ON FREQUENCY RESPONSE OF A HYDRAULIC SERVOMOTOR

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§ 1. Introduction.

The mechanism of a pilot valve controlled hydraulic servomotor can be explained with reference to Fig. 1. The flow of oil induced by a displacement of the pilot valve A causes a similar displacement of the piston B. It is important in the design of an apparatus like this to investigate how faithfully B follows the displacement of A.

One of the methods often used is to investigate the frequency response of an apparatus; i.e. to investigate the motion of B when A is displaced sinusoidally.

Let y denote the displacement of B when the displacement of A is

$x = X \sin \omega t$, X, ω: positive constants, t: time.

Then it is known that y satisfies following differential equation:

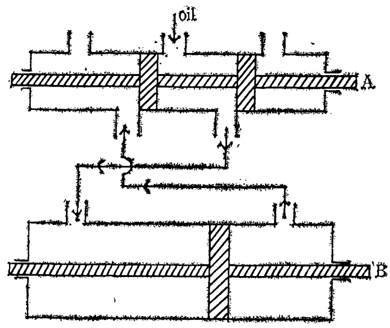

Fig. 1. The mechanism of a pilot valve controlled hydraulic servomotor

$$m\frac{d^2y}{dt^2}+\left(\frac{2A^3}{k^2X^2\sin^2\omega t}+RA^3\right)\left(\frac{dy}{dt}\right)^2-(AP_s-F)=0, \qquad \text{for } \sin\omega t>0,$$

$$m\frac{d^2y}{dt^2}-\left(\frac{2A^3}{k^2X^2\sin^2\omega t}+RA^3\right)\left(\frac{dy}{dt}\right)^2+(AP_s-F)=0, \qquad \text{for } \sin\omega t<0,$$

where m, A, k, R, P_s and F are physical constants determined by the characteristics of the apparatus; cf. [1]. Further explanation of the constants will be omitted.

Put

$$\omega t=\theta, \quad \frac{dy}{d\theta}=u, \quad \frac{2A^3}{mk^2X^2}=a, \quad \frac{RA^3}{m}=b, \quad \frac{AP_s-F}{m\omega^2}=c.$$

Then the above differential equation will be reduced to

$$\text{(1. 1)} \qquad \frac{du}{d\theta}+\left(\frac{a}{\sin^2\theta}+b\right)u^2-c=0, \qquad \text{for } \sin\theta>0,$$

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(1. 2) $$\frac{du}{d\theta}-\left(\frac{a}{\sin^2\theta}+b\right)u^2+c=0, \qquad \text{for } \sin\theta<0.$$

It is known from physical conditions that

(1. 3) $$0<b<a\ll c.$$

The purpose of this paper is to find a solution of (1. 1) and (1. 2) which represents the motion of the piston B.

§2. The symmetry characters of solutions.

If $u=f(\theta)$ is one of the solutions of (1. 1) and if $g(\theta)\equiv f(\theta-\pi)$, then

$$\frac{df(\theta-\pi)}{d(\theta-\pi)}+\left(\frac{a}{\sin^2(\theta-\pi)}+b\right)\{f(\theta-\pi)\}^2-c=0,$$

which can be easily reduced to

$$\frac{dg(\theta)}{d\theta}+\left(\frac{a}{\sin^2\theta}+b\right)\{g(\theta)\}^2-c=0,$$

showing that $u=g(\theta)\equiv f(\theta-\pi)$ is also a solution of (1. 1). This means that when one integral curve of (1. 1) is obtained, it can be translated π units in the θ-direction to obtain another one.

If $h(\theta)\equiv -f(\theta)$, then

$$\frac{dh(\theta)}{d\theta}-\left(\frac{a}{\sin^2\theta}+b\right)\{h(\theta)\}^2+c$$

$$=-\left[\frac{df(\theta)}{d\theta}+\left(\frac{a}{\sin^2\theta}+b\right)\{f(\theta\}^2-c\right]=0,$$

showing that $u=h(\theta)\equiv -f(\theta)$ is a solution of (1. 2) when $u=f(\theta)$ is a solution of (1. 1).

