CONFORMAL MAPPING ONTO POLYGONS BOUNDED BY SPIRAL ARCS

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0. Let G_0 be a simply-connected bounded domain laid on the complex w-plane which is starlike with respect to $w=0$. Suppose that the boundary of G_0 consists of a finite number of circular arcs centred at the origin and of rectilinear segments on rays issuing from the origin. Denote by $w=f(z)$ an analytic function which maps $|z|<1$ univalently onto G_0 and satisfies a normalization $f(0)=0$. Goodman [1] established an integral representation formula of Schwarz-Christoffel type for such a mapping function.

In a recent paper [5], one of the authors has given an alternative proof for Goodman's result. It may be noticed from the proof in [5] that the starlikeness of G_0 as well as its one-sheetedness are not essential. In fact, the whole procedure of this proof remains valid without any substantial modification. Further, as remarked below, it is also admissible that the domain G_0 has infinite protrusions and its boundary contains circular or rectilinear slits.

Now, both (infinitely winding) circumferences centred at the origin and rays issuing from the origin may be regarded as particular extreme members of a wider class of curves which consists of all logarithmic spirals with the asymptotic point at the origin; this is a fact which was ingeniously used by Grunsky [2]. In fact, any logarithmic spiral with the asymptotic point at the origin is expressed by the equation

$$\arg w - \lambda \lg w = c.$$

The parameter λ representing the inclination of the spiral runs over real numbers, while the parameter c may be restricted by $0 \leqq c < 2\pi$. More precisely, the value of χ determined by $\lambda = \tan(\chi/2)$ denotes the constant angle of the tangent at every point of the spiral to the radius vector through the point. A limiting case $\lambda=0$, $\chi=0$ corresponds to a ray starting at the origin, while another limiting case $\lambda=\infty$, $\chi=\pi$ ($\lg|w|=\text{const}$) corresponds to a circumference about the origin.

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1. According to the circumstances explained above, the formula of Goodman will be generalized. Namely, the result to be established states as follows.

THEOREM 1. *Let D_0 be an unramified simply-connected domain which covers a neighborhood of the origin only once but does not cover the point at infinity. Suppose that the boundary of D_0 consists of a finite number of (finite or infinite) arcs of logarithmic spirals with the asymptotic point at the origin, degenerate limiting cases (explained above) being admissible. Denote by $w=f(z)$ an analytic functiou which maps the circle $|z|<1$ onto D_0 with a normalization $f(0)=0$. Let $e^{i\varphi_\mu}$ $(\mu=1, \cdots, m)$ denote the points on $|z|=1$ which correspond to the vertices (i. e., the angular points of the boundary) of D_0 where the jumps of the direction of the tangent vector amount $\alpha_\mu\pi$, respectively.*[1] *Then, the mapping function satisfies the equation*[2]

$$\frac{zf'(z)}{f(z)}=\prod_{\mu=1}^{m}\frac{1}{(1-e^{-i\varphi_\mu}z)^{\alpha_\mu}}; \qquad \sum_{\mu=1}^{m}\alpha_\mu=0.$$

Proof. The proof in [5] remains valid almost in its whole course. First, it will be noted that the domain D_0 may be supposed to be bounded. In fact, otherwise, D_0 is exhaustible by an increasing sequence of its subdomains in the sense of domain-kernel which are obtained by cutting off the unbounded parts of D_0 along circular arcs about the origin. The cuts thus inserted determine sequences defining the prime ends which lie on the point at infinity and are finite in number. These subdomains are bounded by a finite number of spiral arcs and the sequence of functions $\{f_j(z)\}$ mapping $|z|<1$ onto them under a suitable normalization, e. g., $f_j(0)=f(0)(=0)$, $\arg f_j'(0)=\arg f'(0)$, converges to $f(z)$ in the wider sense in $|z|<1$. Now, $f(z)$ being supposed to be bounded, an integral representation obtained in [3] yields

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d(\arg df(e^{i\theta})-\arg f(e^{i\theta})).$$

In view of the boundary character of D_0, the boundary point $f(e^{i\theta})$ satisfies an equation of the form

$$\arg f(e^{i\theta})-\lambda\lg|f(e^{i\theta})|=c$$

along each arc on $|z|=1$ which contains none of the points $e^{i\varphi_\mu}$ $(\mu=1, \cdots, m)$, the

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1) If $\arg f(z)$ has simultaneously a jump at $z=e^{i\varphi_\mu}$, the direction of the tangent vector $\arg df(z)$ must be replaced by $\arg df(z)-\arg f(z)$.

2) Here each factor in the right-hand member denotes the branch which is equal to unity at $z=0$.

values of λ and c depending on an arc. Hence, the quantity

$$\arg df(e^{i\theta})-\arg f(e^{i\theta})=\arg d\lg f(e^{i\theta})$$

$$=\arg(d\lg|f(e^{i\theta})|+i\,d\arg f(e^{i\theta}))=\arg((\lambda^{-1}+i)d\arg f(e^{i\theta}))$$

remains constant along each arc under consideration, since $\arg f(e^{i\theta})$ then varies monotonously with θ. This fact is directly obvious in the degenerate case $\lambda=0$. On the other hand, $\arg df(e^{i\theta})$ possesses a jump with the height $\alpha_\mu\pi$ at $\theta=\varphi_\mu$ while $\arg f(e^{i\theta})$ is continuous there. Consequently, the above integral relation becomes

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\frac{1}{2}\sum_{\mu=1}^{m}\frac{e^{i\varphi_\mu}+z}{e^{i\varphi_\mu}-z}\alpha_\mu.$$

Since the left-hand member vanishes at $z=0$, so does the right-hand member, i. e.,

$$\sum_{\mu=1}^{m}\alpha_\mu=0$$

(cf. also the argument concerning the corresponding relation at the final part in the proof of theorem 2 below). Hence, from the last equation, it follows

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\sum_{\mu=1}^{m}\frac{\alpha_\mu z}{e^{i\varphi_\mu}-z}.$$

Dividing by z and then integrating from 0 to z, the desired representation is obtained.

