A NOTE ON AN ABELIAN COVERING SURFACE, II[1)]

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§1. Preliminaries.

1. We begin with a summary on the general properties of an abelian covering surface which was treated in the previous paper [3]. We should refer the details to [3].

First let R be a closed Riemann surface of genus q. Then there exists a system of $2q$ cycles $\alpha_1, \ldots, \alpha_{2q}$ on R which is called a *canonical homology basis of* R (cf. **3** of [3]).

Next let R be an open Riemann surface. Then there exists a system of cycles $\alpha_1, \alpha_2, \ldots$ on R which is called a *canonical homology basis of R modulo the ideal boundary* $\mathfrak{J}$ of R (cf. **3** of [3]) and further there exists a system of cycles $\beta_{j_1 \cdots j_n}$, $j_n > 1$ being a basis for $\mathfrak{H}_\beta$, which is called a *canonical homology basis of dividing cycles*, where $\mathfrak{H}_\beta$ is the group formed by the homology classes of dividing singular cycles on R (cf. **4** of [3]). A strong homology basis of the open Riemann surface R is formed by the combined system of the cycles α_j ($j=1, 2, \ldots$) and the cycles $\beta_{j_1 \cdots j_n}$, $j_n > 1$ (cf. the lemma 3 of [3]).

Let R be an arbitrary Riemann surface and $\tilde{R}$ be an abelian covering surface of R with its covering transformation group $\mathfrak{G}$. It is one of the most important properties of the abelian covering surface $\tilde{R}$ that a (strong) homology basis of R forms a system of generators of the group $\mathfrak{G}$. Thus α_j ($j=1, 2, \ldots$) or $\beta_{j_1 \cdots j_n}$, $j_n > 1$, can be taken as elements of the group $\mathfrak{G}$ and the whole of them forms a system of generators of $\mathfrak{G}$ (cf. **6** of [3]).[2)]

2. Let $\tilde{R}$ be a Riemann surface admitting a group $\mathfrak{G}$ of one-to-one conformal transformations onto itself which is *free abelian*, finitely generated and properly discontinuous.[3)] Here we assume that no transformation of $\mathfrak{G}$ other than the identity has a fixed point. Let R be a Riemann surface constructed from $\tilde{R}$ by identifying equivalent points modulo $\mathfrak{G}$, denoted by $R \equiv \tilde{R} \pmod{\mathfrak{G}}$. Then, $\tilde{R}$ is an abelian covering surface of R with its covering transformation group $\mathfrak{G}$.

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1) Some results (the theorems 1, 2 and 3, etc.) in the present paper have already been stated by the author in the Shûgakuin Symposium (cf. [2]).

2) $\{\alpha_j\}$ and $\{\beta_{j_1 \cdots j_n}; j_n > 1\}$ may be a finite set and vacuous, respectively, for the case of R of finite genus.

3) For any abelian group $\mathfrak{G}$, let $\mathfrak{T}$ be the torsion group of $\mathfrak{G}$, then the quotient group $\mathfrak{G}/\mathfrak{T}$ is a free abelian group without torsion. Thus, in the present problem there is an essential interest only for the case where $\mathfrak{G}$ is free abelian.

We distinguish several cases by the number of elements of basis of $\mathfrak{G}$ in the following.

I. *The case where* $\mathfrak{G}$ *is generated by a basis consisting of only one element* T.

In the case I., two subcases can be distinguished.

(H) *The case where* $T^{-m}(\tilde{p})$ *and* $T^{m}(\tilde{p})$ $(\tilde{p}\in\tilde{R};\ m=1,2,\ldots)$ *tend to distinct ideal boundary components* γ_1 *and* γ_2 *of* $\tilde{R}$, *respectively. Then* $\tilde{R}$ *will be called the hyperbolic type.*

(P) *The case where both sequences of points* $T^{-m}(\tilde{p})$ *and* $T^{m}(\tilde{p})$ $(\tilde{p}\in\tilde{R};\ m=1,2,\ldots)$ *tend to a common ideal boundary component* γ_0 *of* $\tilde{R}$. *Then* $\tilde{R}$ *will be called the parabolic type.*

II. *The case where* $\mathfrak{G}$ *is generated by a basis consisting of two elements* T_1, T_2.

III. *The case where* $\mathfrak{G}$ *is generated by a basis consisting of three or more elements* $T_1, \ldots, T_N$ $(N\geqq 3)$.

In the cases II. and III. the point sequences $\{T_j^{-m}(\tilde{p})\}_{m=1}^{\infty}$, $\{T_j^{m}(\tilde{p})\}_{m=1}^{\infty}$ $(\tilde{p}\in\tilde{R};\ j=1, 2$ for the case II.; $j=1,\ldots,N$ for the case III.) always tend to a common ideal boundary component γ_0 of $\tilde{R}$ (cf. the lemma 6 of [3]).

In the previous paper [3] we concerned ourselves with only the case I. (H). In the following we shall mainly concern ourselves with the other cases I. (P), II. and III., which was announced in [3].

3. Let $\tilde{R}$ be a Riemann surface admitting the transformation group $\mathfrak{G}$ satisfying the conditions in **2**. In the present paper we shall call a set F consisting of a finite number of closed domains on $\tilde{R}$ a *fundamental region of the covering transformation group* $\mathfrak{G}$ if it satisfies the conditions:

(i) For any point $\tilde{p}\in\tilde{R}$ there exists a point $\tilde{p}^*\in F$ equivalent with $\tilde{p}$ modulo $\mathfrak{G}$, i.e. for any $\tilde{p}\in\tilde{R}$ there exists a transformation $\chi\in\mathfrak{G}$ such that $\tilde{p}^*=\chi(\tilde{p})\in F$;

(ii) Two distinct points $\tilde{p}$, $\tilde{p}^*$ equivalent each other modulo $\mathfrak{G}$ do not simultaneously belong to $(F)^{\circ}$;[4] i.e. $\chi(\tilde{p})\neq\tilde{p}^*$ for any $\tilde{p}$, $\tilde{p}^*\in(F)^{\circ}$ $(\tilde{p}\neq\tilde{p}^*)$ and for any $\chi\in\mathfrak{G}$. Then, $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) is constructed from a fundamental region F of $\mathfrak{G}$ by identifying points of ∂F equivalent modulo $\mathfrak{G}$, where the conformal metric induced from F is taken as one of R. R is uniquely determined by $\tilde{R}$ and $\mathfrak{G}$ (cf. **6** of [3]).

§2. Function-theoritic properties of abelian covering surfaces with finite spherical area.

4. Let R be an arbitrary Riemann surface and f be a meromorphic function on R. We introduce the quantity

$$I(f)=\iint_R \frac{|df/d\zeta|^2}{(1+|f|^2)^2}\,d\xi d\eta,$$

4) The interior of a set E is denoted by $(E)^{\circ}$.

where $\zeta=\xi+i\eta$ is a local uniformizing parameter at a point on R. It expresses the spherical area of the covering surface over the Riemann sphere S which is formed by the image of R under f. We denote by O_{MD} the class of Riemann surfaces R which do not admit any non-constant meromorphic function f with $I(f)<\infty$ (cf. [3]). We say briefly that R *has finite spherical area* if $R\notin O_{MD}$.

By the *valence* $\mathfrak{v}_f$ of f we mean the function on the w-sphere S defined by

$$\mathfrak{v}_f(w)=\sum_{f(p)=w}\mu(p;f), \qquad w\in S,$$

where $\mu(p;f)$ is the multiplicity of f at p. Let $\mathfrak{V}(R)$ be the class of non-constant meromorphic functions of bounded valence on R. We denote by O_V the class of Riemann surfaces R with $\mathfrak{V}(R)=\phi$.