From the above, the following is established:

Obtain all the solutions of (1. 1) for $0\leqq\theta\leqq\pi$. Translate them in the θ-direction by π and change their sign to obtain all the solutions of (1. 2) for $\pi\leqq\theta\leqq 2\pi$. Connect these two sets of integral curves at $\theta=\pi$ and we obtain all the integral curves of the given differential equation for $0\leqq\theta\leqq 2\pi$. Successive translations in the θ-direction by 2π will provide the complete set of solutions of (1. 1) and (1. 2) for $0\leqq\theta<\infty$. Therefore it suffices to know the behavior of integral curves of (1. 1) for $0\leqq\theta\leqq\pi$ to obtain the complete knowledge of the solutions of (1. 1) and (1. 2) for $0\leqq\theta<\infty$.

In constructing the solution curves by the method stated above, there naturally remains some ambiguity since the connection of the integral curves at $\theta=n\pi$ is still left arbitrary. However, physical consideration will show that, to obtain the solution we are seeking for, it is most plausible to choose the possibly smooth connection at $\theta=n\pi$.

Prior to the investigation of the integral curves of

$$\frac{du}{d\theta}+\left(\frac{a}{\sin^2\theta}+b\right)u^2-c=0 \tag{1. 1}$$

in the interval $0\leqq\theta\leqq\pi$, the following fact should be noticed. If $u=f(\theta)$ is one of the solutions of (1. 1) and $-f(\pi-\theta)\equiv\varphi(\theta)$, then

$$\frac{d\varphi(\theta)}{d\theta}+\left(\frac{a}{\sin^2\theta}+b\right)\{\varphi(\theta)\}^2-c$$

$$=\frac{df(\pi-\theta)}{d(\pi-\theta)}+\left\{\frac{a}{\sin^2(\pi-\theta)}+b\right\}\{f(\pi-\theta)\}^2-c=0,$$

showing that $u=\varphi(\theta)\equiv-f(\pi-\theta)$ is also a solution of (1. 1). So if we rotate an integral curve of (1. 1) for $0\leqq\theta\leqq\pi$ by an angle π about a point $\theta=\pi/2$, $u=0$, another integral curve of the same equation is obtained.

§3. Behavior of the solutions at $\theta=0$ and $\theta=\pi$.

As the discussions of §2 have shown, we should naturally lay an emphasis upon the study of solution curves of (1. 1) for $0\leqq\theta\leqq\pi$.

A transformation

$$u=cw\Big/\frac{dw}{d\theta}$$

will reduce (1. 1) to a linear equation

$$\frac{d^2w}{d\theta^2}-c\left(\frac{a}{\sin^2\theta}+b\right)w=0. \tag{3. 1}$$

The indicial equation at a regular singular point $\theta=0$ is

$$\lambda^2-\lambda-ac=0$$

with two roots

$$\lambda_1=\frac{1+\sqrt{1+4ac}}{2},$$

$$\lambda_2 = 1 - \lambda_1 = \frac{1 - \sqrt{1+4ac}}{2}.$$

According to the condition (1. 3), $\lambda_1 > 0$ and $\lambda_2 < 0$. Further we add an assumption that $\lambda_1 - \lambda_2 = \sqrt{1+4ac}$ is not an integer.

In the vicinity of $\theta = 0$, the general solution of (3. 1) is expressed in a form

$$w = \alpha\theta^{\lambda_1}(1+\cdots) + \beta\theta^{\lambda_2}(1+\cdots)$$

where α and β are constants and the unwritten terms inside the brackets are convergent power series of θ without constant terms. From this follows that

$$u = cw \Big/ \frac{dw}{d\theta}$$

$$= c \cdot \frac{\alpha\theta^{\lambda_1}(1+\cdots) + \beta\theta^{\lambda_2}(1+\cdots)}{\alpha\lambda_1\theta^{\lambda_1-1}(1+\cdots) + \beta\lambda_2\theta^{\lambda_2-1}(1+\cdots)}$$

$$= c \cdot \frac{\alpha\theta^{\lambda_1}(1+\cdots) + \beta\theta^{1-\lambda_1}(1+\cdots)}{\alpha\lambda_1\theta^{\lambda_1-1}(1+\cdots) + \beta(1-\lambda_1)\theta^{-\lambda_1}(1+\cdots)}.$$

