma}$$

$$\delta^\mu_{\alpha,\gamma}=0.$$

Hence we get from (1. 2)

$$R_\alpha{}^\beta{}_{\gamma\delta}=R^*{}_\alpha{}^\beta{}_{\gamma\delta},\qquad R_\alpha{}^\lambda{}_{\gamma\delta}=0.$$

Accordingly, if $R_i{}^j{}_{hk}$ are the components of the curvature tensor of γ with respect to u^j of $\mathfrak{X}$ and $R^*_\alpha{}^\beta{}_{\gamma\delta}$ are the ones of the curvature tensor of γ^* with respect to v^α of $\mathfrak{Y}$, then we have

$$R^*{}_\alpha{}^\beta{}_{\gamma\delta}=Y^\beta_j R_i{}^j{}_{hk}\frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^h}{\partial v^\gamma}\frac{\partial u^k}{\partial v^\delta}. \tag{4. 5}$$

This shows the fact that we have to prove.

THEOREM 4. 2. *Let γ be an integrable normal general connection such as in Lemma 2 and $\mathfrak{Y}$ be a maximal integral submanifold of the distribution of the image tangent subspaces of $P=\lambda(\gamma)$. Let λ^* and $(Q\gamma)^*$ be the induced general connections on $\mathfrak{Y}$ from γ and $Q\gamma$ by means of N respectively. If the fundamental group $\pi_1(\mathfrak{Y})$ of $\mathfrak{Y}$ has at most a countable number of elements, the connected components of the homogeneous holonomy group of the affine connection $(Q\gamma)^*$[16] is irreducible and $(Q\gamma)^*$ and $P^*=\lambda(\gamma^*)$ are commutative, then γ^* can be written as an affine connection $\times$ a constant*

Proof. Since $N(Q\gamma)=0$, $Q\gamma$ is contravariantly proper. By means of Theorem 4. 1, the curvature tensor of the affine connection $(Q\gamma)^*$ is induced from the curvature

16) Since $\lambda((Q\gamma)^*)=1$, $(Q\gamma)^*$ is an ffine connection.

tensor of the normal general connection $Q\gamma={}'\gamma$. On the other hand, by means of Lemma 3, (3. 19) holds good for the general connection γ. This follows on $\mathfrak{Y}$

$$P^{*\beta}_{\varepsilon}\,{}'R^{*}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta}={}'R^{*}{}_{\varepsilon}{}^{\beta}{}_{\gamma\delta}P^{*\delta}_{\alpha},$$

where ${}'R^{*}{}_{\alpha}{}^{\varepsilon}{}_{\gamma\delta}$ denote the components of the curvature tensor of $(Q\gamma)^*$.

The assumption that $(Q\gamma)^*$ and P^* are commutative is equivalent to that P^* is covariantly constant with respect to $(Q\gamma)^*$ according to [16], Lemma 1. 1. Hence the assumption regarding to the holonomy group of $(Q\gamma)^*$ and Shur's lemma follow $P^*=c1$, where c is constant.

REMARK. In the theorem, if γ satisfies (ii) $DA=A\omega$ or (iii) $DA=P\omega$ in Lemma 2, the commutativity of $(Q\gamma)^*$ and P^* is equivalent to the one of γ^* and P^*. For by Theorem 4. 1 we have $(Q\gamma)^*=Q^*\gamma^*$ and so $\gamma^*P^*=P^*\gamma^*$.

REFERENCES

[1] CHERN, S. S., Lecture note on differential geometry. Chicago Univ. (1950)-

[2] EHRESMANN, G., Les connexions infinitésimales dans un espace fibré différentiable. Colloque de Topologie (Espaces fibrés) (1950), 29–55.

[3] EHRESMANN, G., Les prolongements d'une variété différentiables I, Calcul des jets, prolongement principal. C. R. Paris **233** (1951), 598–600.

[4] HOUH, C. S., Submanifolds in a Riemannian manifold with general connections. Math. J. Okayama Univ. **12** (1963), 1–37.

[5] ŌTSUKI, T., Geometries of connections. Kyōritsu Shuppan Co. (1957). (in Japanese)

[6] ŌTSUKI, T., On tangent bundles of order 2 and affine connections. Proc. Japan Acad. **34** (1958), 325–330.

[7] ŌTSUKI, T., Tangent bundles of order 2 and general connections. Math. J. Okayama Univ. 8 (1958), 143–179.

[8] ŌTSUKI, T., On general connections, I. Math. J. Okayama Univ. **9** (1960), 99–164.

[9] ŌTSUKI, T., On general connections, II. Math. J. Okayama Univ. **10** (1961), 113–124.

[10] ŌTSUKI, T., On metric general connections. Proc. Japan Acad. **37** (1961), 183–188.

[11] ŌTSUKI, T., On normal general connections. Kōdai Math. Sem. Rep. **13** (1961), 152–166.

[12] ŌTSUKI, T., General connections $A\Gamma A$ and the paralleism of Levi-Civita. Kōdai Math. Sem. Rep. **14** (1962), 40–52.

[13] ŌTSUKI, T., On basic curves in spaces with normal general connections. Kōdai Math. Sem. Rep. **14** (1962), 110–118.

[14] ŌTSUKI, T., A note on metric general connections. Proc. Japan Acad. **38** (1962), 409–413.

[15] ŌTSUKI, T., On curvatures of spaces with normal general connections, I. Kōdai Math. Sem. Rep. **15** (1963), 52–61.

[16] ŌTSUKI, T., On curvatures of spaces with normal general connections, II. Kōdai Math. Sem. Rep. **15** (1963), 184–194.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY.