ON FIBERINGS OF ALMOST CONTACT MANIFOLDS

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Introduction.

We study, in this paper, relations between almost contact structures and the induced complex structures. Throughout this paper, we consider the case in which an almost contact manifold $\tilde{M}$ is the bundle space of a principal fibre bundle over an almost complex manifold M.

We induce, in § 1, an almost complex structure on M.

We consider, in § 2, two sorts of torsions $\tilde{N}$ and $\tilde{\Phi}$ of an almost contact structure defined by Nijenhuis [4] and Sasaki-Hatakeyama [8] respectively, and investigate relations between the integrability of the induced almost complex structure on M and the vanishing of $\tilde{N}$ or $\tilde{\Phi}$.

In § 3 we consider an almost contact metric structure on $\tilde{M}$ and induce an almost Hermitian structure on M and investigate relations between them.

In § 4 we study, as special cases, an almost Sasakian structure and a Sasakian structure on $\tilde{M}$ and induce on M an almost Kähler structure and a Kähler structure.

In the last section we study a relation between the Riemannian connection on $\tilde{M}$ and on M and a relation between the curvature of $\tilde{M}$ and the curvature of M.

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§ 1. Regular almost contact structures.

Let $\tilde{M}$ be a $(2n+1)$-dimensional differentiable manifold. We denote by $\tilde{\mathfrak{X}}$ the Lie algebra of all vector fields on $\tilde{M}$.

An almost contact structure on $\tilde{M}$ is defined by a (1, 1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [7]:

(1. 1) $$\phi(\xi)=0,$$

(1. 2) $$\eta(\phi(\tilde{X}))=0 \qquad \text{for all} \quad \tilde{X}\in\tilde{\mathfrak{X}},$$

(1. 3) $$\eta(\xi)=1,$$

(1. 4) $$\phi^2(\tilde{X})=-\tilde{X}+\eta(\tilde{X})\cdot\xi \qquad \text{for all} \quad \tilde{X}\in\tilde{\mathfrak{X}}.$$

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A differentiable manifold of odd dimension with an almost contact structure is called an almost contact manifold.

DEFINITION 1. 1. A vector field on $\tilde{M}$ is said to be *regular* if each point of $\tilde{M}$ has a regular neighborhood, i.e., a cubical coordinate neighborhood $U(x^1, \cdots, x^{2n+1})$ whose intersection with any integral curve of the vector field can be represented by a single segment $x^1=$const., $\cdots$, $x^{2n}=$const., and is said to be *strictly regular* if all integral curves are homeomorphic to each other.

DEFINITION 1. 2. An almost contact structure (ϕ, ξ, η) is said to be *(strictly) regular* if ξ is a (strictly) regular vector field.

Let (ϕ, ξ, η) be a regular almost contact structure on $\tilde{M}$ and M be the orbit space defined by ξ, then M is a manifold and $\pi(U)(x^1, \cdots, x^{2n})$ is a coordinate neighborhood on M, where π denotes the natural projection of $\tilde{M}$ onto M [6].

If, in particular, $\tilde{M}$ is compact and (ϕ, ξ, η) is a regular almost contact structure on $\tilde{M}$, then any integral curve of ξ must be homeomorphic to the circle S^1 and ξ generates a global action of the circle group S^1 on $\tilde{M}$. An almost contact structure (ϕ, ξ, η) is said to be invariant if ϕ and η is invariant under the action of G, the 1-parameter group generated by ξ.

THEOREM 1. 1. *If (ϕ, ξ, η) is an invariant strictly regular almost contact structure on $\tilde{M}$, then*
(i) *$\tilde{M}$ is a principal G-bundle over M, and*
(ii) *η is a connection form on $\tilde{M}$.*

COROLLARY. *If (ϕ, ξ, η) is an invariant regular almost contact structure on a compact manifold $\tilde{M}$, then*
(i) *$\tilde{M}$ is a principal circle bundle over M, and*
(ii) *η is a connection form on $\tilde{M}$.*

Throughout this paper, we shall only consider invariant strictly regular almost contact structures. We denote by $\mathfrak{X}$ the Lie algebra of all vector fields on M.

