NOTE ON A COUSIN-II DOMAIN OVER C^2

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Dedicated to Professor A. Kobori on his sixtieth birthday

Serre [7] gave a canonical exact sequence

$$0 \to Z \to \mathfrak{O} \to \mathfrak{O}^* \to 0$$

where Z is the additive group of all integers and $\mathfrak{O}$ and $\mathfrak{O}^*$ are, respectively, the sheaves of all germs of holomorphic mappings in a complex plane C and GL(1, C). Therefore we have an exact sequence of cohomology groups

$$H^1(X, Z) \to H^1(X, \mathfrak{O}) \to H^1(X, \mathfrak{O}^*) \to H^2(X, Z) \to H^2(X, \mathfrak{O}).$$

Hence $H^1(X, \mathfrak{O}^*) = H^1(X, Z) = 0$ and $H^1(X, \mathfrak{O}) = H^2(X, Z) = 0$ imply, respectively, $H^1(X, \mathfrak{O}) = 0$ and $H^1(X, \mathfrak{O}^*) = 0$. Taking Cartan [3]-Behnke-Stein [1]'s theorem into account, we see that any domain (D, φ) over C^2 with $H^1(D, \mathfrak{O}^*) = H^1(D, Z) = 0$ is a domain of holomorphy over C^2. Therefore, as we remarked in the previous paper [4], Thullen [9]'s example $E = C^2 - \{(0, 0)\}$ is a Cousin-II domain with $H^1(E, \mathfrak{O}^*) \neq 0$. In the present paper we shall prove that any domain (D, φ) over C^2 satisfies $H^1(D, \mathfrak{O}^*) = 0$ if and only if (D, φ) is a domain of holomorphy over C^2 with $H^2(D, Z) = 0$. Therefore any Cousin-II domain (D, φ) over C^2 which is not a domain of holomorphy over C^2 is always an example of a Cousin-II domain with $H^1(D, \mathfrak{O}^*) \neq 0$.

Let φ be a holomorphic mapping of a complex manifold D in C^n such that φ is locally a biholomorphic mapping. Then (D, φ) is called a *domain over* C^n. Let (D_1, φ_1) and (D_2, φ_2) be domains over C^n. If there exists a holomorphic mapping λ of D_1 in D_2 such that $\varphi_1 = \varphi_2 \circ \lambda$, (D_1, φ_1) is called a *domain over* (D_2, φ_2). Moreover, if there exists a neighbourhood U of x for any $x \in D_2$, such that λ is a biholomorphic mapping of each connected component of $\lambda^{-1}(U)$ onto U, then (D_1, φ_1) is called a *covering manifold of* (D_2, φ_2). For any domain (D, φ) over C^n, we can uniquely construct a covering manifold $(D^\sharp, \varphi^\sharp)$ of (D, φ) such that the fundamental group $\pi_1(D^\sharp)$ of $D^\sharp$ vanishes. This $(D^\sharp, \varphi^\sharp)$ is called a *universal covering manifold of* (D, φ). If (D, φ) coincides with its universal covering manifold, (D, φ) is called *simply connected.*

LEMMA 1. *Let (D, φ) be a domain over C^n and (D', φ') be its covering manifold. Then (D, φ) is a domain of holomorphy over C^n if and only if (D', φ') is a domain of holomorphy over C^n.*

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Proof. The Euclidean distance in C^n induces naturally distances in D and D'. Let $\delta(x)$ and $\delta'(y)$ be, respectively, the distance of $x \in D$ and ∂D and that of $y \in D'$ and $\partial D'$. Since (D', φ') is a covering manifold of (D, φ), we have $\delta' = \delta \circ \lambda$ where λ: $D' \to D$ is the canonical projection. From Oka [5] (D, φ) is a domain of holomorphy over C^n if and only if $-\log \delta$ is plurisubharmonic in D. Since $-\log \delta$ is plurisubharmonic if and only if $-\log \delta'$ is plurisubharmonic, (D, φ) is a domain of holomorphy over C^n if and only if (D', φ') is a domain of holomorphy over C^n. See [8].

