ON THE SIXTH COEFFICIENT OF UNIVALENT FUNCTION

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1 Let S be the family of functions $f(z)$ regular and univalent for $|z|<1$ with expansion at the origin

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

In the present paper we shall prove the following result, which is a partial affirmative answer for the famous Bieberbach conjecture for the sixth coefficient.

THEOREM. *If $f(z)$ belongs to S and if a_2 is real and non-negative, then*

$$\Re a_6 \leqq 6.$$

Equality can occur only for Koebe function $z/(1-z)^2$.

Our starting point is Grunsky's perfect condition for univalency, which consists of an infinite number of inequalities. A part which we shall use contains the freedom of degree two. What utilization is made of this freedom or how to utilize it is our first task and this is the first decisive step of our proof. However it is not independent of several estimations for several quantities in terms of $2-a_2$. Our second task is whether it is possible to construct certain effective estimations of certain quantities or not. Thanks to Jenkins' result [3] and Grunsky's condition of lower degree we can do it. It is the author's opinion that, in the methodological point of view, there would be no way to avoid a somewhat tedious calculation.

2. Grunsky's condition. Let $g(z)$ be a function regular in $1<|z|<\infty$ with Laurent expansion in the neighborhood of infinity given by

$$g(z)=z+\sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

Let $G_\mu(w)$ be its μ th Faber polynomial, which is defined by the relation

$$G_\mu(g(z))=z^\mu+\sum_{n=1}^{\infty} \frac{b_{\mu n}}{z^n}$$

around $z=\infty$. Then it is known that $\nu b_{\mu\nu}=\mu b_{\nu\mu}$. The Grunsky perfect condition for univalency of $g(z)$ is the following system of inequalities

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$$\left|\sum_{n,m=1}^{N} m b_{nm} x_n x_m\right| \leqq \sum_{n=1}^{N} n|x_n|^2 \tag{1}$$

for every finite integer N and for every N-dimensional complex vector $(x_1, \cdots, x_N)$.

Let $g(z)$ be the function $f(1/z^2)^{-1/2}$. Then we have

$$G_1(w)=w,\ G_3(w)=w^3-3b_1w,\ G_5(w)=w^5-5b_1w^3+(5b_1^2-5b_3)w,$$

$$b_{11}=b_1=-\frac{1}{2}a_2,$$

$$b_{31}=3b_{13}=3b_3=-\frac{3}{2}\left(a_3-\frac{3}{4}a_2^2\right),$$

$$b_{33}=-\frac{3}{2}\left(a_4-2a_2a_3+\frac{13}{12}a_2^3\right),$$

$$b_{51}=5b_{15}=-\frac{5}{2}\left(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\right),$$

$$b_{53}=\frac{5}{3}b_{35}=-\frac{5}{2}\left(a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4\right),$$

$$b_{55}=-\frac{5}{2}\left(a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+\frac{21}{4}a_2a_3^2-\frac{59}{8}a_3a_2^3+\frac{689}{320}a_2^5\right).$$

Putting $N=5$, $x_1=\delta$, $x_3=\beta/6$, $x_5=1/5$ and $x_2=x_4=0$ and inserting the values of b_{nm} in (1), we have

$$\begin{aligned}&\Bigg|a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+\frac{21}{4}a_2a_3^2-\frac{59}{8}a_2^3a_3+\frac{689}{320}a_2^5\\ &+\beta\left(a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_2^2a_3-\frac{85}{64}a_2^4\right)+2\delta\left(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\right)\\ &+\frac{\beta^2}{4}\left(a_4-2a_2a_3+\frac{13}{12}a_2^3\right)+\beta\delta\left(a_3-\frac{3}{4}a_2^2\right)+\delta^2a_2\Bigg|\\ &\leqq\frac{2}{5}+\frac{|\beta|^2}{6}+2|\delta|^2.\end{aligned} \tag{2}$$

Putting $N=3$, $x_1=\beta$, $x_2=0$, $x_3=1/3$ in (1), then we have

$$\left|a_4-2a_2a_3+\frac{13}{12}a_2^3+2\beta\left(a_3-\frac{3}{4}a_2^2\right)+\beta^2a_2\right|\leqq\frac{2}{3}+2|\beta|^2. \tag{3}$$

3 Lemmas. In the sequel we shall denote $x_2=2-a_2$, $y+iy'=a_3-3a_2^2/4$ and $\eta+i\eta'=a_4-3a_2a_3/2+5a_2^3/8$. By our assumption on a_2 evidently there hold two inequalities $x \leqq 2$ and $0 \leqq x$.