It is known that if $R\in O_G$, O_G being the class of Riemann surfaces not admitting Green's function, two alternative cases can occur; namely

(i) $\mathfrak{v}_f(w)\equiv\text{const}<\infty$ except for a set of w of capacity zero, and

(ii) $\mathfrak{v}_f(w)\equiv\infty$ except for a set of w of capacity zero.

Thus we can immediately see that, if $R\in O_G$, either R belongs to O_V and O_{MD} simultaneously or not.

In the present chapter we shall state *function-theoritic properties* of abelian covering surfaces of the types I. (P) and II. of **2**, which have finite spherical area and belong to the class O_G.

5. In the present section we assume that $\tilde{R}$ is a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G}=\{T\}$ of the type I. (P) of **2**. Then we have the following theorem similar to the theorem 2 of [3].

THEOREM 1. *Let $\tilde{R}$ be a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G}=\{T\}$ of the type* I. (P) *of* **2**. *If $\tilde{R}$ has finite spherical area there exists a function $f_0\in\mathfrak{V}(\tilde{R})$ uniquely determined except additive constants which satisfies the conditions*

$$f_0\circ T(\tilde{p})=f_0(\tilde{p})+1 \qquad \textit{for any } \tilde{p}\in\tilde{R}$$

and

$$f=g\circ f_0 \qquad \textit{for each } f\in\mathfrak{V}(\tilde{R}),$$

where g is a rational function.

The proof of the present theorem may be performed by the method similar to the theorem 2 of [3]. We omit it.

The function f_0 has minimal local degree d_0 at the ideal boundary component γ_0 of $\tilde{R}$ which is a common limit of both sequences of points $\{T^{-m}(\tilde{p})\}_{m=1}^{\infty}$ and $\{T^m(\tilde{p})\}_{m=1}^{\infty}$ ($\tilde{p}\in\tilde{R}$) (cf. **2**). Then, $\max_w\mathfrak{v}_{f_0}(w)=d_0$ and thus f_0 takes all values on S for d_0-times except for a set of w of capacity zero. Thus we can find a real number l such that, except for $w=\infty$, any point on $\Re w=l$ is neither an exceptional point of f_0 nor the image of a multiple point of f_0. Then the curves $\tilde{C}$ on $\tilde{R}$ defined by $\Re f_0=l$ consist of d_0 simple analytic curves $\tilde{C}_1, \ldots, \tilde{C}_{d_0}$ both ends of each of which tend to the ideal boundary component γ_0 and each of which is univalently mapped on $\Re w=l$ by f_0. By the theorem 1 the closed set F_0, which is not necessarily connected, of $\tilde{R}$ defined by $\{\tilde{p}|l\leqq\Re f_0(\tilde{p})\leqq l+1\}$ gives a fundamental region of the group $\mathfrak{G}$. Then $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) is constructed from F_0 by identifying the equivalent points of $\tilde{C}_j$ with $T(\tilde{C}_j)$ for each $j=1, \ldots, d_0$. Thus $\tilde{R}$ must be conformally equivalent to a covering surface $\tilde{R}^*$ on S which is d_0-sheeted except for a set of $w\in S$ of capacity zero and which is mapped onto itself by the transformation $w|w+1$.

It is immediately seen from the theorem 1 that the differential $f_0'(\tilde{p})d\zeta$ is invariant under the group $\mathfrak{G}$, where $\zeta(\tilde{p})$ is a locally uniformizing parameter at $\tilde{p}\in\tilde{R}$. Hence, we may regard it as an abelian differential of the first kind on the Riemann surface $R\equiv\tilde{R}$ (mod $\mathfrak{G}$).

6. In the present section we assume that $\tilde{R}$ is a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type II. of **2.** Then we have the following theorem.

