ON PARALLEL SLIT MAPPINGS

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Dedicated to Professor K. Kunugi on his sixtieth birthday

1. Let Ω be a plane domain containing ∞. In the family of all the univalent functions f with the expansion $f(z)=z+c/z+\cdots$ about ∞, there exists a function maximizing Re $e^{-2i\alpha}c$. It is known to be unique, which will be denoted by p^{α}. We are interested in the function

$$\varphi=p^{0}+p^{\pi/2},$$

whose importance is well known.

It has been proved that, if Ω is bounded by a finite number of analytic Jordan closed curves, then φ is univalent and, almost simultaneously, every boundary component of its image domain is a *convex analytic Jordan closed curve.* The proof is found in, e.g., Schiffer [5] or Ahlfors-Beurling [1]. People say that their proofs involve a mistake, but it is also universally admitted that a small technical modification makes the proof correct.

For an arbitrary Ω, by an approximation through the exhaustion of the domain, it is easily seen that φ is still univalent. The shape of the boundary of the image domain was discussed by Sario [4], who showed that a boundary component of $\varphi(\Omega)$ is *either a point, or a line segment or else a convex curve.* Unfortunately the latter half of his proof seems to be based on an incomplete discussion.

It is the purpose of the present note to give a complete proof to Sario's theorem.

2. We mean by a *boundary component* Γ of Ω a (connected) component of the boundary of Ω. It is equivalent to the one in the sense of Kerékjártó-Stoïlow. If f is univalent on Ω we see, by the latter definition, a boundary component of $f(\Omega)$ corresponds to Γ canonically, which we shall denote by $f(\Gamma)$. Let us remark that a point w belongs to $f(\Gamma)$ if and only if $w=\lim f(z_n)$ for a sequence of points $z_n\in\Omega$ having limit points only on Γ.

3. For an arbitrary Ω, every boundary component of $p^{\alpha}(\Omega)$ is known to be either a point or a line segment with inclination α. In general this property is not sufficient to characterize p^{α}, but so does if Ω is of finite connectivity. This shows that the following (1) holds for Ω of finite connectivity. The approximation through

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an exhaustion guarantees the validity of the following on an arbitrary Ω:

(1) $$p^{\beta}=e^{i(\beta-\alpha)}(p^{\alpha}\cos(\beta-\alpha)-ip^{\alpha+\pi/2}\sin(\beta-\alpha))$$

for any α and β.

From this relation Sario [4] obtained the following, whose proof we shall repeat here to make this note readable:

LEMMA 1. *For a boundary component Γ of Ω only the following three occur:*

(I) *$p^{\alpha}(\Gamma)$ consists of a single point for all α,*

(II) *$p^{\alpha}(\Gamma)$ consists of a single point for an α, and $p^{\beta}(\Gamma)$ is a line segment for all $\beta\not\equiv\alpha \bmod \pi$,*

(III) *$p^{\alpha}(\Gamma)$ is a line segment for all α.*

Proof. Suppose neither (I) nor (III) occurs, and let $p^{\alpha}(\Gamma)=\{w_0\}$. If $p^{\alpha+\pi/2}(\Gamma)$ consists of a single point, then $p^{\beta}(\Gamma)$ must consist of a single point for all β, contradicting the assumption; for, if $p^{\alpha+\pi/2}(\Gamma)=\{w_0'\}$, then $\lim p^{\alpha}(z_n)=w_0$ and $\lim p^{\alpha+\pi/2}(z_n)=w_0'$ for every sequence of points $z_n\in\Omega$ having limit points only on Γ, so that $\lim p^{\beta}(z_n)=e^{i(\beta-\alpha)}(w_0\cos(\beta-\alpha)-iw_0'\sin(\beta-\alpha))$ showing that $p^{\beta}(\Gamma)$ consists of a single point. Therefore $p^{\alpha+\pi/2}(\Gamma)$ is a line segment, say $\{tw_0'+(1-t)w_0'';\ 0\leqq t\leqq 1\}$, $w_0'\neq w_0''$. Then again by the argument similar to the above, we see that $p^{\beta}(\Gamma)=\{e^{i(\beta-\alpha)}[w_0\cos(\beta-\alpha)-i(tw_0'+(1-t)w_0'')\sin(\beta-\alpha)];\ 0\leqq t\leqq 1\}$, being a line segment for all $\alpha\not\equiv\beta$ mod π.

