# ON ALMOST CONTACT MANIFOLDS ADMITTING AXIOM OF PLANES OR AXIOM OF FREE MOBILITY

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A Riemannian manifold is said to admit the axiom of planes if there exists a 2-dimensional totally geodesic submanifold tangent to any 2-dimensional section at every point of the manifold, and is said to admit the free mobility if there exists an isometry which carries any point and any frame attached to the point to any other point and any other frame attached to the point [1].

It is well known that a Riemannian manifold admits the axiom of planes or the free mobility if and only if it is of constant curvature [1]. Yano and Mogi [8] proved a similar result in a complex manifold. If the holomorphic sectional curvature at every point of a Kähler manifold does not depend on the holomorphic section at the point, then it is constant on the manifold and the curvature tensor has the form

$$K_{kji}{}^{h}=\frac{k}{4}[(\delta_k^h g_{ji}-\delta_j^h g_{ki})+(F_k{}^h F_{ji}-F_j{}^h F_{ki})-2F_{kj}F_i{}^h],$$

where $F$ denotes the complex structure. They proved that such a Kähler manifold admits the so-called axiom of holomorphic planes and the holomorphic free mobility, and conversely.

Recently, Sasaki [4] defined a structure called $(\phi, \xi, \eta, g)$-structure on an odd dimensional manifold. The structure is analogous to an almost Hermitian structure on an even dimensional manifold.

In this paper, we consider, in a Sasakian manifold, the problems corresponding to those of Yano and Mogi.

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## §1. Identities.

A $(\phi, \xi, \eta, g)$-structure is defined on a $(2n+1)$-dimensional differentiable manifold $M$ by a (1, 1)-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying the following conditions [4]:

(1. 1) $$\xi^i\eta_i=1,$$

(1. 2) $$\text{rank }(\phi_j{}^i)=2n,$$

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(1. 3) $$\phi_j{}^i\xi^j=0,$$

(1. 4) $$\phi_j{}^i\eta_i=0,$$

(1. 5) $$\phi_j{}^i\phi_k{}^j=-\delta_k^i+\xi^i\eta_k,$$

(1. 6) $$g_{ji}\xi^j=\eta_i,$$

(1. 7) $$g_{ji}\phi_k{}^j\phi_h{}^i=g_{kh}-\eta_k\eta_h.$$

If this is the case, $\phi_{ji}=g_{ia}\phi_j{}^a$ is skew-symmetric and the 2-form $\Phi=\phi_{ji}dx^j\wedge dx^i$ is called the fundamental 2-form. When the above structure satisfies an additional condition

(1. 8) $$\Phi=d\eta$$

it is called an almost Sasakian structure.

Moreover, we define a (1, 2)-tensor field $N$ by

(1. 9) $$N_{ji}{}^h=\phi_j{}^a(\partial_a\phi_i{}^h-\partial_i\phi_a{}^h)-\phi_i{}^a(\partial_a\phi_j{}^h-\partial_j\phi_a{}^h)+\eta_j\partial_i\xi^h-\eta_i\partial_j\xi^h.$$

A Sasakian structure is a $(\phi, \xi, \eta, g)$-structure with additional conditions

(1. 10) $$\begin{cases}\Phi=d\eta,\\ N=0,\end{cases}$$

which are equivalent to

(1. 11) $$\phi_j{}^i=\nabla_j\xi^i \quad \text{or} \quad \phi_{ji}=\nabla_j\eta_i$$

and

(1. 12) $$\nabla_k\phi_j{}^i=\eta_j\delta_k^i-\xi^i g_{kj} \quad \text{or} \quad \nabla_k\phi_{ji}=\eta_j g_{ki}-\eta_i g_{kj},$$

where $\nabla$ denotes the covariant differentiation with respect to the Riemannian connection determined by $g$. In this case, vector field $\xi$ is a unit Killing vector field.