THEOREM 2. *Let $\tilde{R}$ be a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type* II. *of* **2.** *If $\tilde{R}$ has finite spherical area there exists a function $f_0\in\mathfrak{B}(\tilde{R})$ uniquely determined except additive constants which satisfies the conditions*

$$f_0\circ T_1(\tilde{p})=f_0(\tilde{p})+1,\quad f_0\circ T_2(\tilde{p})=f_0(\tilde{p})+\lambda \quad \textit{for any } \tilde{p}\in\tilde{R},$$

and

$$f=g\circ f_0 \qquad \textit{for each } f\in\mathfrak{B}(\tilde{R}),$$

where λ is a non-real constant uniquely determined by $\tilde{R}$ and $\mathfrak{G}$, and g is a rational function.

Proof. Let $\mathfrak{G}_j$ $(j=1, 2)$ be the covering transformation groups of $\tilde{R}$ generated by the elements T_j of the basis of the group $\mathfrak{G}$, respectively. By the lemma 6 of [3], the groups $\mathfrak{G}_j$ $(j=1, 2)$ are of the type I. (P) of **2.** Then, by the theorem 1, there exist the functions f_0 and f_0^* of $\mathfrak{B}(\tilde{R})$ uniquely determined except additive constants which satisfy the conditions

$$f_0\circ T_1(\tilde{p})=f_0(\tilde{p})+1, \tag{1}$$

$$f_0^*\circ T_2(\tilde{p})=f_0^*(\tilde{p})+1 \qquad \text{for any } \tilde{p}\in\tilde{R} \tag{2}$$

and

$$f=g\circ f_0,\qquad f=g^*\circ f_0^* \qquad \text{for each } f\in\mathfrak{B}(\tilde{R}),$$

where g and g^* are rational functions. In particular there exist the rational functions g_0 and g_0^* such that

$$f_0^*=g_0\circ f_0,\qquad f_0=g_0^*\circ f_0^*.$$

Then we see that $g_0=g_0^{*-1}$ is a one-to-one map of the w-plane S onto itself with at least a fixed point $w=\infty$ and thus it has the form

$$g_0(w)=\frac{w}{\lambda}+c,$$

λ, c being finite constants ($\lambda\neq 0$). Thus we have

$$f_0^*=\frac{1}{\lambda}f_0+c,$$

and by (2) we have

$$\frac{1}{\lambda}f_0\circ T_2(\tilde{p})+c=\frac{1}{\lambda}f_0(\tilde{p})+c+1$$

or

(3) $$f_0\circ T_2(\tilde{p})=f_0(\tilde{p})+\lambda \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

Now we show that λ cannot be real. First if λ were a rational number:

$$\lambda=\frac{m_1}{m_2} \qquad (m_1,\ m_2\text{: integers}),$$

then by (1) and (3) we would have

(4) $$f_0\circ T_1^{-m_1}\circ T_2^{m_2}(\tilde{p})=f_0(\tilde{p}) \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

If we put $\chi=T_1^{-m_1}\circ T_2^{m_2}$ for the simplicity then (4) has the form

(5) $$f_0\circ\chi(\tilde{p})=f_0(\tilde{p}) \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

Further, by (5) we have that

$$f_0\circ\chi^j(\tilde{p})=f_0(\tilde{p}) \qquad \text{for any integer } j.$$

Thus there exists an integer m $(1\leqq m\leqq d_0;\ d_0=\max_w \mathfrak{v}_{f_0}(w))$ such that

(6) $$\chi^m(\tilde{p})=\tilde{p},$$

where the integer m depends on the point $\tilde{p}\in\tilde{R}$. If there holds (6) for an integer m, then we have that

(7) $$\chi^{\nu m}(\tilde{p})=\tilde{p} \qquad \text{for any integer } \nu.$$

In fact, if there holds $\chi^{\nu m}(\tilde{p})=\tilde{p}$, then we have

$$\chi^{(\nu+1)m}(\tilde{p})=\chi^m\circ\chi^{\nu m}(\tilde{p})=\chi^m(\tilde{p})=\tilde{p}$$

and

$$\chi^{-\nu m}(\tilde{p})=\chi^{-\nu m}(\chi^{\nu m}(\tilde{p}))=I(\tilde{p})=\tilde{p},$$

where I is the identity map of $\tilde{R}$ onto itself. By (6) and (7) we have that

$$\chi^{d_0!}(\tilde{p})=I(\tilde{p}) \qquad \text{for any} \quad \tilde{p}\in\tilde{R}$$

or

$$T_1^{d_0!m_1}=T_2^{d_0!m_2}$$

which shows that T_1 and T_2 are linearly dependent to each other. This fact contradicts that T_1 and T_2 form a basis of $\mathfrak{G}$.