First we prove the following

THEOREM 1. 2. *If we define a (1,1)-tensor field J on M as follows:*

$$J_p(X)=d\pi(\phi_{\tilde{p}})(X^*_{\tilde{p}})), \quad p\in M, \quad \tilde{p}\in\tilde{M}, \quad p=\pi(\tilde{p}), \tag{1. 5}$$

where $X^*_{\tilde{p}}$ denotes the lift of $X\in\mathfrak{X}$ at $\tilde{p}$ with respect to the connection η, then J is an almost complex structure on M.

Proof. First of all, J is well defined, since, for any $a\in G$ we have $\phi_{\tilde{p}a}(X^*_{\tilde{p}a}) =\phi_{\tilde{p}a}(dR_a X^*_{\tilde{p}})=(R^*_a\phi_{\tilde{p}a})(X^*_{\tilde{p}})=\phi_{\tilde{p}}(X^*_{\tilde{p}})$, where we denote the right translation in $\tilde{M}$ by $a\in G$ by R_a and the induced map by R^*_a.

Next, $J^2=-1$. In fact, for any $X\in\mathfrak{X}$ we see that

$$J^2(X)=J(JX)=J(d\pi\phi(X^*))=d\pi\phi(d\pi\phi(X^*))^*=d\pi\phi\phi(X^*)$$
$$=d\pi(-X^*+\eta(X^*)\xi)=-d\pi(X^*)=-X.$$

Hence J is an almost complex structure on M. Q.E.D.

Thus a manifold with an invariant strictly regular almost contact structure is a principal fibre bundle over a manifold with almost complex structure.

We shall call J the induced almost complex structure on M.

§ 2. Integrability and normality.

DEFINITION 2. 1. Let (ϕ, ξ, η) be an almost contact structure on $\tilde{M}$ and J be the induced almost complex structure on M. We define tensor fields $\tilde{N}$ and $\tilde{\Phi}$ on $\tilde{M}$ and N on M as follows [4], [8]:

$$\tilde{N}(\tilde{X}, \tilde{Y})=[\phi\tilde{X}, \phi\tilde{Y}]-\phi[\phi\tilde{X}, \tilde{Y}]-\phi[\tilde{X}, \phi\tilde{Y}]+\phi^2[\tilde{X}, \tilde{Y}], \tag{2. 1}$$

$$\tilde{\Phi}(\tilde{X}, \tilde{Y})=\tilde{N}(\tilde{X}, \tilde{Y})+2d\eta(\tilde{X}, \tilde{Y})\cdot\xi \qquad \text{for all} \quad \tilde{X}, \tilde{Y}\in\tilde{\mathfrak{X}} \tag{2. 2}$$

and

$$N(X, Y)=[JX, JY]-J[JX, Y]-J[X, JY]+J^2[X, Y] \qquad \text{for all} \quad X, Y\in\mathfrak{X}. \tag{2. 3}$$

We now prove the following

LEMMA 2. 1. *We have*

$$N(X, Y)=d\pi\tilde{N}(X^*, Y^*)$$

and

$$N(X, Y)=d\pi\tilde{\Phi}(X^*, Y^*)$$

for any $X, Y\in\mathfrak{X}$, *where* X^* *denotes the lift of* $X\in\mathfrak{X}$ *with respect to the connection* η.