From (1. 11) and (1. 12), we have

$$\nabla_k\nabla_j\xi^i=\eta_j\delta_k^i-\xi^i g_{kj} \quad \text{or} \quad \nabla_k\nabla_j\eta_i=\eta_j g_{ki}-\eta_i g_{kj},$$

using the Ricci formula,

(1. 13) $$K_{kji}{}^a\eta_a=\eta_k g_{ji}-\eta_j g_{ki}$$

or

(1. 13)′ $$K_{kjia}\xi^a=\eta_k g_{ji}-\eta_j g_{ki},$$

where $K$ denotes the curvature tensor of the given connection. The equation (1. 13) is equivalent to

(1. 14) $$K_{aji}{}^{h}\xi^{a}=\xi^{h}g_{ji}-\eta_{i}\delta_{j}^{h}$$

or

(1. 15) $$K_{kja}{}^{h}\xi^{a}=\eta_{j}\delta_{k}^{h}-\eta_{k}\delta_{j}^{h}.$$

From (1. 12) and the Ricci formula, we get

(1. 16) $$\phi_{k}{}^{b}\phi_{j}{}^{a}K_{bai}{}^{h}=K_{kji}{}^{h}+\phi_{k}{}^{h}\phi_{ji}-\phi_{j}{}^{h}\phi_{ki}-\delta_{k}^{h}g_{ji}+\delta_{j}^{h}g_{ki}$$

or

(1. 16)′ $$\phi_{k}{}^{b}\phi_{j}{}^{a}K_{baih}=K_{kjih}+\phi_{kh}\phi_{ji}-\phi_{jh}\phi_{ki}-g_{kh}g_{ji}+g_{jh}g_{ki}.$$

Moreover we have

(1. 17) $$\phi_{i}{}^{d}\phi_{k}{}^{b}K_{dhbj}=\phi_{k}{}^{d}\phi_{i}{}^{b}K_{ajbh}.$$

From (1. 16)′, we have

$$\phi_{m}{}^{b}\phi_{j}{}^{a}K_{balh}=K_{mjlh}+\phi_{mh}\phi_{jl}-\phi_{jh}\phi_{ml}-g_{mh}g_{jl}+g_{jh}g_{ml}$$

from which, transvecting with $\phi_{i}{}^{m}\phi_{k}{}^{l}$, we get

(1. 18) $$\phi_{k}{}^{d}\phi_{h}{}^{b}K_{ajbi}-\phi_{k}{}^{d}\phi_{i}{}^{b}K_{djbh}=g_{jh}g_{ki}-g_{kh}g_{ji}+\eta_{k}\eta_{h}g_{ji}-\eta_{k}\eta_{i}g_{jh}-\phi_{jh}\phi_{ki}+\phi_{kh}\phi_{ji}.$$

From

$$\phi_{k}{}^{d}\phi_{j}{}^{b}K_{aib}{}^{h}=-\phi_{k}{}^{d}\phi_{j}{}^{b}(K_{ibd}{}^{h}+K_{bdi}{}^{h})=\phi_{j}{}^{b}\phi_{k}{}^{d}K_{bid}{}^{h}-\phi_{j}{}^{b}\phi_{k}{}^{d}K_{bdi}{}^{h}$$

and (1. 16), we get

(1. 19) $$\phi_{k}{}^{d}\phi_{j}{}^{b}K_{dib}{}^{h}-\phi_{j}{}^{d}\phi_{k}{}^{b}K_{dib}{}^{h}=K_{kji}{}^{h}+\phi_{k}{}^{h}\phi_{ji}-\phi_{j}{}^{h}\phi_{ki}-\delta_{k}^{h}g_{ji}+\delta_{j}^{h}g_{ki}$$

or

(1. 19)′ $$\phi_{k}{}^{d}\phi_{j}{}^{b}K_{dibh}-\phi_{j}{}^{d}\phi_{k}{}^{b}K_{dibh}=K_{kjih}+\phi_{kh}\phi_{ji}-\phi_{jh}\phi_{ki}-g_{kh}g_{ji}+g_{jh}g_{ki}.$$

## § 2. *C*-holomorphic sectional curvature.

Let $p$ be any point of $M$, and $V_{p}$ be the subspace of $T_{p}(M)$, the tangent space of $M$ at $p$, whose elements are orthogonal to $\xi$, i.e.

$$V_{p}=\{v\in T_{p}(M)|g(\xi, v)=0\}.$$

Let $u$ be any unit vector of $V_{p}$. By $\xi$-section, we mean the section determined by $\xi$ and $u$.