To the solutions of (3. 1) with $\beta = 0$ corresponds a solution of (1. 1):

$$u = c\frac{\alpha\theta^{\lambda_1}(1+\cdots)}{\alpha\lambda_1\theta^{\lambda_1-1}(1+\cdots)} = c\frac{\theta(1+\cdots)}{\lambda_1(1+\cdots)}$$

such that

$$\lim_{\theta\to 0}\frac{u}{\theta} = \frac{c}{\lambda_1} > 0.$$

Except this one, the solutions of (1. 1) are written in the form

$$u = c\frac{k\theta^{\lambda_1}(1+\cdots) + \theta^{1-\lambda_1}(1+\cdots)}{k\lambda_1\theta^{\lambda_1-1}(1+\cdots) + (1-\lambda_1)\theta^{-\lambda_1}(1+\cdots)}$$

$$= c\frac{k\theta^{2\lambda_1}(1+\cdots) + \theta(1+\cdots)}{k\lambda_1\theta^{2\lambda_1-1}(1+\cdots) + (1-\lambda_1)(1+\cdots)}$$

where we have put $\alpha/\beta = k$. Since $2\lambda_1 - 1 = \sqrt{1+4ac} > 0$, and $\lambda_1 > 1$, we have

$$\lim_{\theta\to 0}\frac{u}{\theta} = \frac{c}{1-\lambda_1} < 0.$$

It is thus concluded that all the integral curves of (1. 1) tend to zero as $\theta \to 0$, and only one among them is tangent to the line

$$u=\frac{c}{\lambda_1}\theta \qquad \left(\frac{c}{\lambda_1}>0\right),$$

while the others are all tanget to the line

$$u=\frac{c}{1-\lambda_1}\theta \qquad \left(\frac{c}{1-\lambda_1}<0\right).$$

The behavior of the integral curves at $\theta=\pi$ can be easily inferred by a remark mentioned at the end of §2. Therefore $u\to 0$ as $\theta\to\pi$ and only one of the integral curves is tangent to the line

$$u=\frac{c}{\lambda_1}(\theta-\pi)$$

while the others are all tangent to the line

$$u=\frac{c}{1-\lambda_1}(\theta-\pi).$$

§4. Fundamental properties of integral curves.

Putting $u=0$ in (1. 1), we get

$$\frac{du}{d\theta}=c>0 \qquad \text{for} \quad u=0.$$

Hence:

1) The integral curve has a positive inclination when it crosses the θ-axis.

Rewriting (1. 1) in a form

$$0\leqq\left(\frac{a}{\sin^2\theta}+b\right)u^2=c-\frac{du}{d\theta},$$

we immediately have:

2) $du/d\theta$ is bounded above. Therefore it never happens that $u\to+\infty$ as θ increases from 0 to π.

Differentiating both sides of (1. 1) with respect to θ, we get

$$\frac{d^2u}{d\theta^2}-\frac{2a\cos\theta}{\sin^3\theta}u^2+2\left(\frac{a}{\sin^2\theta}+b\right)u\frac{du}{d\theta}=0.$$

Put $du/d\theta=0$ in this and get

$$\frac{d^2u}{d\theta^2}=\frac{2a\cos\theta}{\sin^3\theta}u^2\begin{cases}\geqq 0 & \text{for } 0<\theta\leqq\pi/2,\\ \leqq 0 & \text{for } \pi/2\leqq\theta<\pi.\end{cases}$$

Therefore

3) u does not attain its maximum between $0<\theta<\pi/2$, and does not attain its minimum between $\pi/2<\theta<\pi$.

Our further investigation is essentially based on those properties stated above.

§ 5. Definition of the domains S, S_1, S_2.