Proof. From (1. 5) we get

$$\begin{aligned} N(X, Y)&=[JX, JY]-J[JX, Y]-J[X, JY]+J^2[X, Y] \\ &=[d\pi\phi X^*, d\pi\phi Y^*]-d\pi\phi[d\pi\phi X^*, d\pi Y^*]^*-d\pi\phi[d\pi X^*, d\pi\phi Y^*]^*+d\pi\phi(d\pi\phi[X, Y]^*)^* \\ &=d\pi[\phi X^*, \phi Y^*]-d\pi\phi[\phi X^*, Y^*]-d\pi\phi[X^*, \phi Y^*]+d\pi\phi^2[X, Y]^* \\ &=d\pi\{[\phi X^*, \phi Y^*]-\phi[\phi X^*, Y^*]-\phi[X^*, \phi Y^*]+\phi^2[X^*, Y^*]\} \\ &=d\pi\tilde{N}(X^*, Y^*), \end{aligned}$$

where we have used the facts that $d\pi$ is an isomorphism and $\phi[X, Y]^*=\phi h[X^*, Y^*]=\phi[X^*, Y^*]$; h denotes the horizontal component with respect to the connection η. The second relation is clear from the first and the fact that ξ is vertical. Q.E.D.

The following result is the direct consequence of this Lemma.

THEOREM 2. 1. *We have* $N(X, Y)=0$ *if and only if* $\tilde{N}(X^*, Y^*)$ (*or* $\tilde{\Phi}(X^*, Y^*)$) *is vertical for all* $X, Y\in\mathfrak{X}$.

DEFINITION 2. 2. An almost contact structure is said to be *integrable* if $\tilde{N}=0$ and is said to be *normal* if $\tilde{\Phi}=0$.

The following is easily seen from Lemma 2. 1.

PROPOSITION 2. 1. *If an almost contact structure* (ϕ, ξ, η) *is integrable* (*or normal*), *then the induced almost complex structure* J *is integrable, i.e.,* J *is a complex structure.*

Now we prove the following Lemma for later use.

LEMMA 2. 2. *We have* $2d\eta(\tilde{X}, \tilde{Y})=-\eta([\tilde{X}, \tilde{Y}])$ *for any horizontal vector fields* $\tilde{X}$ *and* $\tilde{Y}$.

Proof. This is an immediate consequence from the identity $2d\eta(\tilde{X}, \tilde{Y})=\tilde{X}\cdot\eta(\tilde{Y})-\tilde{Y}\cdot\eta(\tilde{X})-\eta([\tilde{X}, \tilde{Y}])$ and the fact that η is a connection form. Q.E.D.

The connection η is said to be involutive if $[\tilde{X}, \tilde{Y}]$ is horizontal for all horizontal vector fields $\tilde{X}$ and $\tilde{Y}$.

THEOREM 2. 2. *The almost contact structure* (ϕ, ξ, η) *on* $\tilde{M}$ *is integrable if and only if*

(i) *the induced almost complex structure* J *on* M *is integrable, and*

(ii) *the connection* η *is involutive.*

Proof. By Theorem 2. 1, $N(X, Y)=0$ implies that $\tilde{N}(X^*, Y^*)$ is vertical for all $X, Y\in\mathfrak{X}$. On the other hand, from (2. 1) we have $\eta(\tilde{N}(X^*, Y^*))=\eta([\phi X^*, \phi Y^*])=0$ since ϕX^* and ϕY^* are horizontal. This shows that $\tilde{N}(X^*, Y^*)$ is horizontal. Hence we have $\tilde{N}(X^*, Y^*)=0$. Now it is clear that $[\xi, X^*]=0$ since X^* is invariant under the action of G. We see also $[\xi, \phi X^*]=[\xi, (JX)^*]=0$. Hence we can easily verify that $\tilde{N}(\xi, X^*)=0$. We have thus proved that $\tilde{N}(\tilde{X}, \tilde{Y})=0$ is valid for the lifts of vector fields on M or the vertical vector fields. Since $\tilde{N}$ is a tensor field, $\tilde{N}(\tilde{X}, \tilde{Y})=0$ holds for any vector fields $\tilde{X}$ and $\tilde{Y}$.

The converse is clear. Q.E.D.

Similarly we have

THEOREM 2. 3 (Morimoto [3]). *The almost contact structure* (ϕ, ξ, η) *on* $\tilde{M}$ *is normal if and only if*

(i) *the induced almost complex structure* J *on* M *is integrable, and*

(ii) $\Sigma(JX, JY)=\Sigma(X, Y)$ *for all* $X, Y\in\mathfrak{X}$, *where* $d\eta=\pi^*\Sigma$.