LEMMA 1. $$y \leqq 3x - \frac{15}{4} x^2 + \frac{10}{3} x^2.$$

Proof. Jenkins [3] proved the following result: There holds an inequality

$$\Re\left\{e^{-2i\phi}\left(a_3 - \frac{3}{4} a_2^2\right)\right\} \leqq 1 + \frac{3}{8}\tau^2 - \frac{\tau^2}{4}\log\frac{\tau}{4} + \frac{1}{4}\Re\left\{e^{-2i\phi}a_2^2\right\} + \tau\Re\left\{e^{-i\phi}a_2\right\}$$

for every real ϕ and for every real τ satisfying $0 \leqq \tau \leqq 4$. Putting $\phi = \pi$, there holds

$$y \leqq 1 + \frac{3}{8}\tau^2 - \frac{\tau^2}{4}\log\frac{\tau}{4} + \frac{1}{4}(2-x)^2 - \tau(2-x).$$

Further, putting $\tau = 4e^{-s}$, there holds

$$y \leqq 2 - 8e^{-s} + 6e^{-2s} + 4se^{-2s} - x + \frac{x^2}{4} + 4xe^{-s}$$

for every s in $[0, \infty)$. Choosing s as x, there holds

$$y \leqq 2 - 8e^{-x} + 6e^{-2x} + 4xe^{-2x} - x + \frac{x^2}{4} + 4xe^{-x}$$

for $0 \leqq x \leqq 2$. Let $h(x)$ be the function

$$4x - 4x^2 + \frac{10}{3}x^3 - 2 + 8e^{-x} - 6e^{-2x} - 4xe^{-2x} - 4xe^{-x}.$$

Then $h'''(x) = 20(1-e^{-x}) + 4x(e^{-x} + 8e^{-2x}) > 0$ for $0 < x \leqq 2$ and $h''(0) = 0$. Thus $h''(x) > 0$ there. Since $h'(0) = 0$, there holds $h'(x) > 0$ there. By $h(0) = 0$, there holds $h(x) > 0$ there. Since

$$h(x) \leqq 3x - \frac{15}{4}x^2 + \frac{10}{3}x^3 - y,$$

we have the desired result.

LEMMA 2. $5(\eta^2 + \eta'^2) + 3(y^2 + y'^2) \leqq 4x - x^2$.

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 3. *If* $\eta \geqq 0$, *then* $y \leqq 1.62\, x - 0.56\, x^2$ *for* $0 \leqq x \leqq 300/287$.

Proof. By (3) there holds

$$\eta - \frac{1}{2}(2-x)y - x + \frac{x^2}{2} - \frac{x^3}{12} + 2\beta y - \beta^2 x \leqq 0$$

for every real β. Thus its discriminant satisfies

$$D_x(y)=y^2-x\left(x-\frac{x^2}{2}+\frac{x^3}{12}+\frac{1}{2}(2-x)y-\eta\right)\leqq 0,$$

which implies that $\omega_1\leqq y\leqq\omega_2$, where ω_1 and ω_2 are two zero points of the function $D_x(y)$. Since $\eta\geqq 0$, we have

$$y\leqq\omega_2\leqq\frac{1}{2}\left[x-\frac{x^2}{2}+\left(\frac{x^2}{4}(2-y)^2+4x^2-2x^3+\frac{x^4}{3}\right)^{1/2}\right]$$
$$=\frac{x}{4}\left[2-x+2\sqrt{5}\left(1-\frac{3}{5}x+\frac{7}{60}x^2\right)^{1/2}\right].$$

For $0\leqq x\leqq 300/287$, we have

$$\sqrt{1-\frac{3}{5}x+\frac{7}{60}x^2}\leqq 1-\frac{7}{25}x.$$

Therefore we have

$$y\leqq\frac{x}{4}\left[2-x+2\sqrt{5}\left(1-\frac{7}{25}x\right)\right]$$
$$=\frac{1+\sqrt{5}}{2}x-\frac{25+14\sqrt{5}}{100}x^2\leqq 1.62x-0.56x^2.$$

LEMMA 4. $|y|\leqq 1+2e^{-6},\ |y'|\leqq 1+2e^{-6}.$

In the sequel we shall make use of Lemma 4 under the forms $|y|\leqq 1.005$, $|y'|\leqq 1.005$.

