SUPPLEMENT TO CONFORMAL MAPPING ONTO POLYGONS BOUNDED BY SPIRAL ARCS

By Yûsaku Komatu

In a previous paper [2], Nishimiya and the present author have derived an integral representation formula for analytic functions which map the unit circle onto polygons bounded by arcs of logarithmic spirals with the asymptotic point at the origin. This formula generalizes a result of Goodman [1] in which the image polygons are bounded by concentric circular arcs and radial rectilinear segments. An alternative proof of Goodman's original formula was given previously by Nishimiya [3], where moreover an analogue of Goodman's formula was derived in doubly-connected case. This doubly-connected result has been also generalized correspondingly in the paper [2].

Now, a question arises in what extent the formulas derived in the paper [2] are characteristic for mappings of a circle or an annulus onto simply- or doubly-connected polygons bounded by spiral arcs. The purpose of the present paper is to show that the converse of these theorems in [2] is also true.

Theorem 1′. *Let φ_μ $(\mu=1, \cdots, m)$ be a set of distinct real numbers satisfying $0\leqq\varphi_\mu<2\pi$ and α_μ $(\mu=1, \cdots, m)$ a set of real numbers satisfying $\sum_{\mu=1}^{m}\alpha_\mu=0$. Then, any function $f(z)$ defined by*

$$\frac{zf'(z)}{f(z)}=\prod_{\mu=1}^{m}\frac{1}{(1-e^{-i\varphi_\mu}z)^{\alpha_\mu}}$$

maps $|z|<1$ onto a domain bounded by logarithmic spiral arcs in quite such a manner as stated in theorem 1 *in the previous paper* [2].

Proof. From the equation defining $f(z)$ it follows that $zf'(z)/f(z)$ is regular in $|z|<1$ and is equal to 1 at $z=0$. Hence, $f(z)$ has a simple zero at $z=0$ and is itself regular throughout $|z|<1$. Since $zf'(z)/f(z)$ vanishes nowhere in $|z|<1$, $f'(z)$ does not vanish there, that is, the image domain D_0 of $|z|<1$ by $w=f(z)$ is unramified. In view of $\sum_{\mu=1}^{m}\alpha_\mu=0$, the boundary relation

$$d\lg f(e^{i\theta})=\frac{f'(e^{i\theta})}{f(e^{i\theta})}ie^{i\theta}d\theta$$

Received August 31, 1964.

$$=i\,d\theta\prod_{\mu=1}^{m}\frac{1}{(1-e^{-i(\varphi_\mu-\theta)})^{\alpha_\mu}}=i\,d\theta\prod_{\mu=1}^{m}\left(\frac{e^{i(\varphi_\mu-\theta)/2}}{2i\sin((\varphi_\mu-\theta)/2)}\right)^{\alpha_\mu}$$

$$=i\,d\theta\exp\left(\frac{i}{2}\sum_{\mu=1}^{m}\alpha_\mu\varphi_\mu\right)\prod_{\mu=1}^{m}\operatorname{cosec}^{\alpha_\mu}\frac{\varphi_\mu-\theta}{2}$$

holds good. Consequently, $\arg d\lg f(e^{i\theta})$ remains constant along any arc on $|z|=1$ which contains none of the points $e^{i\varphi_\mu}$ $(\mu=1,\cdots,m)$. This shows in turn

$$\frac{d\arg f(e^{i\theta})}{d\lg|f(e^{i\theta})|}=\tan\arg d\lg f(e^{i\theta})=\lambda,$$

$$\arg f(e^{i\theta})-\lambda\lg|f(e^{i\theta})|=c$$

along such an arc, λ and c being real constants; here λ may be eventually equal to infinity and the last relation is then replaced by $|f(e^{i\theta})|=\text{const.}$ Thus, the image domain D_0 is bounded surely by arcs of logarithmic spirals with the asymptotic point at the origin. Further, from the above expression for $d\lg f(e^{i\theta})$ it follows that the quantity

$$\arg df(e^{i\theta})-\arg f(e^{i\theta})=\arg d\lg f(e^{i\theta})$$

possesses a jump with the height $\alpha_\mu\pi$ at $\theta=\varphi_\mu$ $(\mu=1,\cdots,m)$. In particular, if $\arg f(e^{i\theta})$ is continuous there, this jump results purely from a turn of the tangent vector at the vertex.

THEOREM 2′. *Let φ_μ $(\mu=1,\cdots,m)$ and ψ_ν $(\nu=1,\cdots,n)$ be two sets of real numbers which are distinct within each set and satisfy $0\leqq\varphi_\mu<2\pi$, $0\leqq\psi_\nu<2\pi$ and let α_μ $(\mu=1,\cdots,m)$ and β_ν $(\nu=1,\cdots,n)$ be two sets of real numbers satisfying $\sum_{\mu=1}^{m}\alpha_\mu=\sum_{\nu=1}^{n}\beta_\nu=0$. Then, any function $f(z)$ defined by*

$$\frac{zf'(z)}{f(z)}=Cz^{ic^*}\prod_{\mu=1}^{m}\frac{1}{\sigma(i\lg z+\varphi_\mu)^{\alpha_\mu}}\prod_{\nu=1}^{n}\sigma_3(i\lg z+\psi_\nu)^{\beta_\nu},$$

where C is a non-vanishing constant and c^* a real constant determined by

$$c^*=\frac{\eta_1}{\pi}\left(\sum_{\mu=1}^{m}\alpha_\mu\varphi_\mu-\sum_{\nu=1}^{n}\beta_\nu\psi_\nu\right)$$

maps the annulus $q<|z|<1$ onto a ring domain bounded by logarithmic spiral arcs in quite such a manner as stated in theorem 2 in the previous paper [2].