The sectional curvature determined by $\xi$-section is said to be $\xi$-sectional curvature. Denoting it by $k(\xi, u)$, we get

$$k(\xi, u)=-K_{kjih}u^{k}\xi^{j}u^{i}\xi^{h},$$

or, using (1. 13)′,

$$k(\xi, u) = -(\eta_k g_{ji} - \eta_j g_{ki}) u^k \xi^j u^i.$$

By assumption,

$$g_{ji}\xi^j u^i = 0 \quad \text{and} \quad g_{ki}u^k u^i = 1,$$

thus we have

$$k(\xi, u) = 1,$$

hence we obtain [2]

THEOREM 1. *In Sasakian manifold, $\xi$-sectional curvature is always equal to* 1.

Next we consider the sectional curvature determined by two orthogonal vectors in $V_p$.

If $v$ be any unit vector, not necessarily in $V_p$, then $\phi v$ lies in $V_p$, and any element of $V_p$ can be written in this form.

So we consider the section determined by $\phi v$ and $\phi^2 v$, which are orthogonal to each other. We call this a $C$-holomorphic section, and the sectional curvature determined by such a section is said to be the $C$-holomorphic sectional curvature:

$$k = -\frac{K_{dcba}(\phi v)^d(\phi^2 v)^c(\phi v)^b(\phi^2 v)^a}{g_{dc}(\phi v)^d(\phi v)^c g_{ba}(\phi^2 v)^b(\phi^2 v)^a}, \tag{2. 1}$$

from which, using (1. 13)′, we have

$$k = -\frac{(\phi_k{}^d\phi_i{}^b K_{djbh} + \eta_j\eta_h(g_{ki} - \eta_k\eta_i))v^k v^j v^i v^h}{(g_{kj} - \eta_k\eta_j)(g_{ih} - \eta_i\eta_h)v^k v^j v^i v^h}. \tag{2. 2}$$

Now if we assume $k$ is independent of the choice of $C$-holomorphic section at $p \in M$, then (2. 2) or

$$[\phi_k{}^d\phi_i{}^b K_{djbh} + \eta_j\eta_h(g_{ki} - \eta_k\eta_i) + k(g_{kj} - \eta_k\eta_j)(g_{ih} - \eta_i\eta_h)]v^k v^j v^i v^h = 0 \tag{2. 3}$$

should be satisfied for any $v$, from which we get

$$\begin{aligned}
&2(\phi_k{}^d\phi_i{}^b K_{djbh} + \phi_k{}^d\phi_h{}^b K_{dibj} + \phi_k{}^d\phi_j{}^b K_{dhbi}) + 2(\phi_{hj}\phi_{ki} + \phi_{kj}\phi_{ih} + \phi_{hk}\phi_{ji}) \\
&+ (\eta_j\eta_i g_{kh} + \eta_j\eta_h g_{ki} + \eta_i\eta_h g_{kj} - \eta_k\eta_j g_{ih} - \eta_k\eta_i g_{jh} - \eta_k\eta_h g_{ji}) \\
&+ (1-2k)(\eta_k\eta_j g_{ih} + \eta_k\eta_i g_{jh} + \eta_k\eta_h g_{ji} + \eta_j\eta_i g_{kh} + \eta_j\eta_h g_{ki} + \eta_i\eta_h g_{kj}) \\
&+ 2k(g_{kj}g_{ih} + g_{ki}g_{jh} + g_{kh}g_{ji}) + 6(k-1)\eta_k\eta_j\eta_i\eta_h = 0
\end{aligned}$$

by virtue of (1. 17) and (1. 18).