4. In the first place we put $\beta=2a_2$ in (2). Then we have, taking the real part and rearranging the terms,

$$\Re a_6\leqq\frac{2}{5}+\frac{2}{3}(2-x)^2+\frac{11}{120}(2-x)^5+\frac{7}{4}(2-x)y^2-\frac{7}{4}(2-x)y'^2+3y\eta$$
$$-3y'\eta'+\frac{5}{4}(2-x)^2\eta+\frac{11}{8}(2-x)^3y+x\delta^2-2\delta\eta-2(2-x)y\delta.$$

For every positive number α we have

$$-3y'\eta'=-\frac{3}{2}\left(\frac{y'}{\sqrt{\alpha}}+\sqrt{\alpha}\,\eta'\right)^2+\frac{3}{2\alpha}y'^2+\frac{3\alpha}{2}\eta'^2$$
$$\leqq\frac{3}{2\alpha}y'^2+\frac{3\alpha}{2}\eta'^2.$$

By Lemma 2 we have

$$-3y'\eta'\leqq\left(\frac{3}{2\alpha}-\frac{9\alpha}{10}\right)y'^2+\frac{3\alpha}{10}(4x-x^2)-\frac{9\alpha}{10}y^2-\frac{3\alpha}{2}\eta^2.$$

Thus we have

$$\Re a_6 \leqq \frac{2}{5}+\frac{2}{3}(2-x)^2+\frac{11}{120}(2-x)^5+\frac{3\alpha}{10}(4x-x^2)+\left(\frac{7}{2}-\frac{9\alpha}{10}-\frac{7}{4}x\right)y^2$$

$$+\left(\frac{3}{2\alpha}-\frac{9\alpha}{10}-\frac{7}{2}+\frac{7}{4}x\right)y'^2+\frac{11}{8}(2-x)^3y+\frac{5}{4}(2-x)^2\eta+3y\eta$$

$$-2\delta\eta-2(2-x)y\delta+x\delta^2-\frac{3\alpha}{2}\eta^2.$$

Here we put $\delta=5(2-x)^2/8+7y/8$ and $\alpha=1/2$. Then there holds an inequality

$$\Re a_6 \leqq P(x)+Q(x, y, y', \eta),$$

$$P(x)=\frac{2}{5}+\frac{2}{3}(2-x)^2+\frac{11}{120}(2-x)^5+\frac{3}{20}(4x-x^2)+\frac{25}{64}x(2-x)^4, \tag{4}$$

$$Q(x, y, y', \eta)=\frac{y}{8}(2-x)^2\left(2+\frac{31}{4}x\right)+\left(\frac{49}{64}x-\frac{9}{20}\right)y^2+\left(\frac{7}{4}x-\frac{19}{20}\right)y'^2+\frac{5}{4}y\eta-\frac{3}{4}\eta^2.$$

Since there holds an inequality

$$-\frac{25}{48}y^2+\frac{5}{4}y\eta-\frac{3}{4}\eta^2 \leqq 0,$$

we have

$$Q(x, y, y', \eta) \leqq F(y, y'),$$

$$F(y, y')=\frac{y}{8}(2-x)^2\left(2+\frac{31}{4}x\right)+\left(\frac{49}{64}x+\frac{17}{240}\right)y^2+\left(\frac{7}{4}x-\frac{19}{20}\right)y'^2$$

and we have

$$\Re a_6 \leqq P(x)+F(y, y'). \tag{5}$$

We shall divide our proof into four cases: (I) $y\geqq 0$, $\eta\geqq 0$, (II) $y\geqq 0$, $\eta\leqq 0$, (III) $y\leqq 0$, $\eta\geqq 0$ and (IV) $y\leqq 0$, $\eta\leqq 0$.

5. Case (I). We shall start from the inequality (5) in this case. We shall divide this case into several subcases.

If $0\leqq x\leqq 19/35$, then

$$\Re a_6 \leqq P(x)+F(y, y'),$$

$$F(y, y') \leqq \frac{y}{8}(2-x)^2\left(2+\frac{31}{4}x\right)\cdot\left(\frac{49}{64}x+\frac{17}{240}\right)y^2.$$

Here we make use of Lemmas 2 and 3. Then we finally have

$$F(y, y')\leqq\frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)(1.62\,x-0.56\,x^2)+\left(\frac{49}{64}x+\frac{17}{240}\right)\frac{4x-x^2}{3}.$$

Therefore we have

$$\Re a_6\leqq 6-1.43\dot{5}x+0.4447\dot{2}x^2-2.029375x^3+1.391041\dot{6}x^4-0.243541\dot{6}x^5.$$

Evidently there holds $\Re a_6\leqq 6$ in this case and equality can occur only for $x=0$.