Next, if λ were an irrational number, then by (1) and (3) we would have

$$f_0\circ T_1^{m_1}\circ T_2^{m_2}(\tilde{p})=f_0(\tilde{p})+m_1+m_2\lambda \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

Now, for any small positive number ε, we can take the integers m_1, m_2 such that

$$0<f_0\circ T_1^{m_1}\circ T_2^{m_2}(\tilde{p})-f_0(\tilde{p})=m_1+m_2\lambda<\frac{\varepsilon}{d_0\,!} \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

Then we have

$$0<f_0\circ T_1^{d_0!m_1}\circ T_2^{d_0!m_2}(\tilde{p})-f_0(\tilde{p})=d_0!(m_1+m_2\lambda)<\varepsilon \qquad \text{for any} \quad \tilde{p}\in\tilde{R},$$

and thus we can verify by the method similar to the previous case that, for an arbitrary point $\tilde{p}_0\in\tilde{R}$, the point $T_1^{d_0!m_1}\circ T_2^{d_0!m_2}(\tilde{p}_0)$ belongs to the neighborhood of $\tilde{p}_0$ which is the component of $\{\tilde{p}\,|\,|f(\tilde{p})-f(\tilde{p}_0)|<\varepsilon\}$ containing $\tilde{p}_0$. This fact contradicts that $\mathfrak{G}$ is properly discontinuous (cf. **2**).

The uniqueness of f_0 and λ is obvious.

7. If we remove the assumption that $\tilde{R}$ has finite spherical area, we can show, for an arbitrary real number λ, an example of the Riemann surface $\tilde{R}$ which admits a conformal transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type II. of **2** and which admits a function f_0 satisfying the condition

$$(8) \qquad f_0\circ T_1(\tilde{p})=f_0(\tilde{p})+1, \qquad f_0\circ T_2(\tilde{p})=f_0(\tilde{p})+\lambda \qquad \text{for any} \quad \tilde{p}\in\tilde{R}.$$

First, let λ be a real number not being an integer. Let

$$l_{j,k}=\{\Re z=j+k\lambda,\ \ 0\leqq\Im z\leqq1\} \qquad (j,\ k=0,\ \pm1,\ \ldots),$$

and

$$\tilde{F}_k=\{|z|<\infty\}-\bigcup_{j=-\infty}^{+\infty} l_{j,k}-\bigcup_{j=-\infty}^{+\infty} l_{j,k+1} \qquad (k=0,\ \pm1,\ \ldots).$$

We connect crosswise $\tilde{F}_k$ with $\tilde{F}_{k+1}$ along all the slits $l_{j,k+1}$ ($j=0, \pm1, \ldots$) with the common projection on the z-plane for each k ($k=0, \pm1, \ldots$). Let $\tilde{R}$ be the Riemann surface constructed from $\tilde{F}_k$ ($k=0, \pm1, \ldots$) by this process, which covers the z-plane infinitely often, and let f_0 be the projection map of $\tilde{R}$ onto the z-plane. Then there exists the conformal transformation T_1 of $\tilde{R}$ onto itself which transforms an arbitrary point $\tilde{p}$ on $\tilde{F}_k$ ($k=0, \pm1, \ldots$) to the point $\tilde{p}^*$ on the *same* $\tilde{F}_k$ such that

$$f_0(\tilde{p}^*)=f_0(\tilde{p})+1,$$

and also there exists the conformal transformation T_2 of $\tilde{R}$ onto itself which transforms an arbitrary point $\tilde{p}$ on $\tilde{F}_k$ $(k=0, \pm 1, \dots)$ to the point $\tilde{p}^*$ on $\tilde{F}_{k+1}$ such that

$$f_0(\tilde{p}^*)=f_0(\tilde{p})+\lambda.$$

Then the conformal transformation group $\mathfrak{G}=\{T_1, T_2\}$ generated by T_1 and T_2 is of the type II. of **2**, and f_0 satisfies the condition (8).