Moreover using (1. 18) and (1. 19)′, we have

$$3\phi_m{}^d\phi_l{}^b K_{djbh} - K_{jlmh} + K_{hjml} + 3(g_{jh}g_{ml} - g_{mj}g_{lh}) + k(g_{mj}g_{lh} + g_{ml}g_{jh} + g_{mh}g_{lj})$$

$$+(1-k)(\eta_m\eta_j g_{lh}+\eta_m\eta_h g_{jl}+\eta_j\eta_l g_{mh}+\eta_l\eta_h g_{mj})-(2+k)\eta_m\eta_l g_{jh}$$

$$+3\phi_{mh}\phi_{jl}+3(k-1)\eta_m\eta_j\eta_l\eta_h=0.$$

Transvecting this equation with $\phi_k{}^m\phi_i{}^l$ and using (1. 13)′ and (1. 16)′, we get

$$\phi_k{}^a\phi_i{}^b K_{ajbh}=3K_{khji}-K_{jhki}+g_{kh}g_{ji}+2g_{kj}g_{ih}-(k+3)g_{ki}g_{hj}+k\eta_k\eta_i g_{jh}$$
$$+(k-1)\eta_j\eta_h g_{ki}-(k-2)\phi_{kh}\phi_{ij}+(k-1)\phi_{kj}\phi_{hi}-(k-1)\eta_k\eta_j\eta_i\eta_h.$$

Taking the skew-symmetric part of this equation with respect to $i$ and $h$, and using (1. 18), we obtain

(2. 4)
$$4K_{kjih}=(k+3)(g_{kh}g_{ji}-g_{ki}g_{jh})$$
$$+(k-1)(\eta_k\eta_i g_{jh}+\eta_j\eta_h g_{ki}-\eta_k\eta_h g_{ji}-\eta_j\eta_i g_{kh}+\phi_{ki}\phi_{hj}-\phi_{kh}\phi_{ij}+2\phi_{kj}\phi_{hi}).$$

Transvecting this with $g^{kh}$, we have

(2. 5)
$$2K_{ji}=[n(k+3)+k-1]g_{ji}-(n+1)(k-1)\eta_j\eta_i.$$

Sasakian manifold with the Ricci tensor of the above form is called $\eta$-Einstein manifold [3]. Moreover transvecting this with $g_{ji}$, we have

(2. 6)
$$2K=n(2n+1)(k+3)+n(k-1).$$

On the other hand, from the Bianchi identity, we get

$$2\nabla_a K_j{}^a-\nabla_j K=0.$$

Substituting (2. 5) and (2. 6) into the last equation, we get

(2. 7)
$$(n-1)\nabla_j k+\eta_j\xi^a\nabla_a k=0.$$

Transvecting this with $\xi^j$, we have

$$\xi^a\nabla_a k=0$$

and hence we have from (2. 7)

$$\nabla_j k=0 \qquad (n\neq 1).$$

Thus, the $C$-holomorphic sectional curvature $k$ is constant on the manifold. Hence we have proved

THEOREM 2. *If, in a Sasakian manifold, $C$-holomorphic sectional curvature is independent of $C$-holomorphic section at a point, then the curvature tensor has form* (2. 4), *where $k$ is a constant if $n\neq 1$.*

A Sasakian manifold is said to be of constant $C$-holomorphic curvature, if the $C$-holomorphic sectional curvature is constant on the manifold.

## § 3. Axiom of $C$-holomorphic planes.

Let $p$ be any point of $M$. When a section is given at $p$, we consider a 2-dimensional totally geodesic submanifold passing through this point and being tangent to the given section.

If we represent such a submanifold by parametric equations

$$x^h = x^h(y^\lambda) \qquad (\lambda, \mu, \nu = 1, 2), \tag{3. 1}$$

then the fact that the submanifold is totally geodesic is represented by the equations

$$\frac{\partial^2 x^h}{\partial y^\nu \partial y^\mu} + \frac{\partial x^j \partial x^i}{\partial y^\nu \partial y^\mu}\left\{ {h \atop j\ i} \right\} - \frac{\partial x^h}{\partial y^\lambda}\left\{ {\lambda \atop \nu\ \mu} \right\}' = 0, \tag{3. 2}$$

where $\{{}_\nu{}^\lambda{}_\mu\}'$ are Christoffel symbols formed with the naturally induced Riemannian metric

$$g_{ji} \frac{\partial x^j}{\partial y^\nu} \frac{\partial x^i}{\partial y^\mu}$$

of the submanifold.