If $19/35\leqq x\leqq 300/287$, then, using Lemmas 2 and 3, we have

$$F(y, y')\leqq\frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)(1.62x-0.56x^2)$$

$$+\left(\frac{49}{64}x+\frac{17}{240}\right)\frac{4x-x^2}{3}+\left(\frac{7}{4}x-\frac{19}{20}\right)\frac{4x-x^2}{3}.$$

Then we have

$$\Re a_6\leqq 6-2.70\dot{2}x+3.0947\dot{2}x^2-2.612708\dot{3}x^3+1.391041\dot{6}x^4-0.243541\dot{6}x^5.$$

Since $-2.70\dot{2}+3.0947\dot{2}x-x^2<0$ for every x by the negativity of its discriminant and $-1.612708\dot{3}+1.391041\dot{6}x-0.243541\dot{6}x^2<0$ for $0\leqq x\leqq 1.1$, there holds $\Re a_6<6$ in this case.

If $1\leqq x\leqq 1.4$, then, using Lemma 4,

$$P(x)=1+\frac{31}{60}(2-x)^2+\frac{11}{120}(2-x)^5+\frac{25}{64}(2-x)^4x,$$

$$F(y, y')\leqq\frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)1.005+(1.005)^2\left(\frac{49}{64}x+\frac{7}{4}x+\frac{17}{240}-\frac{19}{20}\right).$$

Evidently we have

$$P(x)\leqq P(1)\leqq 1.9990,\quad F(y, y')\leqq 1.2249+2.6693.$$

Thus we have

$$\Re a_6\leqq 1.9990+1.2249+2.6693=5.8932<6.$$

If $1.4\leqq x\leqq 1.7$, then

$$P(x)\leqq P(1.4)\leqq 1.2641,\quad F(y, y')\leqq 1.0719+3.4323.$$

Thus we have

$$\Re a_6\leqq 5.7683<6.$$

If $1.7\leqq x\leqq 2$, then

$$P(x)\leqq P(1.7)\leqq 1.0471,\quad F(y, y')\leqq 0.2010+4.1941.$$

Thus we have

$$\Re a_6\leqq 5.4422<6.$$

6. Case (II). We shall start from the inequality (4). In this case evidently there holds

$$\Re a_6 \leqq P(x)+Q(x,\, y,\, y',\, \eta),\quad Q(x,\, y,\, y',\, \eta)\leqq G(y,\, y'),$$

$$G(y,\, y')=\frac{y}{8}(2-x)^2\left(2+\frac{31}{4}x\right)+\left(\frac{49}{64}x-\frac{9}{20}\right)y^2+\left(\frac{7}{4}x-\frac{19}{20}\right)y'^2.$$

If $0\leqq x\leqq 19/35$, then we have

$$G(y,\, y')\leqq \frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)y.$$

Using Lemma 1, we have

$$\Re a_6\leqq 6-0.15x+0.225x^2-12.61458\dot{3}x^3+23.875x^4-15.4171875x^5+3.2291\dot{6}x^6.$$

Since the discriminants of two quadratic polynomials

$$-0.15+0.225x-0.105x^2,\quad -12.50958\dot{3}+23.875x-11.3117x^2$$

are negative, both of two are negative in $0\leqq x\leqq 2$. Further there holds

$$-4.1054875x^5+3.2291\dot{6}x^6\leqq 0$$

for $0\leqq x\leqq 1.2$. Thus we can conclude that $\Re a_6\leqq 6$ for $0\leqq x\leqq 19/35$ and equality can occur only for $x=0$.

If $19/35\leqq x\leqq 0.95$, then we have, using Lemmas 1 and 2,

$$\begin{aligned} G(y,\, y')&\leqq \frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)\left(3x-\frac{15}{4}x^2+\frac{10}{3}x^3\right)\\ &\quad+\left(\frac{7}{4}x-\frac{19}{20}\right)\left(\frac{4x-x^2}{3}-y^2\right)+\left(\frac{49}{64}x-\frac{9}{20}\right)y^2\\ &=\frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)\left(3x-\frac{15}{4}x^2+\frac{10}{3}x^3\right)\\ &\quad+\left(\frac{7}{4}x-\frac{19}{20}\right)\frac{4x-x^2}{3}+y^2\left(-\frac{63}{64}x+\frac{1}{2}\right). \end{aligned}$$