Proof. By Theorem 2. 1, $N(X, Y)=0$ implies that $\tilde{\Phi}(X^*, Y^*)$ is vertical for all $X, Y\in\mathfrak{X}$. On the other hand, from (2. 2) we have

$$\eta(\tilde{\Phi}(X^*, Y^*))=\eta([\phi X^*, \phi Y^*])+2d\eta(X^*, Y^*)=-2d\eta(\phi X^*, \phi Y^*)+2d\eta(X^*, Y^*)$$

$$=-2(\pi^*\Sigma)(\phi X^*, \phi Y^*)+2(\pi^*\Sigma)(X^*, Y^*)=-2\Sigma(JX, JY)+2\Sigma(X, Y)=0,$$

which shows that $\tilde{\Phi}(X^*, Y^*)$ is horizontal. Hence we have $\tilde{\Phi}(X^*, Y^*)=0$. Now it is clear that $d\eta(\xi, X^*)=0$ since

$$2d\eta(\xi, X^*)=\xi\cdot\eta(X^*)-X^*\cdot\eta(\xi)-\eta([\xi, X^*]).$$

Hence we have

$$\tilde{\Phi}(\xi, X^*)=\tilde{N}(\xi, X^*)+2d\eta(\xi, X^*)\cdot\xi=0.$$

We have thus proved that $\tilde{\Phi}(\tilde{X}, \tilde{Y})=0$ is valid for the lifts of vector fields on M or the vertical vector fields. Since $\tilde{\Phi}$ is a tensor field, $\tilde{\Phi}(\tilde{X}, \tilde{Y})=0$ holds for any vector fields $\tilde{X}$ and $\tilde{Y}$.

The converse is clear. Q.E.D.

§3. Almost contact metric structures.

We now suppose that G-invariant Riemannian metric $\tilde{g}$ is given on $\tilde{M}$ which admits an invariant strictly regular almost contact structure (ϕ, ξ, η).

If we define a tensor field g of type (0,2) on M by

(3. 1) $$g(X, Y)=\tilde{g}(X^*, Y^*) \qquad \text{for all } X, Y\in\mathfrak{X},$$

then g is a Riemannian metric on M. We call g the induced Riemannian metric on M.

Definition 3. 1 ([7]). $(\phi, \xi, \eta, \tilde{g})$ is called an *almost contact metric structure* on $\tilde{M}$ if (ϕ, ξ, η) is an almost contact structure and, in addition,

(3. 2) $$\tilde{g}(\xi, \tilde{X})=\eta(\tilde{X}),$$

and

(3. 3) $$\tilde{g}(\phi\tilde{X}, \phi\tilde{Y})=\tilde{g}(\tilde{X}, \tilde{Y})-\eta(\tilde{X})\cdot\eta(\tilde{Y}) \qquad \text{for all } \tilde{X}, \tilde{Y}\in\tilde{\mathfrak{X}}.$$

Proposition 3. 1. *If $(\phi, \xi, \eta, \tilde{g})$ is an almost contact metric structure on $\tilde{M}$, then ξ is a Killing vector field.*

Theorem 3. 1. *If $(\phi, \xi, \eta, \tilde{g})$ is an almost contact metric structure on $\tilde{M}$, then the induced structure (J, g) on M is an almost Hermitian structure.*

Proof. It suffices to show that g is a Hermitian metric on M. For any X, $Y\in\mathfrak{X}$, we have

$$g(JX, JY)=\tilde{g}((JX)^*, (JY)^*)=\tilde{g}(\phi X^*, \phi Y^*)=\tilde{g}(X^*, Y^*)-\eta(X^*)\eta(Y^*)$$
$$=\tilde{g}(X^*, Y^*)=g(X, Y)$$

which shows that g is a Hermitian metric. Q.E.D.