4. We have need of seeing some properties of the parallel slit domain $p^{\alpha}(\Omega)$. For simplicity we discuss them for $\alpha=0$.

The domain $p^0(\Omega)$ is characterized as a *minimal* horizontal parallel slit domain. In terms of extremal length this is expressed as follows (Jenkins [3]):

Let S_L be the square $\{w;\ |\mathrm{Re}\,w|<L,\ |\mathrm{Im}\,w|<L\}$ and let $\{\gamma\}$ be the family of all the curves in $S_L\cap p^0(\Omega)$ joining the pair of vertical sides of S_L. Then, for any L with $p^0(\Omega)^c\subset S_L$, the extremal length $\lambda\{\gamma\}$ of $\{\gamma\}$ is equal to 1.

We apply this to prove the following:

LEMMA 2. *Let w_1 and w_2 be distinct points in $p^0(\Omega)$ with $\mathrm{Im}\,w_1=\mathrm{Im}\,w_2$. Let R_δ be the rectangle $\{w;\ \mathrm{Re}\,w_1<w<\mathrm{Re}\,w_2,\ |\mathrm{Im}(w-w_1)|<\delta\}$. Then, for any $\delta>0$, w_1 and w_2 can be joined by a curve in $R_\delta\cap p^0(\Omega)$.*

Proof.[1] We suppose that there are w_1, w_2, and R_δ which do not have the mentioned property. Take the S_L with L so large that $\bar{R}_\delta\subset S_L$. We are going to show that $\lambda\{\gamma\}>1$, being a contradiction.

Since w_1 and w_2 can not be joined within R_δ, there exists $\varepsilon_1>0$ with the pro-

1) The authors are indepted to Professor M. Ohtsuka, Hiroshima University, for the discussions we have had to prove Lemma 2.

perty that any curve in S_L joining w_1 and w_2 has length not smaller than $|w_2-w_1|+\varepsilon_1$. Let $w_1'=-L+i\,\mathrm{Im}\,w_1$, $w_2'=L+i\,\mathrm{Im}\,w_1$, then there exists $\varepsilon_2>0$ with the property that any curve in S_L joining w_1' and w_2' has length $\geqq 2L+\varepsilon_2$. Next let σ be the right vertical side of S_L, i.e., $\sigma=\{w;\ \mathrm{Re}\,w=L,\ |\mathrm{Im}\,w|\leqq L\}$. It is possible to find $\varepsilon_3>0$ such that any curve in S_L joining w_1' with σ has length $\geqq 2L+\varepsilon_3$. We may assume that ε_3 is chosen so small that $\varepsilon_3<L-|\mathrm{Im}\,w_1'|$ and $\{w;\ |w-w_1'|\leqq\varepsilon_3\}\subset p^0(\Omega)$. Then any curve in S_L joining the arc $S_L\cap\{w;\ |w-w_1'|=\varepsilon_3\}$ and σ has length $\geqq 2L$.

To estimate $\lambda\{\gamma\}$, we consider the density function

$$\rho(w)=\begin{cases} 1, & w\in S_L-\{w;\ |w-w_1'|<\varepsilon_3\},\\ 0, & w\in S_L\cap\{w;\ |w-w_1'|<\varepsilon_3\}.\end{cases}$$

Then

$$\int_\gamma \rho ds\geqq 2L$$

holds no matter whether γ meet $\{w;\ |w-w_1'|\leqq\varepsilon_3\}$ or not. Thus we obtain the desired conclusion

$$\lambda\{\gamma\}\geqq\frac{4L^2}{4L^2-\pi\varepsilon_3^2/2}>1.$$

We remark that a similar theorem is found in Ahlfors-Beurling [1], p. 120.