Since $-63x/64+1/2\leqq 0$ for $x\geqq 32/63$ and $19/35>32/63$, we have

$$\Re a_6\leqq P(x)+G(y,\, y'),$$

$$G(y,\, y')\leqq \frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)\left(3x-\frac{15}{4}x^2+\frac{10}{3}x^3\right)+\left(\frac{7}{4}x-\frac{19}{20}\right)\frac{4x-x^2}{3}.$$

Therefore we have

$$\Re a_6\leqq 6-1.41\dot{6}x+2.875x^2-13.19791\dot{6}x^3+23.875x^4-15.4171875x^5+3.2291\dot{6}x^6.$$

Since

$$-1.41\dot{6}+2.875x-1.4589x^2<0,\quad -11.73901\dot{6}+23.875x-12.2x^2<0$$

for every x by the negativity of their discriminants and

$$3.2171875x^5+3.2291\dot{6}x^6\leqq 0$$

for $0\leqq x\leqq 0.95$, we have $\Re a_6<6$ for $19/35\leqq x\leqq 0.95$.

If $0.95\leqq x\leqq 1.5$, then, using Lemmas 2 and 4,

$$P(x)=1+\frac{31}{60}(2-x)^2+\frac{11}{120}(2-x)^5+\frac{25}{64}(2-x)^4x,$$

$$G(y, y')\leqq \frac{1}{8}(2-x)^2\left(2+\frac{31}{4}x\right)1.005+\left(\frac{7}{4}x-\frac{19}{20}\right)\frac{4x-x^2}{3}.$$

Then we have

$$P(x)\leqq P(0.9)\leqq 2.28752,\ G(y, y')\leqq 1.36426+2.09375.$$

Thus we have $\Re a_6<6$ for $0.9\leqq x\leqq 1.5$.

If $1.5\leqq x\leqq 1.6$, then we have, using Lemma 4,

$$G(y, y')\leqq \frac{1.005}{8}(2-x)^2\left(2+\frac{31}{4}x\right)+\left(\frac{7}{4}x-\frac{19}{20}\right)1.005^2+\left(\frac{49}{64}x-\frac{9}{20}\right)1.005^2$$

$$<0.42792+1.86855+0.78277=3.07924,$$

$$P(x)\leqq P(1.5)<1.16866.$$

Therefore we have $\Re a_6<6$ for $1.5\leqq x\leqq 1.6$.

If $1.6\leqq x\leqq 2$, then we have, using Lemma 4,

$$G(y, y')\leqq \frac{1.005}{8}(2-x)^2\left(2+\frac{31}{4}x\right)+(1.005)^2\left(\frac{49}{64}x+\frac{7}{4}x-\frac{14}{10}\right)$$

$$\leqq 0.2895+3.6680$$

and

$$P(x)\leqq P(1.6)\leqq 1.1081.$$

Thus we have $\Re a_6\leqq 5.0656<6$ for $1.6\leqq x\leqq 2$.

7. Case (III). This case can be reduced to the case (II). By (4) we have

$$\Re a_6\leqq P(x)\leqq Q(x, y, y', \eta),\ Q(x, y, y'\ \eta)\leqq H(y, y'),$$

$$H(y, y')=\left(\frac{49}{64}x-\frac{9}{20}\right)y^2+\left(\frac{7}{4}x-\frac{19}{20}\right)y'^2.$$

Thus there is no need to repeat the discussions.

8. Case (IV). This case can be reduced to the case (I). By (5) we have

$$\Re a_6\leqq P(x)+\mathrm{I}(y, y'),$$

$$\mathrm{I}(y, y')=\left(\frac{49}{64}x+\frac{17}{240}\right)y^2+\left(\frac{7}{4}x-\frac{19}{20}\right)y'^2;$$

Thus there is no need to repeat the discussions.

Therefore we have completed our proof.

9. Remarks. Lemmas 1 and 3 have played the central role in our proof. Unfortunately the former one is beyond the elementary level. It seems to the present author that there would be another way of proof being on the elementary level in principle. This may be somewhat interesting problem.

Beside the above problem there happens a sort of problems, for which we shall list only two. Seek for the quantities

$$\overline{\lim_{x\to 0}}\frac{y}{x} \qquad \text{and} \qquad \overline{\lim_{x\to 0}}\frac{\eta}{x}.$$

Jenkins [3] gave the perfect solution of the first one implicitely. And we have

$$\overline{\lim_{x\to 0}}\frac{y}{x}=3.$$

To the second one it is easy to prove, under our assumption on a_2,

$$\overline{\lim_{x\to 0}}\frac{\eta}{x}\leqq\frac{5}{4}.$$

References

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