As was shown in § 3, $\lim_{\theta\to0}u=0$ and $\lim_{\theta\to\pi}u=0$ for all the solutions of (1. 1) and only one of the integral curves is tangent at $\theta=0$ to the straight line $u=(c/\lambda_1)\theta$. This solution will be denoted by C_1: $u=f_1(\theta)$. Also the one particular solution which is tangent at $\theta=\pi$ to the line $u=(c/\lambda_1)(\theta-\pi)$ is denoted by C_2: $u=f_2(\theta)$.

PROPOSITION 1. *As θ increases from 0 to π, $f_1(\theta)$ increases at first, reaches its maxtmum at $\theta=\theta_0>\pi/2$, and decreases monotonically thereafter to have the limiting values*

$$\lim_{\theta\to\pi}f_1(\theta)=0 \qquad \textit{and} \qquad \lim_{\theta\to\pi}f_1'(\theta)=\frac{c}{1-\lambda_1}.$$

As θ decreases from π to 0, $f_2(\theta)$ decreases at first, reaches its minimum at $\theta=\pi-\theta_0<\pi/2$ and then increases to have the limiting values

$$\lim_{\theta\to0}f_2(\theta)=0 \qquad \textit{and} \qquad \lim_{\theta\to0}f_2'(\theta)=\frac{c}{1-\lambda_1}.$$

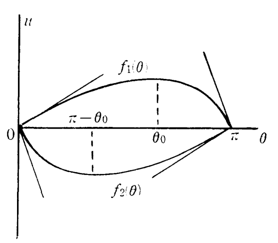

Fig. 2. The curves C_1 and C_2

Proof. As $\lim_{\theta\to0}f_1'(\theta)=c/\lambda_1>0$, $f_1(\theta)$ increases in the vicinity of $\theta=0$.

As θ increases from 0 to π, $f_1(\theta)$ is bounded above as was shown in 2) of § 4 and the curve C_1 cannot cross the θ-axis with a negative inclination according to 1) of § 4. Therefore $0<f_1(\theta)<\infty$ for $0<\theta<\pi$. In other words, the curve C_1 lies in some bounded area of the upper half plane. In addition, $\lim_{\theta\to0}f_1(\theta)=\lim_{\theta\to\pi}f_1(\theta)=0$. Hence $f_1(\theta)$ attains its maxima somewhere between $0<\theta<\pi$. Let the smallest of the θ's that make $f_1(\theta)$ maximal be θ_0. Then $\theta_0>\pi/2$ by 3) of § 4. Then, owing also to 3) of § 4, $f_1(\theta)$ has no minima for $\theta>\theta_0>\pi/2$. Therefore $f_1(\theta)$ is monotonically decreasing for $\theta>\theta_0$ and $\lim_{\theta\to\pi}f_1(\theta)=0$. Moreover $f_1(\theta)$ being positive for $0<\theta<\pi$, $\lim_{\theta\to\pi}f_1'(\theta)$ must be negative. As the value of $du/d\theta$ at $\theta=\pi$ is either $c/\lambda_1>0$ or $c/(1-\lambda_1)<0$, it is proved that

$$\lim_{\theta\to\pi} f_1'(\theta)=\frac{c}{1-\lambda_1}.$$

This proves the truth of our statement concerning the behavior of $f_1(\theta)$.

According to the remark stated at the end of § 2, the curve

$$u=-f_1(\pi-\theta)$$

obtained by rotating the curve C_1 by an angle π about a point $(\pi/2, 0)$ is also one of the integral curves of (1. 1) and is tangent at $\theta=\pi$ to the line $u=(c/\lambda_1)(\theta-\pi)$. However, the only integral curve which is tangent at $\theta=\pi$ to the line $u=(c/\lambda_1)(\theta-\pi)$ being C_2, the statement about the behavior of C_2 described in Proposition 1 can be immediately derived from above.

We divide the strip

$$0\leqq\theta\leqq\pi,\quad -\infty<u<\infty$$

into three domains—namely the domain enclosed by C_1 and C_2, the domain above C_1 and the domain below C_2. These domains will be named S, S_1 and S_2 respectively.

§ 6. Behavior of the integral curves in *S*.