2. In the paper [5] cited before, an analogue of Goodman's result has been derived for doubly-connected domains. Corresponding to theorem 1 in the present paper, this analogue can be also extended to the form as stated in the following theorem.

THEOREM 2. *Let D_q be an unramified ring domain with the modulus* $\lg q^{-1}$ *which does not cover the origin and the point at infinity. Suppose that the boundary of D_q consists of a finite number of (finite or infinite) arcs of logarithmic spirals with the asymptotic point at the origin, degenerate limiting cases (explained before) being admissible. Denote by $w=f(z)$ a single-valued analytic function which maps the annulus $q<|z|<1$ onto D_q such that $|z|=1$ corresponds to the exterior boundary component of D_q. Let $e^{i\varphi_\mu}$ $(\mu=1, \cdots, m)$ and $qe^{i\psi_\nu}$ $(\nu=1, \cdots, n)$ denote the points which correspond to the vertices of D_q where the jumps of the direction of the tangent* *vector amount* $\alpha_\mu \pi$ *and* $\beta_\nu \pi$, *respectively.*[3] *Then, the mapping function satisfies the equation*

$$\frac{zf'(z)}{f(z)}=Cz^{ic^*}\prod_{\mu=1}^{m}\frac{1}{\sigma(i\lg z+\varphi_\mu)^{\alpha_\mu}}\prod_{\nu=1}^{n}\sigma_3(i\lg z+\psi_\nu)^{\beta_\nu};$$

$$\sum_{\mu=1}^{m}\alpha_\mu=\sum_{\nu=1}^{n}\beta_\nu=0.$$

The symbols of sigma-functions depend on Weierstrassian theory of elliptic functions with the primitive periods $2\omega_1=2\pi$ *and* $2\omega_3=2i\lg q^{-1}$. *In the equation, C is a non-vanishing constant and* c^* *a real constant determined by*

$$c^*=\frac{\eta_1}{\pi}\left(\sum_{\mu=1}^{m}\alpha_\mu\varphi_\mu-\sum_{\nu=1}^{n}\beta_\nu\psi_\nu\right).$$

Proof. Based on a similar reason as mentioned in the proof of theorem 1, the ring domain D_q may be supposed bounded from ∞ and 0. The proof of the original theorem in [5] is then transferred with slight modification. In fact, an integral representation derived in [4] yields

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}$$

$$=\frac{1}{\pi i}\int_0^{2\pi}\zeta(i\lg z+\theta)\,d(\arg df(e^{i\theta})-\arg f(e^{i\theta}))$$

$$-\frac{1}{\pi i}\int_0^{2\pi}\zeta_3(i\lg z+\theta)\,d(\arg df(qe^{i\theta})-\arg f(qe^{i\theta}))+ic^*,$$

where c^* is a real constant given by

$$c^*=\frac{\eta_1}{\pi^2}\int_0^{2\pi}\theta\,d(\arg df(e^{i\theta})-\arg f(e^{i\theta}))$$

$$-\frac{\eta_1}{\pi^2}\int_0^{2\pi}\theta\,d(\arg df(qe^{i\theta})-\arg f(qe^{i\theta})).$$

In the same reason as in the proof of theorem 1, the quantity $\arg df(z)-\arg f(z)$

3) Here the supporting point of the tangent vector moves on the boundary of D_q in such a manner that its antecedent moves on $|z|=1$ or $|z|=q$ in the direction of increasing $\arg z$. Cf. also the footnote 1).

remains constant along each arc on $|z|=1$ and $|z|=q$ which contains none of the points $e^{i\varphi_\mu}$ ($\mu=1, \cdots, m$) and $qe^{i\psi_\nu}$ ($\nu=1, \cdots, n$). On the other hand, $\arg df(z)$ possesses jump with the height $\alpha_\mu\pi$ or $\beta_\nu\pi$ at $z=e^{i\varphi_\mu}$ or $z=qe^{i\psi_\nu}$, respectively, while $\arg f(z)$ is continuous throghout each boundary component. Consequently, the above integral relation for $f(z)$ and the expression for c^* reduce to

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}$$

$$=\frac{1}{i}\sum_{\mu=1}^{m}\zeta(i\lg z+\varphi_\mu)\alpha_\mu-\frac{1}{i}\sum_{\nu=1}^{n}\zeta_3(\lg z+\psi_\nu)\beta_\nu+ic^*$$

and

$$c^*=\frac{\eta_1}{\pi}\left(\sum_{\mu=1}^{m}\alpha_\mu\varphi_\mu-\sum_{\nu=1}^{n}\beta_\nu\psi_\nu\right).$$

Dividing the first relation by z and then integrating with respect to z, the desired representation is obtained. Finally, from the above argument, it follows

$$\int_0^{2\pi} d(\arg df(e^{i\theta})-\arg f(e^{i\theta}))=\pi\sum_{\mu=1}^{m}\alpha_\mu,$$

$$\int_0^{2\pi} d(\arg df(qe^{i\theta})-\arg f(qe^{i\theta}))=\pi\sum_{\nu=1}^{n}\beta_\nu.$$

On the other hand, both boundary components of D_q are by assumption homologous to a simple contour oriented positively with respect to the origin. Hence, the left-hand members of the last two relations must vanish, whence follows

$$\sum_{\mu=1}^{m}\alpha_\mu=\sum_{\nu=1}^{n}\beta_\nu=0.$$

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