5. LEMMA 3. *Let w_0 be a boundary point of $p^\alpha(\Omega)$. Assume further that, if the boundary component containing w_0 is not a single point, w_0 is an end point of the component being a line segment. Then, for any $\varepsilon>0$, it is possible to find an open set U such that $w_0\in U\subset\{w;\ |w-w_0|<\varepsilon\}$ and $U\cap p^\alpha(\Omega)$ is connected.*

Proof. Without loss of generality we may consider the case where $\alpha=0$ and w_0 is the left end point of the segment. It is possible to find a $w_0'\in p^0(\Omega)$ such that $\mathrm{Re}\,w_0-\varepsilon<\mathrm{Re}\,w_0'<\mathrm{Re}\,w_0$, $\mathrm{Im}\,w_0'=\mathrm{Im}\,w_0$. Take an $\varepsilon_0>0$ with $\{w;\ |w-w_0'|<\varepsilon_0\}\subset\Omega$. Then

$$U=\{w;\ |w-w_0'|<\varepsilon_0\}\cup\{w;\ |w-w_0|<\varepsilon,\ \mathrm{Re}\,w_0'<\mathrm{Re}\,w,\ |\mathrm{Im}\,(w-w_0)|<\varepsilon_0\}$$

is the desired, because, by Lemma 2, every point of $U\cap p^0(\Omega)$ can be joined with the segment $\{w;\ \mathrm{Re}\,w=w_0',\ |\mathrm{Im}(w-w_0')|<\varepsilon_0\}$ by a curve in $U\cap p^0(\Omega)$.

6. We now come back to our function $\varphi=p^0+p^{\pi/2}$. By (1) it can also be expressed as

$$\varphi=p^\alpha+p^{\alpha+\pi/2}. \tag{2}$$

For any boundary component Γ of Ω, there exists a uniquely determined component of $\varphi(\Omega)^c$ whose boundary coincides with $\varphi(\Gamma)$. We denote it by Δ_Γ. We denote further the end points of $p^\alpha(\Gamma)$ by w^α and w'^α ($w^\alpha=w'^\alpha$ may happen); namely

$$p^\alpha(\Gamma)=\{tw^\alpha+(1-t)w'^\alpha;\ 0\leqq t\leqq 1\}.$$

Consider also the following closed rectangle with inclination α:

$$Q^\alpha{}_\Gamma=\{w+w';\ w\in p^\alpha(\Gamma),\ w'\in p^{\alpha+\pi/2}(\Gamma)\};$$

it may degenerate to a line segment or a single point.

THEOREM. *Let $\varphi=p^0+p^{\pi/2}$ on an arbitrary Ω ($\infty\in\Omega$). Then, for any boundary component Γ of Ω, the component Δ_Γ of the $\varphi(\Omega)^c$ corresponding to $\varphi(\Gamma)$ is a (compact) convex set.*

More precisely, one of the following three occurs:

(I) *Δ_Γ is a single point and coincides with $Q^\alpha{}_\Gamma$ for all α,*

(II) *Δ_Γ is a line segment and coincides with $Q^\alpha{}_\Gamma$ for an α, and for any $\beta\neq\alpha$ mod π, Δ_Γ is a diagonal of the (non-degenerate) rectangle $Q^\beta{}_\Gamma$.*

(III) *Δ_Γ is a convex set with an interior point and, for any α, $Q^\alpha{}_\Gamma$ is the smallest rectangle with inclination α which contains Δ_Γ.*

7. We shall prove (III) under the assumption of the (III) of Lemma 1. In a similar way we can prove (I) and (II) under the assumption of (I) and (II) of Lemma 1, respectively.

First we show the latter half of (III). For this purpose, it suffices to show that $\Delta_\Gamma\subset Q^\alpha{}_\Gamma$ and $\sigma\cap\varphi(\Gamma)\neq\phi$ for any side σ (including end points) of $Q^\alpha{}_\Gamma$. This implies further that Δ_Γ contains an interior point whenever the convexity of Δ_Γ is guaranteed.