First the behavior of the integral curves starting from the point $\theta=0$, $u=0$ into the area S will be investigated. S being surrounded by two integral curves C_1 and C_2, these curves cannot go out of S and they stay in S until they reach the point $\theta=\pi$, $u=0$. Among these solutions, C_1 is the only curve which is tangent to the straight line $u=(c/\lambda_1)\theta$, and the others are all tangent to the line $u=(c/(1-\lambda_1))\theta$. In other words, if one of the latter is denoted by $u=f(\theta)$, it naturally follows that $\lim_{\theta\to 0}f'(\theta)=c/(1-\lambda_1)<0$. So $f(\theta)$ decreases and $f(\theta)<0$ in the vicinity of $\theta=0$.

On the other hand, as $\theta\to\pi$, C_2 is the only integral curve such that $\lim_{\theta\to\pi}du/d\theta =c/\lambda_1>0$. Therefore for all the other curves $u=f(\theta)$, $\lim_{\theta\to\pi}f'(\theta)=c/(1-\lambda_1)<0$ and thus we are lead to the conclusion that $f(\theta)>0$ in the vicinity of $\theta=\pi$. Therefore as θ increases from 0 to π, $f(\theta)$ decreases at first and then attains a minimum at $\theta=\theta_1<\pi$. From 3) of § 4 follows that $\theta_1<\pi/2$. As $f(\theta)>0$ in the vicinity of $\theta=\pi$, $f(\theta)$ must attain its maximum at some point $\theta=\theta_2>\theta_1$. According to 3) of § 4, $\theta_2>\pi/2$ and $f(\theta)$ cannot have a minimum for $\theta>\theta_2>\pi/2$. So $f(\theta)$ decreases monotonically for $\theta>\theta_2$ and tends to the point $(\pi, 0)$. Thus we have reached the following

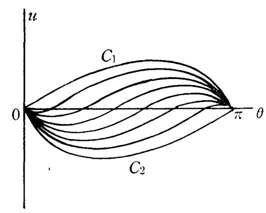

Fig. 3. The integral curves in S

PROPOSITION 2. *Any one of the integral curves starting from the point* (0, 0) *into the domain* S (C_1 *and* C_2 *excluded*) *is tangent at* (0, 0) *to the curve* C_2, *decreases at first, attains its minimum at* $\theta=\theta_1<\pi/2$, *increases thereafter, attains its maximum at* $\theta=\theta_2>\pi/2$ *and decreases to reach* $(\pi, 0)$ *where it is tangent to* C_1. *Therefore the domain* S *is covered with integral curves whose shapes are shown in Fig.* 3.

§ 7. Behavior of the integral curves in S_1 and S_2.

Next the integral curves starting from (0, 0) into the domain S_2 will be investigated. As they must remain in S_2, they cannot remain bounded until they reach the line $\theta=\pi$. Because, if so, they must tend to the point $(\pi, 0)$ as $\theta\to\pi$ where they must naturally be tangent to C_2. However C_2 is the only integral curve with this property, such a situation can never arise. Therefore such solutions must tend to $-\infty$ as $\theta\to\theta_3-0$ for some $\theta_3<\pi$.

To obtain the integral curves in S_1, we have only to rotate the integral curves in S_2 by an angle π about a point $(\pi/2, 0)$ owing to the remark at the end of § 2.

§ 8. Determination of the desired solution.

The above investigation clearly indicates the behavior of the totality of the integral curves of (1. 1) for $0\leqq\theta\leqq\pi$. Then, according to the result of § 2, the integral curves of (1. 2) for $\pi\leqq\theta\leqq 2\pi$ can be constructed by translating these curves by π units in the θ-direction and changing their sign.

Thus all the integral curves of the given equation for $0\leqq\theta\leqq 2\pi$ are obtained. In order to get all the solutions for $0\leqq\theta<\infty$, it is only necessary to translate them by 2π in the θ-direction repeatedly. Integral curves thus obtained are shown in Fig. 4.