Combining Proposition 2. 1 and Theorem 3. 1, we get

COROLLARY. *If* $(\phi, \xi, \eta, \tilde{g})$ *is an integrable (or normal) almost contact metric structure on* $\tilde{M}$, *then the induced structure* (J, g) *is a Hermitian structure on* M.

§4. Almost Sasakian structures and Sasakian structures.

In this section, we consider properties of the so-called almost Sasakian structures and Sasakian structures.

DEFINITION 4.1. Let $(\phi, \xi, \eta, \tilde{g})$ be an almost contact metric structure on $\tilde{M}$, and (J, g) the induced almost Hermitian structure on M. We define a 2-form Θ on $\tilde{M}$ and a 2-form Ω on M as follows:

$$\Theta(\tilde{X}, \tilde{Y})=\tilde{g}(\phi\tilde{X}, \tilde{Y}) \qquad \text{for all} \quad \tilde{X}, \tilde{Y}\in\tilde{\mathfrak{X}}, \tag{4.1}$$

and

$$\Omega(X, Y)=g(JX, Y) \qquad \text{for all} \quad X, Y\in\mathfrak{X}. \tag{4.2}$$

We call Θ and Ω the fundamental 2-form of the almost contact metric structure and of the induced almost Hermitian structure respectively.

LEMMA 4.1. $$\pi^*\Omega=\Theta.$$

Proof. For any $X, Y\in\mathfrak{X}$ we get

$$(\pi^*\Omega)(X^*, Y^*)=\Omega(d\pi X^*, d\pi Y^*)=\Omega(X, Y)$$

$$=g(JX, Y)=\tilde{g}((JX)^*, Y^*)=\tilde{g}(\phi X^*, Y^*)=\Theta(X^*, Y^*). \qquad \text{Q.E.D.}$$

DEFINITION 4.2 ([9]). An almost contact metric structure is called an *almost Sasakian structure* if $\Theta=d\eta$, and is called a *Sasakian structure* if $\Theta=d\eta$ and $\tilde{\Phi}=0$.

THEOREM 4.1. *If* $(\phi, \xi, \eta, \tilde{g})$ *is an almost Sasakian structure, then the induced structure* (J, g) *is an almost Kähler structure.*

Proof. Since π^* is an isomorphism, we conclude from $\pi^*d\Omega=d\pi^*\Omega=d\Theta=dd\eta=0$ that $d\Omega=0$ which means that (J, g) is an almost Kähler structure. Q.E.D.

Combining Proposition 2.1 and Theorem 4.1 we get

COROLLARY. *If* $(\phi, \xi, \eta, \tilde{g})$ *is a Sasakian structure, then the induced structure* (J, g) *is a Kähler structure.*

THEOREM 4, 2. *The fundamental 2-form* Θ *of an almost Sasakian structure is the curvature form of the connection* η.

Proof. For any $\tilde{X}, \tilde{Y}\in\tilde{\mathfrak{X}}$, we have

$$D\eta(\tilde{X}, \tilde{Y})=d\eta(h\tilde{X}, h\tilde{Y})=\Theta(h\tilde{X}, h\tilde{Y})=(\pi^*\Omega)(h\tilde{X}, h\tilde{Y})$$
$$=\Omega(d\pi h\tilde{X}, d\pi h\tilde{Y})=\Omega(d\pi\tilde{X}, d\pi\tilde{Y})=(\pi^*\Omega)(\tilde{X}, \tilde{Y})=\Theta(\tilde{X}, \tilde{Y})$$

which means that $\Theta=D\eta$. Q.E.D.

THEOREM 4. 3. *Let* $(\phi, \xi, \eta, \tilde{g})$ *be an almost Sasakian structure on* $\tilde{M}$ *and* (J, g) *be the induced almost Kähler structure on* M. *Then* (J, g) *is a Kähler structure if and only if* $(\phi, \xi, \eta, \tilde{g})$ *is a Sasakian structure.*

Proof. From the definition of almost Sasakian structure and Lemma 4. 1, we get $d\eta=\pi^*\Omega$. Moreover it is clear that Ω satisfies the second condition of Theorem 2. 3, i.e., $\Omega(JX, JY)=\Omega(X, Y)$ for all $X, Y\in\mathfrak{X}$. Thus our assertion is easy to see from Theorem 2. 3. Q.E.D.