LEMMA 2. *Let (D, φ) be a domain over C^n with $H^1(D, \mathfrak{O}^*)=0$, $(D^\sharp, \varphi^\sharp)$ be its universal covering manifold and λ: $D^\sharp \to D$ be the canonical mapping. Then for any $(n-1)$-dimensional analytic plane H in C^n and for any holomorphic function h in $D \cap \varphi^{-1}(H)$, the holomorphic function $h \circ \lambda$ in $D^\sharp \cap \varphi^{\sharp-1}(H)$ is a trace of a holomorphic function f in $D^\sharp$.*

Proof. Without loss of generality we may suppose that $H = \{z = (z_1, z_2, \cdots, z_n); z_1 = 0\}$. There exists a neighbourhood V of $D \cap \varphi^{-1}(H)$ such that h is a trace of a holomorphic function h' in V. We take another open subset U of D such that $\mathfrak{U} = \{U, V\}$ is an open covering of D and $U \cap \varphi^{-1}(H) = \phi$. We put

$$g = e^{h'/z_1 \circ \varphi}$$

in $U \cap V$. Then $\{(g, U \cap V)\}$ is a 1-cocycle of $\mathfrak{U}$ with value in $\mathfrak{O}^*$. Since $H^1(D, \mathfrak{O}^*) = 0$ implies $H^1(\mathfrak{U}, \mathfrak{O}^*) = 0$, there exist $f_1 \in H^0(U, \mathfrak{O}^*)$ and $f_2 \in H^0(V, \mathfrak{O}^*)$ such that

$$f_1 / f_2 = e^{h'/z_1 \circ \varphi}$$

in $U \cap V$. We put

$$F = f_1$$

in U and

$$F = f_2 e^{h'/z_1 \circ \varphi}$$

in $V - D \cap \varphi^{-1}(H)$. Then we have $F \in H^0(D - D \cap \varphi^{-1}(H), \mathfrak{O}^*)$. Hence any function element obtained by $(z_1 \circ \varphi^\sharp) \log F \circ \lambda$ is analytically continued along any Jordan curve in $D^\sharp - D^\sharp \cap \varphi^{\sharp-1}(H)$ for any branch of logarithmus. Since it can also be analytically continued at any point of $D^\sharp$ which is simply connected,

$$f = (z_1 \circ \varphi^\sharp) \log F \circ \lambda$$

gives a uniform and holomorphic function in $D^\sharp$ if we take a fixed branch. Moreover we have

$$f = h \circ \lambda$$

in $D^\sharp \cap \varphi^{\sharp-1}(H)$.

LEMMA 3. *Under the assumption of Lemma 2, if each connected component of* $D\cap\varphi^{-1}(H)$ *is a domain of holomorphy over* H *for any* $(n-1)$*-dimensional analytic plane* H *in* C^n, *then* (D, φ) *is a domain of holomorphy over* C^n.

Proof. Let $(D^{\#}, \varphi^{\#})$ be the universal covering manifold of (D, φ). From Lemma 2 each point of $\partial D^{\#}$ has the frontier property in the sense of Bochner-Martin [2]. Hence there exists a holomorphic function f in $D^{\#}$ which is unbounded at each point of $\partial D^{\#}$. Let (D', φ') be the domain of holomorphy of f and λ: $D^{\#}\to D'$ be the canonical mapping. We shall prove that $(D^{\#}, \varphi^{\#})$ is a covering manifold of (D', φ'). Let $K=\{x=x(t);\ 0\leqq t\leqq 1\}$ be a curve in D' such that $\lambda(y_0)=x(0)$ for $y_0\in D^{\#}$. Let τ be the supremum of t' such that $\lambda(y(t))=x(t)$ $(0\leqq t\leqq t')$ for a curve $\{y=y(t);\ 0\leqq t\leqq t'\}$ in $D^{\#}$ with $y_0=y(0)$. Obviously $0<\tau$. Suppose that $\tau<1$. There exists a semiopen curve $K_\tau=\{y=y(t);\ 0\leqq t<\tau\}$ such that $\lambda(y(t))=x(t)$ $(0\leqq t<\tau)$ and $y_0=y(0)$. Then K_τ defines a point y_τ of $\partial D^{\#}$. Since f is unbounded at y_τ, the image $x(\tau)$ of y_τ by the canonical continuous extension of λ does not belong to D'. But this is a contradiction. Hence we have $\tau=1$. In the same way we can prove the existence of a curve $\{y=y(t);\ 0\leqq t\leqq 1\}$ in $D^{\#}$ such that $\lambda(y(t))=x(t)$ $(0\leqq t\leqq 1)$ and $y_0=y(0)$. Therefore $(D^{\#}, \varphi^{\#})$ is a covering manifold of (D', φ'). From Lemma 1 $(D^{\#}, \varphi^{\#})$ is a domain of holomorphy over C^n. Again from Lemma 1 (D, φ) itself is a domain of holomorphy over C^n.