Next, let $\lambda=m$ be an integer. We construct the Riemann surface $\tilde{R}$ for $\lambda=1/2$ by the above procedure, and we take T_1, $T_1^{-1}\circ T_2^2$ for the case $m=0$ and T_1, T_2^{2m} for the case $m\neq 0$ in place of T_1, T_2, as the basis of the conformal transformation group $\mathfrak{G}$ of $\tilde{R}$. Then (8) is satisfied.

It follows by the lemma 5 given later that the Riemann surface $\tilde{R}$ has infinite spherical area.

8. In the theorem 2 the function f_0 has minimal local degree d_0 at the ideal boundary component γ_0 which is the common limit of the four point sequences $\{T_1^{-m}(\tilde{p})\}_{m=1}^{\infty}$, $\{T_1^{m}(\tilde{p})\}_{m=1}^{\infty}$, $\{T_2^{-m}(\tilde{p})\}_{m=1}^{\infty}$ and $\{T_2^{m}(\tilde{p})\}_{m=1}^{\infty}$ for a point $\tilde{p}\in\tilde{R}$ (cf. the lemma 6 of [3]). Then, $\max_w \mathfrak{v}_{f_0}(w)=d_0$ and thus f_0 takes all values on the w-plane S for d_0-times except for a set of w of capacity zero and never takes 0 and ∞. Thus we can find real constants l_1 and l_2 such that, except for $w=\infty$, any point on $\mathfrak{I}w=l_1$ and $\mathfrak{I}(e^{-i\alpha}w)=l_2$ $(\alpha=\arg\lambda)$ is neither an exceptional point of f_0 nor the image of a multiple point of f_0. Then, by the theorem 2 the closed set F_0, which is not necessarily connected (e.g. cf. the proof of the lemma 11), of $\tilde{R}$ defined by $\{\tilde{p}\,|\,l_1\leqq\mathfrak{I}f_0(\tilde{p})\leqq l_1+|\mathfrak{I}\lambda|,\ l_2\leqq\mathfrak{I}(e^{-i\alpha}f_0(\tilde{p}))\leqq l_2+|\sin\alpha|\}$ gives a fundamental region of the group $\mathfrak{G}$ and the relatively compact boundary $\tilde{C}=\partial F_0$ consists of a finite number of simple closed analytic curves $\tilde{C}_1, \dots, \tilde{C}_\kappa$ $(\kappa\leqq d_0)$. Then $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) is constructed from F_0 by identifying the points of $\tilde{C}$ equivalent modulo $\mathfrak{G}$. Thus $\tilde{R}$ must be conformally equivalent with a d_0-sheeted covering surface $\tilde{R}^*$ on the w-plane S which is d_0-sheeted except for a set of $w\in S$ of capacity zero and which is mapped onto itself by the transformations $w|w+1$ and $w|w+\lambda$.

9. Let $\tilde{R}$ be a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type II. of **2**, and let $R\equiv\tilde{R}$ (mod $\mathfrak{G}$). Then a strong homology basis of R forms a system of generators of the group $\mathfrak{G}$ (cf. **1**).[5] Further we assume that $\tilde{R}$ has finite spherical area. Then by the lemmas 5 and 6 given later, no dividing cycle on R can be a non-trivial generator of $\mathfrak{G}$ and only a finite number of elements of a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$ can be non-trivial generators of $\mathfrak{G}$.[6] Let $\alpha_1, \alpha_2, \dots$ be an arbitrary canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$ defined in

5) In the following, the argument will be done only for the case where R is of infinite genus because for the other cases it is done similarly and more easily

6) By a non-trivial generator of $\mathfrak{G}$ we mean an element of $\mathfrak{G}$ which is not the identity transformation.