§5. Curvature tensor fields.

In this section we consider some properties of curvature tensor fields.
First we prove the following

PROPOSITION 5. 1. *Let* $(\phi, \xi, \eta, \tilde{g})$ *be an almost contact metric structure on* $\tilde{M}$ *and* (J, g) *the induced almost Hermitian structure on* M. *Then*

$$\tilde{\nabla}_{X^*}Y^*=(\nabla_X Y)^*+\frac{1}{2}([X^*, Y^*])\cdot\xi \tag{5. 1}$$

for all $X, Y\in\mathfrak{X}$, *where* ∇ (*resp.* $\tilde{\nabla}$) *denotes the covariant differentiation with respect to the Riemannian connection determined by* g (*resp.* $\tilde{g}$).

Proof. By definition, ∇ and $\tilde{\nabla}$ are characterized respectively by [2]:

$$2g(\nabla_X Y, Z)=X\cdot g(Y, Z)+Y\cdot g(X, Z)-Z\cdot g(X, Y)+g([X, Y], Z)+g([Z, X], Y)-g(X, [Y, Z])$$

for all X, Y and $Z\in\mathfrak{X}$, and

$$2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z})=\tilde{X}\cdot\tilde{g}(\tilde{Y}, \tilde{Z})+\tilde{Y}\cdot\tilde{g}(\tilde{X}, \tilde{Z})-\tilde{Z}\cdot\tilde{g}(\tilde{X}, \tilde{Y})+\tilde{g}([\tilde{X}, \tilde{Y}], \tilde{Z})+\tilde{g}([\tilde{Z}, \tilde{X}], \tilde{Y})-\tilde{g}(\tilde{X}, [\tilde{Y}, \tilde{Z}])$$

for all $\tilde{X}, \tilde{Y}$ and $\tilde{Z}\in\tilde{\mathfrak{X}}$.

First we prove that the horizontal component of $\tilde{\nabla}_{X^*}Y^*$ is given by $(\nabla_X Y)^*$. We have

$$2g(d\pi\tilde{\nabla}_{X^*}Y^*, Z)=2\tilde{g}(\tilde{\nabla}_{X^*}Y^*, Z^*)$$
$$=X^*\cdot\tilde{g}(Y^*, Z^*)+Y^*\cdot\tilde{g}(X^*, Z^*)-Z^*\cdot\tilde{g}(X^*, Y^*)+\tilde{g}([X^*, Y^*], Z^*)+\tilde{g}([Z^*, X^*], Y^*)$$
$$-\tilde{g}(X^*, [Y^*, Z^*])$$

$=X\cdot g(Y, Z)+Y\cdot g(X, Z)-Z\cdot g(X, Y)+g([X, Y], Z)+g([Z, X], Y)-g(X, [Y, Z])$

$=2g(\nabla_X Y, Z),$

which shows that $d\pi\tilde{\nabla}_{X^*}Y^*=\nabla_X Y$, i.e., $h\tilde{\nabla}_{X^*}Y^*=(\nabla_X Y)^*$.

On the other hand, we have

$$2\tilde{g}(\tilde{\nabla}_{X^*}Y^*, \xi)=X^*\cdot\tilde{g}(Y^*, \xi)+Y^*\cdot\tilde{g}(X^*, \xi)-\xi\cdot\tilde{g}(X^*, Y^*)$$
$$+\tilde{g}([X^*, Y^*], \xi)+\tilde{g}([\xi, X^*], Y^*)-\tilde{g}(X^*, [Y^*, \xi])$$
$$=\tilde{g}([X^*, Y^*], \xi),$$

which is equivalent to

$$2\eta(\tilde{\nabla}_{X^*}Y^*)=\eta([X^*, Y^*]). \tag{5. 2}$$

Hence the vertical component of $\tilde{\nabla}_{X^*}Y^*$ is given by $(1/2)\eta([X^*, Y^*])\cdot\xi$. Q.E.D.