The integrability conditions of the differential equations (3. 2) are

$$B_\nu{}^k B_\mu{}^j B_\lambda{}^i K_{kji}{}^h = B_\kappa{}^h K'_{\nu\mu\lambda}{}^\kappa \tag{3. 3}$$

where

$$B_\lambda{}^i = \frac{\partial x^i}{\partial y^\lambda},$$

and $K'_{\nu\mu\lambda}{}^\kappa$ is the curvature tensor of the submanifold.

(3. 3) means that $B_\nu{}^k B_\mu{}^j B_\lambda{}^i K_{kji}{}^h$ must be a linear combination of $B_1$ and $B_2$. We first consider a $\xi$-section.

If we put $B_1 = \xi$, $B_2 = u$, from (1. 14) we have

$$\xi^k u^j \xi^i K_{kji}{}^h = -u^h, \qquad \xi^k u^j u^i K_{kji}{}^h = \xi^h,$$

hence conditions (3. 3) are satisfied, that is, there always exists a 2-dimensional totally geodesic submanifold tangent to $\xi$-section.

Next, we assume that the manifold admits the axiom of $C$-holomorphic planes; that is, for any $C$-holomorphic section at $p$, there always exists a 2-dimensional totally geodesic submanifold tangent to it.

If we put $B_1 = \phi v$, $B_2 = \phi^2 v$, then from (3. 3) we have

$$\begin{cases} (\phi_s{}^n v^s)(\phi_q{}^m \phi_j{}^q v^j)(\phi_p{}^l \phi_i{}^p v^i) K_{nml}{}^h = \alpha \phi_j{}^h v^j + \beta \phi_r{}^h \phi_i{}^r v^i \\ (\phi_s{}^n v^s)(\phi_q{}^m \phi_j{}^q v^j)(\phi_i{}^l v^i) K_{nml}{}^h = \gamma \phi_i{}^h v^i + \delta \phi_r{}^h \phi_i{}^r v^i. \end{cases} \tag{3. 4}$$

From the first equation of (3. 4), we have

$$(\phi_s{}^n K_{nji}{}^h + \phi_{si} \xi^h \eta_j - \phi_s{}^h \eta_j \eta_i) v^s v^j v^i = (\alpha g_{sj} \phi_i{}^h - \beta g_{sj} \delta_i^h + \beta g_{sj} \xi^h \eta_i) v^s v^j v^i$$

or

$$(\phi_s{}^n K_{njih}+\phi_{si}\eta_h\eta_j-\phi_{sh}\eta_j\eta_i)v^s v^j v^i=(\alpha g_{sj}\phi_{ih}-\beta g_{sj}g_{ih}+\beta g_{sj}\eta_h\eta_i)v^s v^j v^i.$$

Transvecting the last equation with $v^h$ and using the skew-symmetry of $K_{njih}$ and $\phi_{si}$, we get $\beta=0$.

If we put

$$k=\frac{1}{||\phi v||}\alpha=\frac{g_{sj}v^s v^j}{(g_{sj}-\eta_s\eta_j)v^s v^j}\alpha,$$

then we have

$$(\phi_s{}^n K_{nji}{}^h+\phi_{si}\xi^h\eta_j-\phi_s{}^h\eta_j\eta_i)v^s v^j v^i=k(g_{sj}-\eta_s\eta_j)\phi_i{}^h v^s v^j v^i,$$

from which we get

$$\phi_s{}^n K_{nji}{}^h+\phi_s{}^n K_{nij}{}^h+\phi_j{}^n K_{nis}{}^h+\phi_j{}^n K_{nsi}{}^h+\phi_i{}^n K_{nsj}{}^h+\phi_i{}^n K_{njs}{}^h$$
$$-2(\phi_s{}^h\eta_j\eta_i+\phi_j{}^h\eta_i\eta_s+\phi_i{}^h\eta_s\eta_j)$$
$$=2k(g_{sj}\phi_i{}^h+g_{ji}\phi_s{}^h+g_{is}\phi_j{}^h-\eta_s\eta_j\phi_i{}^h-\eta_j\eta_i\phi_s{}^h-\eta_i\eta_s\phi_j{}^h).$$