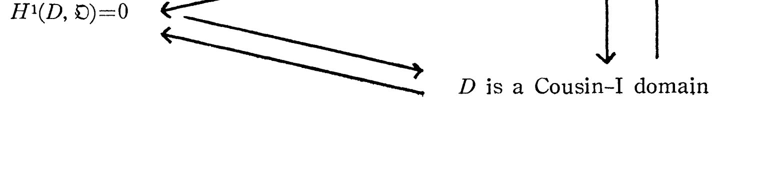
PROPOSITION 1. *Any domain* (D, φ) *over* C^2 *satisfies* $H^1(D, \mathfrak{D}^*)=0$ *if and only if* (D, φ) *is a domain of holomorphy over* C^2 *with* $H^2(D, Z)=0$.

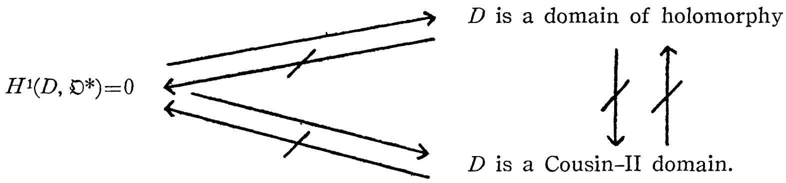
Proof. If (D, φ) satisfies $H^1(D, \mathfrak{D}^*)=0$, (D, φ) is a domain of holomorphy over C^2 from Lemma 3. Since $H^2(D, \mathfrak{D})=0$, from the exact sequence

$$H^1(D, Z)\to H^1(D, \mathfrak{D})\to H^1(D, \mathfrak{D}^*)\to H^2(D, Z)\to H^2(D, \mathfrak{D}),$$

we have $H^2(D, Z)=0$. Conversely if (D, φ) is a domain of holomorphy over C^2 with $H^2(D, Z)=0$, we have $H^1(D, \mathfrak{D}^*)=0$ from the above exact sequence as $H^1(D, \mathfrak{D})=0$.

For a domain (D, φ) over C^2 we have the following diagram where $A\to B$ means that A implies B and $A\not\to B$ means that A does not imply B:





Serre [7] proved that $H^1(X, \mathfrak{O})=H^2(X, Z)=0$ implies $H^1(X, \mathfrak{O}^*)=0$ for any complex manifold X. $D=C^n-\{(0, 0, \cdots, 0)\}$ satisfies $H^1(D, \mathfrak{O})=H^2(D, Z)=0$ from Scheja [6] for $n\geqq 3$. Hence there exists a domain D in C^n with $H^1(D, \mathfrak{O}^*)=0$ which is not a domain of holomorphy for $n\geqq 3$. But we can prove the following proposition by induction with respect to $n\geqq 3$ making use of Lemma 3.

PROPOSITION 2. *Let (D, φ) be a domain over C^n with $H^1(D, \mathfrak{O}^*)=0$ such that $H^1(D\cap\varphi^{-1}(H),\mathfrak{O}^*)=0$ for any l-dimensional analytic plane H in C^n $(2\leqq l\leqq n-1)$. Then (D, φ) is a domain of holomorphy over C^n.*

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