REMARK 5. 1. The equation (5. 1) can also be written as follows:

$$(\nabla_X Y)^*=\tilde{\nabla}_{X^*}Y^*-\eta(\tilde{\nabla}_{X^*}Y^*)\cdot\xi=-\phi^2\tilde{\nabla}_{X^*}Y^*. \tag{5. 3}$$

REMARK 5. 2. The equation (5. 1) shows that $\tilde{\nabla}_{X^*}Y^*=(\nabla_X Y)^*$ if and only if the connection η is involutive.

It is well known that the curvature tensor fields R and $\tilde{R}$ can be expressed, in terms of covariant differentiation, as follows:

$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X, Y]}Z$ for all X, X and $Z\in\mathfrak{X}$, and

$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}=\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}-\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z}-\tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}\tilde{Z}$ for all $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}\in\tilde{\mathfrak{X}}$.

PROPOSITION 5. 2.

$$(R(X, X)Z)^*=-\phi^2\Big\{\tilde{R}(X^*, Y^*)Z^*-\frac{1}{2}\eta([Y^*, Z^*])\cdot\tilde{\nabla}_{X^*}\xi$$
$$+\frac{1}{2}\eta([X^*, Z^*])\cdot\tilde{\nabla}_{Y^*}\xi+\eta([X^*, Y^*])\cdot\tilde{\nabla}_{\xi}Z^*\Big\} \tag{5. 4}$$

for all X, Y *and* $Z\in\mathfrak{X}$.

Proof. From Proposition 5. 1, we get

$$(\nabla_X\nabla_Y Z)^*=-\phi^2\tilde{\nabla}_{X^*}(\nabla_Y Z)^*=-\phi^2\tilde{\nabla}_{X^*}\Big(\tilde{\nabla}_{Y^*}Z^*-\frac{1}{2}\eta([Y^*, Z^*])\cdot\xi\Big)$$
$$=-\phi^2\Big\{\tilde{\nabla}_{X^*}\tilde{\nabla}_{Y^*}Z^*-\frac{1}{2}(X^*\cdot\eta([Y^*, Z^*])\cdot\xi+\eta([Y^*, Z^*])\cdot\tilde{\nabla}_{X^*}\xi)\Big\}$$

$$=-\phi^2\left\{\tilde{\nabla}_{X^*}\tilde{\nabla}_{Y^*}Z^*-\frac{1}{2}\eta([Y^*, Z^*])\cdot\tilde{\nabla}_{X^*}\xi\right\}$$

since $\phi(\xi)=0$. Similarly we get

$$(\nabla_Y\nabla_X Z)^*=-\phi^2\left\{\tilde{\nabla}_{Y^*}\tilde{\nabla}_{X^*}Z^*-\frac{1}{2}\eta([X^*, Z^*])\cdot\tilde{\nabla}_{Y^*}\xi\right\}.$$

Moreover we get

$$(\nabla_{[X, Y]}Z)^*=-\phi^2\tilde{\nabla}_{[X, Y]^*}Z^*=-\phi^2\{\tilde{\nabla}_{[X^*, Y^*]}Z^*-\eta([X^*, Y^*])\cdot\tilde{\nabla}_\xi Z^*\},$$

since $[X^*, X^*]=[X, Y]^*+\eta([X^*, Y^*])\cdot\xi$. Hence we obtain (5. 4). Q.E.D.

DEFINITION 5. 1 (Yano and Mogi [10]). Manifold with Kähler structure (J, g) is said to be of *constant holomorphic curvature* if the curvature tensor field is given by

(5. 5) $4R(X, Y)Z=k\{g(Y, Z)X-g(X, Z)Y-\Omega(X, Z)\cdot JY+\Omega(Y, Z)\cdot JX-2\Omega(X, Y)\cdot JZ\}$

where k is a constant.