We shall do it for $\alpha=0$. Because of (2) it does not mean the loss of generality. For any $w_0\in\varphi(\Gamma)$ it is possible to find a sequence of points $z_n\in\Omega$ having accumulation points on Γ only and such that $w_0=\lim\varphi(z_n)$. We can find its subsequence such that $\lim p^0(z_{n_k})=w_1$ and $\lim p^{\pi/2}(z_{n_k})=w_2$ exist. $w_0=w_1+w_1'$, $w_1\in p^0(\Gamma)$, $w_1'\in p^{\pi/2}(\Gamma)$, therefore $w_0\in Q^0{}_\Gamma$, thus $\Delta_\Gamma\subset Q^0{}_\Gamma$.

Since (III) of Lemma 1 is assumed. $Q^0{}_\Gamma$ is a non-degenerate rectangle. Let σ be, e.g., its left vertical side. Let w^0 be the left end point of $p^0(\Gamma)$. Then there exists a sequence of points $z_n\in\Omega$ having accumulation points on Γ only and such that $\lim p^0(z_n)=w^0$. Take such a subsequence that $\lim p^{\pi/2}(z_{n_k})=w'\in p^{\pi/2}(\Gamma)$ exists. Then $\lim\varphi(z_{n_k})=w^0+w'\in\varphi(\Gamma)$ and, on the other hand, $w^0+w'\in\sigma$. We conclude that $\sigma\cap\varphi(\Gamma)\neq\phi$.

8. Next we prove the convexity of Δ_Γ under the assumption of (III) of Lemma 1.

If Δ_Γ is not convex, we can find a line l and points w_0, w_0', w_0^* with the following property: Δ_Γ is in one of the (closed) half plane determined by l, w_0 and w_0' are different and belong to Δ_Γ, w_0^* does not belong to Δ_Γ and belongs to the line segment determined by w_0 and w_0'. The proof is carried out by a standard argument (cf. Eggleston [2]), for instance, as in the following lines. Namely, if Δ_Γ is not convex, its convex hull does not coincide with Δ_Γ. Since the complement of Δ_Γ is connected, it is possible to find a boundary point w_0^* of the hull such that $w_0^* \notin \Delta_\Gamma$. Through this point there is a supporting line l of Δ_Γ, on which we can find the desired w_0 and w_0'.

Let the inclination of l be α. By (2), we may again assume that $\alpha=0$, and that Δ_Γ is in the upper-half plane determined by l.

Take sequences z_n, $z_n' \in \Omega$ which have accumulation points only on Γ and are such that

$$\lim \varphi(z_n)=w_0, \qquad \lim \varphi(z_n')=w_0'.$$

It is easily verified that

$$\lim p^{\pi/2}(z_n)=\lim p^{\pi/2}(z_n')=w^{\pi/2},$$

where $w^{\pi/2}$ is the lower end point of the segment $p^{\pi/2}(\Gamma)$. Therefore

$$\lim p^0(z_n)=w_1 \in p^0(\Gamma) \text{ and } \lim p^0(z_n')=w_1' \in p^0(\Gamma)$$

exist. We have $w_1+w^{\pi/2}=w_0$, $w_1'+w^{\pi/2}=w_0'$, and therefore $w_1 \neq w_1'$. Let $w_1^* \in p^0(\Gamma)$ be the point defined by $w_1^*+w^{\pi/2}=w_0^*$. Let l^* be the line passing through w_1^* and being perpendicular to $p^0(\Gamma)$.

Let ε_n be such that $|p^{\pi/2}(z_n)-w^{\pi/2}|<\varepsilon_n$, $|p^{\pi/2}(z_n')-w^{\pi/2}|<\varepsilon_n$, $\varepsilon_n \downarrow 0$. According to Lemma 3, we can find a curve in $p^{\pi/2}(\Omega) \cap \{w;\ |w-w^{\pi/2}|<\varepsilon_n\}$ joining $p^{\pi/2}(z_n)$ and $p^{\pi/2}(z_n')$. Let its inverse image under $p^{\pi/2}$ be γ_n, which clusters only to Γ. For sufficiently large n, the points $p^0(z_n)$ and $p^0(z_n')$ lie in different sides of l^*, therefore, $p^0(\gamma_n) \cap l^* \neq \phi$. Thus we can find $z_n^* \in \gamma_n$ with $p^0(z_n^*) \in l^*$. Accumulation points of z_n^* lies on Γ only, so that $p^0(z_n^*)$ clusters to $p^0(\Gamma)$ only. We conclude that $\lim p^0(z_n^*)=w_1^*$. Of course $\lim p^{\pi/2}(z_n^*)=w^{\pi/2}$, therefore $w_0^*=\lim \varphi(z_n^*) \in \varphi(\Gamma)$, contradicting the assumption $w_0^* \notin \Delta_\Gamma$. We thus conclude that Δ_Γ is convex.