Proof. It is readily seen as in the proof of theorem 1′ that $f(z)$ is regular in $q<|z|<1$,[1] and both $f(z)$ and $f'(z)$ do not vanish there. Along the circumference $|z|=1$, the boundary relation

1) Here the single-valuedness of $f(z)$ is supposed implicitly. However, it may be perhaps deduced from the defining equation of $f(z)$.

$$d \lg f(e^{i\theta})=i\, d\theta\, Ce^{-c^*\theta} \prod_{\mu=1}^{m} \frac{1}{\sigma(\varphi_\mu-\theta)^{\alpha_\mu}} \prod_{\nu=1}^{n} \sigma_3(\psi_\nu-\theta)^{\beta_\nu}$$

holds good. Since the sigma-functions depend on the primitive periods $2\omega_1=2\pi$ and $2\omega_3=2i \lg q^{-1}$, the values of $\sigma(\varphi_\mu-\theta)$ and $\sigma_3(\psi_\nu-\theta)$ are real. Moreover, $\sigma(\varphi_\mu-\theta)$ vanishes for $0\leqq\theta<2\pi$ only at $\theta=\varphi_\mu$ and really simply, while $\sigma_3(\psi_\nu-\theta)$ does never vanish there. Hence, as in the proof of theorem 1′, it is concluded that the image of $|z|=1$ by $w=f(z)$ consists of logarithmic spiral arcs with the asymptotic point at the origin which are connected in turn at $f(e^{i\varphi_\mu})$ $(\mu=1, \cdots, m)$ in the assigned manner. Next, along the circumference $|z|=q$, the relation

$$d \lg f(qe^{i\theta})=i\, d\theta\, Cq^{ic^*}e^{-c^*\theta} \prod_{\mu=1}^{m} \frac{1}{\sigma(\varphi_\mu-\theta+i \lg q)^{\alpha_\mu}} \prod_{\nu=1}^{n} \sigma_3(\psi_\nu-\theta+i \lg q)^{\beta_\nu}$$

holds. By making use of the well-known formulas

$$\sigma(u-\omega_3)=-\sigma(\omega_3)e^{-\eta_3 u}\sigma_3(u), \qquad \sigma_3(u-\omega_3)=\sigma(\omega_3)^{-1}e^{-\eta_3(u-\omega_3)}\sigma(u)$$

and remembering the assumed relation $\sum_{\mu=1}^{m}\alpha_\mu=\sum_{\nu=1}^{n}\beta_\nu=0$, this becomes

$$d \lg f(qe^{i\theta})=i\, d\theta\, Cq^{ic^*}e^{-c^*\theta} \prod_{\nu=1}^{n} \sigma(\psi_\nu-\theta)^{\beta_\nu} \prod_{\mu=1}^{m} \frac{1}{\sigma_3(\varphi_\mu-\theta)^{\alpha_\mu}}.$$

Comparing the last relation with the previous one for $d \lg f(e^{i\theta})$, it is seen that the expression for $d \lg f(qe^{i\theta})$ results from that for $d \lg f(e^{i\theta})$ by replacing $\{\varphi_\mu\}_{\mu=1}^{m}$, $\{\psi_\nu\}_{\nu=1}^{n}$, $\{\alpha_\mu\}_{\mu=1}^{m}$, $\{\beta_\nu\}_{\nu=1}^{n}$ and C by $\{\psi_\nu\}_{\nu=1}^{n}$, $\{\varphi_\mu\}_{\mu=1}^{m}$, $\{-\beta_\nu\}_{\nu=1}^{n}$, $\{-\alpha_\mu\}_{\mu=1}^{m}$ and Cq^{ic^*} respectively, the quantity c^* remaining invariant. However, according to the circumstances, $d \lg f(qe^{i\theta})$ is to be understood more precisely as $d \lg f((q+0)e^{i\theta})$, though $d \lg f(e^{i\theta})$ has been understood as $d \lg f((1-0)e^{i\theta})$. Therefore, the argument of the factor $\sigma(\psi_\nu-\theta)^{\beta_\nu}$ in the expression of $d \lg f(qe^{i\theta})$ varies from 0 at $\theta=\psi_\nu-0$ to $\beta_\nu\pi$ at $\theta=\psi_\nu+0$, while that of $\sigma(\varphi_\mu-\theta)^{\alpha_\mu}$ in $d \lg f(e^{i\theta})$ has varied from 0 at $\theta=\varphi_\mu-0$ to $-\alpha_\mu\pi$ at $\theta=\varphi_\mu+0$. Consequently, the assertion with respect to the image of the boundary component $|z|=q$ is also verified.

References

[1] Goodman, A. W., Conformal mapping onto certain curvilinear polygons. Univ. Nac. Tucumán Rev. (A) **13** (1960), 20–26.

[2] Komatu, Y., and H. Nishimiya, Conformal mapping onto polygons bounded by spiral arcs. Kōdai Math. Sem. Rep. **16** (1964), 243–248.

[3] Nishimiya, H., Conformal mapping onto gear-like domains. Sci. Rep. Saitama Univ. (A) **5** (1964), 1–4.

Department of Mathematics, Tokyo Institute of Technology.