DEFINITION 5. 2 (Ogiue [5]). Manifold with Sasakian structure $(\phi, \xi, \eta, \tilde{g})$ is said to be of *constant C-holomorphic curvature* if the curvature tensor field is given by

$$\begin{aligned}4\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}=&(k+3)\{\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X}-\tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}\}\\&+(k-1)\{\eta(\tilde{X})\eta(\tilde{Z})\tilde{Y}-\eta(\tilde{Y})\eta(\tilde{Z})\tilde{X}+\tilde{g}(\tilde{X}, \tilde{Z})\eta(\tilde{Y})\cdot\xi-\tilde{g}(\tilde{Y}, \tilde{Z})\eta(\tilde{X})\cdot\xi\\&-\Theta(\tilde{X}, \tilde{Z})\phi\tilde{Y}+\Theta(\tilde{Y}, \tilde{Z})\phi\tilde{X}-2\Theta(\tilde{X}, \tilde{Y})\phi\tilde{Z}\}\end{aligned}\tag{5. 6}$$

where k is a constant.

Now we can prove the following

THEOREM 5. 1. *If a manifold $\tilde{M}$ with Sasakian structure $(\phi, \xi, \eta, \tilde{g})$ is of constant C-holomorphic curvature, then the manifold M with the induced Kähler structure (J, g) is of constant holomorphic curvature.*

Proof. By Definition 4. 2, the conditions for $\tilde{M}$ to be a Sasakian manifold are given by $\Theta=d\eta$ and $\tilde{\Phi}=0$. These are equivalent to

$$\phi(\tilde{X})=\tilde{\nabla}_{\tilde{X}}\xi \quad\text{and}\quad (\tilde{\nabla}_{\tilde{X}}\phi)(\tilde{Y})=\eta(\tilde{Y})\cdot\tilde{X}-\tilde{g}(\tilde{X}, \tilde{Y})\cdot\xi \qquad \text{(Tashiro [9]).}$$

On the other hand, we have $\tilde{\nabla}_{X^*}\xi=\tilde{\nabla}_\xi X^*$ since $[\xi, X^*]=0$. Moreover from Lemma 2. 2 we get $\eta([X^*, Y^*])=-2d\eta(X^*, Y^*)=-2\Theta(X^*, Y^*)$. Hence from (5. 4) we have

$$(4R(X, Y)Z)^* = -\phi^2\{4R(X^*, Y^*)Z^* - 2\eta([Y^*, Z^*])\cdot\tilde{\nabla}_{X^*}\xi + 2\eta([X^*, Z^*])\cdot\tilde{\nabla}_{Y^*}\xi$$
$$+4\eta([X^*, Y^*])\cdot\tilde{\nabla}_{\xi}Z^*\}$$
$$= -\phi^2[(k+3)\{\tilde{g}(Y^*, Z^*)X^* - \tilde{g}(X^*, Z^*)Y^*\}$$
$$+(k-1)\{-\Theta(X^*, Z^*)\phi Y^* + \Theta(Y^*, Z^*)\phi X^* - 2\Theta(X^*, Y^*)\phi Z^*\}$$
$$+4\Theta(Y^*, Z^*)\phi X^* - 4\Theta(X^*, Z^*)\phi Y^* - 8\Theta(X^*, Y^*)\phi Z^*]$$
$$= -\phi^2(k+3)\{\tilde{g}(Y^*, Z^*)X^* - \tilde{g}(X^*, Z^*)Y^*$$
$$-\Theta(X^*, Z^*)\phi Y^* + \Theta(Y^*, Z^*)\phi X^* - 2\Theta(X^*, Y^*)\phi Z^*\}.$$

Thus we obtain

$$4R(X, Y)Z = (k+3)\{g(Y, Z)X - g(X, Z)Y - \Omega(X, Z)\cdot JY + \Omega(Y, Z)\cdot JX - 2\Omega(X, Y)\cdot JZ\},$$

which shows that M is of constant holomorphic curvature. Q.E.D.

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