Transvecting this with $\phi_k{}^s$, we obtain

$$-2K_{kji}{}^h-2K_{kij}{}^h+\phi_j{}^n\phi_k{}^s K_{nis}{}^h+\phi_i{}^n\phi_k{}^s K_{njs}{}^h-\delta_j{}^h g_{ki}-\delta_i{}^h g_{kj}+2\delta_k{}^h g_{ji}$$
$$+2g_{ij}\xi^h\eta_k-\delta_j{}^h\eta_k\eta_i-\delta_i{}^h\eta_k\eta_j+2\delta_k{}^h\eta_j\eta_i-2\xi^h\eta_k\eta_j\eta_i+\phi_j{}^h\phi_{ki}+\phi_i{}^h\phi_{kj}$$
$$=2k(\phi_{kj}\phi_i{}^h-g_{ji}\delta_k{}^h+g_{ji}\xi^h\eta_k+\phi_{ki}\phi_j{}^h+\delta_k{}^h\eta_j\eta_i-\xi^h\eta_k\eta_j\eta_i)$$

from which, taking the skew-symmetric part with respect to $k$ and $j$, and using the relations (1. 19) and

$$\phi_i{}^n\phi_k{}^s K_{njs}{}^h-\phi_i{}^n\phi_j{}^s K_{nks}{}^h$$
$$=-K_{kji}{}^h+2(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})+\delta_j{}^h\eta_i\eta_k-\delta_k{}^h\eta_i\eta_j+2\phi_{kj}\phi_i{}^h$$

we find

$$4K_{kji}{}^h=(k+3)(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})$$
$$+(k-1)(\delta_j{}^h\eta_k\eta_i-\delta_k{}^h\eta_j\eta_i+g_{ki}\xi^h\eta_j-g_{ji}\xi^h\eta_k-\phi_{ki}\phi_j{}^h+\phi_{ji}\phi_k{}^h-2\phi_{kj}\phi_i{}^h).$$

Conversely, if the curvature tensor has the above form, it is easily seen that (3. 3) is satisfied, hence we have proved

THEOREM 3. *If a Sasakian manifold admits the axiom of C-holomorphic planes, then the manifold is of constant C-holomorphic curvature. The converse is also true.*

## §4. *C*-holomorphic free mobility.

If a Sasakian manifold admits a family of isometries which leave $\xi$ (or $\eta$) invariant and carry any $C$-holomorphic section to any other $C$-holomorphic section, then we say that the manifold admits a $C$-holomorphic free mobility.

Now, we assume the manifold admits a $C$-holomorphic free mobility.

At any point $p\in M$, we consider two arbitrary $C$-holomorphic sections, then by assumption there always exists a transformation which carries one to the other. Since $p$ is arbitrary, the manifold must be of constant $C$-holomorphic curvature and consequently the curvature tensor has the form

$$4K_{kji}{}^{h}=(k+3)(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki})$$
$$+(k-1)(\delta_{j}^{h}\eta_{k}\eta_{i}-\delta_{k}^{h}\eta_{j}\eta_{i}+g_{ki}\xi^{h}\eta_{j}-g_{ji}\xi^{h}\eta_{k}-\phi_{ki}\phi_{j}{}^{h}+\phi_{ji}\phi_{k}{}^{h}-2\phi_{kj}\phi_{i}{}^{h}),$$

where $k$ is a constant.

Conversely, we assume the curvature tensor of the manifold has the above form.

If $v$ denotes an infinitesimal transformation, then the fact that this is an infinitesimal isometry is represented by

$$L_{v}g=0 \tag{4.1}$$

and the fact that this leaves $\xi$ invariant and carries any $C$-holomorphic section to any other $C$-holomorphic section is represented by

$$L_{v}\phi=0 \tag{4.2}$$

and

$$L_{v}\eta=0. \tag{4.3}$$

From (4. 1), we get

$$L_{v}\left\{{}_{j}{}^{h}{}_{i}\right\}=\nabla_{j}\nabla_{i}v^{h}+K_{kji}{}^{h}v^{k}=0. \tag{4.4}$$

If this is the case, $L_{v}$ commutes with $\nabla$, hence we have

$$L_{v}\phi_{ji}=L_{v}\nabla_{j}\eta_{i}=\nabla_{j}L_{v}\eta_{i},$$

thus the equation (4. 2) is not essential.