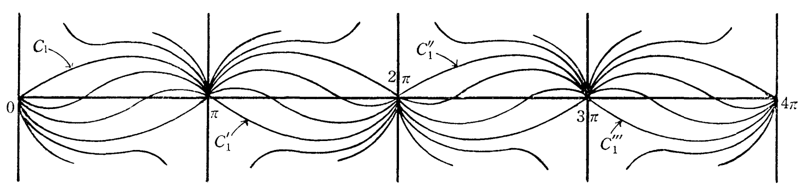

Fig. 4. Totality of the integral curves

In connecting a curve between $(n-1)\pi\leqq\theta\leqq n\pi$ $(n=1, 2, 3, \cdots)$ with a curve between $n\pi\leqq\theta\leqq(n+1)\pi$, there always occurs discontinuity of respective derivatives at $\theta=n\pi$, whatever integral curves are selected. From a physical point of view, it is reasonable to suppose that actual connection will take place so that the jump of the derivatives at $\theta=n\pi$ is minimized. It may be said that a curve C obtained by connecting $C_1, C_1', C_1'', \cdots$ is physically stable. Here C_1' is constructed by translating C_1 in the θ-direction by π and changing its sign, C_1'' is constructed by translation of C_1 in the θ-direction by 2π, C_1''' is constructed by translation of C_1' in the θ-direction by 2π and so on. Whatever solution curve is chosen at $\theta=0$ (excluding the unbounded ones), this solution will finally be connected to C as can be easily seen from Fig. 4.

Therefore, if $u=\varphi(\theta)$ is the equation of the curve C_1, then the desired solution will be given by

$$u=\varphi(\theta) \quad \text{for} \quad 0\leqq\theta\leqq\pi,$$

$$u=-\varphi(\theta-\pi) \quad \text{for} \quad \pi\leqq\theta\leqq 2\pi,$$

$$u=\varphi(\theta-2\pi) \quad \text{for} \quad 2\pi\leqq\theta\leqq 3\pi,$$

$$u=-\varphi(\theta-3\pi) \quad \text{for} \quad 3\pi\leqq\theta\leqq 4\pi,$$

....

§9. Analytical expression of $\varphi(\theta)$.

Finally the explicit analytical expression of the solution will be given.
By putting $\sin^2(\theta/2)=z$, (3. 1) is transformed into

$$z(1-z)\frac{d^2w}{dz^2}+\frac{1}{2}(1-2z)\frac{dw}{dz}-c\left[\frac{a}{4z(1-z)}+b\right]w=0. \tag{9. 1}$$

Again by putting $w=z^{\lambda_1/2}(1-z)^{(1-\lambda_1)/2}\cdot W$, (9. 1) is reduced to

$$z(1-z)\frac{d^2W}{dz^2}+\left(\lambda_1+\frac{1}{2}-2z\right)\frac{dW}{dz}-\frac{1+4bc}{4}W=0. \tag{9. 2}$$

This is a well-known Gauss' hypergeometric differential equation.
Since $u=\varphi(\theta)$ which represents a curve C_1 is of the form

$$u=\frac{c\theta}{\lambda_1}+\cdots,$$

corresponding w can be expressed in a form

$$w=\text{const}\times\theta^{\lambda_1}(1+\cdots)=\text{const}\times z^{\lambda_1/2}(1+\cdots)$$

in the vicinity of $\theta=0$. Thus, in turn, corresponding W should be a solution of (9. 2) such that

$$W=1+\cdots$$

in the vicinity of $z=0$ where the terms not explicitly written are power series of z. Such a solution of (9. 2) is obviously given by a hypergeometric function

$$W=F(\alpha, \beta, \gamma;\ z),$$

$$\alpha=\frac{1}{2}+i\sqrt{bc}\,,\quad \beta=\frac{1}{2}-i\sqrt{bc}\,,\quad \gamma=\frac{2+\sqrt{1+4ac}}{2}.$$

Thus we immediately have

$$u=\varphi(\theta)=c\sin\theta\left[\frac{1+\sqrt{1+4ac}}{2}-\sin^2\frac{\theta}{2}\right.$$

$$+2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\frac{F'(\alpha, \beta, \gamma; \sin^2(\theta/2))}{F(\alpha, \beta, \gamma; \sin^2(\theta/2))}\Bigg]^{-1}.$$

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