9. Unlike the case where Ω is bounded by a finite number of analytic curves, we can not expect $\varphi(\Gamma)$ to be an analytic curve even in the case (III). We give an *example of Ω, Γ of case* (III) *for which there exists an α with*

$$\Delta_\Gamma=Q^\alpha{}_\Gamma.$$

In the w-plane prepare the vertical line segments $\sigma_n=\{w;\ \mathrm{Re}\, w=1/n,\ |\mathrm{Im}\, w|\leqq 1\}$, $n=\pm1, \pm2, \cdots$ and $\sigma_0=\{w;\ \mathrm{Re}\, w=0,\ |\mathrm{Im}\, w|\leqq 1\}$. Map the domain $(\cup_{n=-\infty}^{\infty}\sigma_n)^c$ by $z=f(w)$, being the extremal horizontal parallel slit map p^0 of this domain. Let $s_n=f(\sigma_n)$, being a boundary component of the image domain, $n=0, \pm1, \cdots$. Because of the symmetry of the domain, f is symmetric about the real and the imaginary axes, and, therefore, s_n is a line segment on the real axis symmetric with s_{-n} about the imaginary axis ($n=\pm 1, \pm 2, \cdots$). Further $s_0=\{0\}$; it is seen from the boundary correspondence of the upper-half plane being simply connected.

Rotate s_n about the origin by $\pi/2$ and let the resulting be s_n' ($n=0, \pm 1, \cdots$). Note that $f^{-1}(s_n')$ is a line segment on the imaginary axis ($n=\pm 1, \pm 2, \cdots$). Then $\Omega=(\bigcup_{n=-\infty}^{\infty}(s_n \cup s_n'))^c$ and $\Gamma=\{0\}$ are the desired.

In fact, $f^{-1}(\Omega)$ is a minimal vertical parallel slit domain, therefore the restriction of f^{-1} on Ω is the $p^{\pi/2}$ of Ω. The domain Ω is symmetric about the real and the imaginary axes, so that we see

$$p^0(z)=-ip^{\pi/2}(iz).$$

Therefore $p^0(\Gamma)=[-1, 1]$, $p^{\pi/2}(\Gamma)=[-i, i]$, and $Q^0{}_\Gamma$ is the square with verteces $\pm 1 \pm i$; here the blackets express the line segments determined by the indicated points. Because of the convexity of Δ_Γ, we conclude that

$$\Delta_\Gamma=Q^0{}_\Gamma$$

whenever $\pm 1 \pm i \in \varphi(\Gamma)$ is shown.

For instance, take $1+i$. We remark that the first quadrant corresponds under p^0 and $p^{\pi/2}$. Draw a line segment in $p^0(\Omega) \cap$ (1st quad.) with an end point at $w=1$. Its image under $(p^0)^{-1}$ is a curve in $\Omega \cap$ (1st quad.) terminating at $z=0$, and its image under $p^{\pi/2} \circ (p^0)^{-1}$ is a curve in $p^{\pi/2}(\Omega) \cap$ (1st quad.) clustering to $[0, i]$. On the last curve it is possible to find a sequence of points w_n converging to $w=i$. Thus we get $(p^{\pi/2})^{-1}(w_n)=z_n \in \Omega$ such that $\lim z_n=0$, $\lim \varphi(z_n)=\lim p^0 \circ (p^{\pi/2})^{-1}(w_n)+\lim w_n=1+i$ and conclude that $1+i \in \varphi(\Gamma)$.

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