It is well known that the integrability conditions of the differential equation (4. 4) are given by

$$L_{v}K_{kji}{}^{h}=v^{l}\nabla_{l}K_{kji}{}^{h}-K_{kji}{}^{l}\nabla_{l}v^{h}+K_{lji}{}^{h}\nabla_{k}v^{l}+K_{kli}{}^{h}\nabla_{j}v^{l}+K_{kjl}{}^{h}\nabla_{i}v^{l}=0. \tag{4.5}$$

Let $p$ be an arbitrary but fixed point of $M$.

It is easily seen that the condition (4. 5) are always satisfied by any $v$ for which (4. 1), (4. 2) and (4. 3) are satisfied at any point, and that the differential equations (4. 4) have solutions.

Moreover, using the identities

$$\nabla_k L_v g_{ji} = (L_v\{{}_k{}^h{}_j\})g_{hi} + (L_v\{{}_k{}^h{}_i\})g_{jh}$$

and

$$L_v\{{}_j{}^h{}_i\} = \frac{1}{2} g^{ha}(\nabla_j L_v g_{ia} + \nabla_i L_v g_{ja} - \nabla_a L_v g_{ji}),$$

we see that (4. 4) are equivalent to

$$\nabla_k L_v g_{ji} = 0$$

and consequently, (4. 1) is satisfied at every point of $M$ by any solutions of (4. 4) for which the initial conditions are satisfied.

On the other hand, from (1. 12) we get

$$L_v \nabla_j \phi_{ih} = (L_v \eta_i) g_{jh} - (L_v \eta_h) g_{ji}$$

by virtue of $L_v g = 0$, and the left hand side of the last equation is equal to $\nabla_j \nabla_i L_v \eta_h$, hence we have

$$\nabla_j \nabla_i L_v \eta_h = (L_v \eta_i) g_{jh} - (L_v \eta_h) g_{ji} \tag{4. 6}$$

which can be regarded as homogeneous differential equations with respect to $L_v \eta_h$.

The initial conditions for (4. 6) are given by $(L_v \eta_h)_p = 0$ and $(\nabla_i L_v \eta_h)_p = 0$, since $\nabla_i L_v \eta_h = L_v \phi_{ih}$.

Hence the solutions of (4. 6) are $L_v \eta_h = 0$, and consequently (4. 3) is satisfied by any solutions of (4. 6) for which the initial conditions are satisfied.

Thus (4. 3) is satisfied, hencc (4. 2) is also satisfied.

The conditions (4. 1), (4. 2) and (4. 3) are equivalent to

$$\nabla_j v_i + \nabla_i v_j = 0, \tag{4. 7}$$

$$v^l \nabla_l \eta_i + \xi^l \nabla_i v_l = 0 \tag{4. 8}$$

and

$$v^l \nabla_l \phi_{ji} - \phi_i{}^l \nabla_j v_l + \phi_j{}^l \nabla_i v_l = 0 \tag{4. 9}$$

which can be regarded as linear equations with unknown $v^i$ and $\nabla_j v_i$.

We first consider the solutions of above equations satisfying $v_p = 0$, then (4. 8) and (4. 9) show that $\nabla_j v_i$ leave $\xi$ and $\phi$ invariant and that the totality of such solutions $(0, \nabla_j v_i)$ is isomorphic to the Lie algebra of $U(n)$, hence a $C$-holomorphic section at $p$ is transformed to any other $C$-holomorphic section at $p$.

On the other hand, we consider the solutions such that $v_p \neq 0$. There exist $2n+1$ independent solutions $(v^i, \nabla_j v_i)$ which transform $p$ to any point $p'$ of a neighborhood of $p$ and any $C$-holomorphic section at $p$ to $C$-holomorphic section at $p'$.

Thus the manifold admits a $C$-holomorphic free mobility.

Summarizing the above results, we have

THEOREM 4. *The necessary and sufficient condition that a Sasakian manifold admits a C-holomorphic free mobility is that the manifold is of constant C-holomorphic curvature.*

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