3 of [3]. Then, a system of a finite number of $\alpha_1, \alpha_2, \ldots$ forms a system of generators of $\mathfrak{G}$ and thus we may assume that there exists a number κ such that

$$\alpha_{2j-1} \neq I \quad \text{or} \quad \alpha_{2j} \neq I \qquad \text{for each} \quad j=1, \ldots, \kappa,$$

and

$$\alpha_j = I \qquad \text{for all} \quad j \geqq 2\kappa+1,$$

if necessary, by a suitable change of the subindices of α_j, where I is the identical transformation of $\tilde{R}$ onto itself. Then, $\alpha_1, \alpha_2, \ldots$ have the expressions

$$\alpha_j = T_1^{m_{1,j}} \circ T_2^{m_{2,j}} \qquad (j=1, 2, \ldots), \tag{9}$$

for T_1, T_2 are the basis of $\mathfrak{G}$, where $m_{1,j} = m_{2,j} = 0$ for $j \geqq 2\kappa+1$. On the other hand, we have the expressions

$$T_1 = \prod_{j=1}^{2\kappa} \alpha_j^{n_{1,j}}, \quad T_2 = \prod_{j=1}^{2\kappa} \alpha_j^{n_{2,j}}, \tag{10}$$

for $\alpha_1, \ldots, \alpha_{2\kappa}$ are a system of generators of $\mathfrak{G}$. On substituting (9) in (10), we have the relation

$$\begin{pmatrix} m_{11} & \cdots & m_{1,2\kappa} \\ m_{21} & \cdots & m_{2,2\kappa} \end{pmatrix} \begin{pmatrix} n_{11} & n_{21} \\ \vdots & \vdots \\ n_{1,2\kappa} & n_{2,2\kappa} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

Conversely, if the system of integers $m_{1,j}, m_{2,j}$ $(j=1, \ldots, 2\kappa)$ satisfies the condition (11) for a system of integers $n_{1,j}, n_{2,j}$ $(j=1, \ldots, 2\kappa)$, then we see immediately that $\alpha_j = T_1^{m_{1,j}} \circ T_2^{m_{2,j}}$ $(j=1, \ldots, 2\kappa)$ forms a system of generators of $\mathfrak{G}$. Thus we can obtain

LEMMA 1. *$\alpha_j = T_1^{m_{1,j}} \circ T_2^{m_{2,j}}$ $(j=1, \ldots, 2\kappa)$ form a system of generators of $\mathfrak{G}$ if and only if the system of integers $m_{1,j}, m_{2,j}$ $(j=1, \ldots, 2\kappa)$ satisfies the condition* (11) *for a system of integers $n_{1,j}, n_{2,j}$ $(j=1, \ldots, 2\kappa)$.*

10. We shall continue from the previous section. Let $\zeta = \zeta(\tilde{p})$ be a local uniformizing parameter at $\tilde{p} \in \tilde{R}$. If $\tilde{R}$ has finite spherical area, by the theorem 2 the differential $f_0' d\zeta$ is invariant under the group $\mathfrak{G}$ and thus we may regard it as an abelian differential of the first kind on the Riemann surface $R \equiv \tilde{R} \pmod{\mathfrak{G}}$. We can easily verify that it has a finite Dirichlet integral over R.

It is known (see the theorem 2 of [9]) that there exists a system of analytic abelian differentials dw_j of the first kind with finite Dirichlet integrals on R such that

$$\int_{\alpha_{2k-1}} dw_j = \delta_{j,k} \quad (j, k=1, 2, \ldots),$$

where $\delta_{j,k}$ is the Kronecker symbol. We shall put

$$\int_{\alpha_{2k}} dw_j = \tau_{j,k} \quad (j, k=1, 2, \ldots).$$

Then, we have

$$f_0' d\zeta = \sum_{j=1}^{\kappa} c_j dw_j, \tag{12}$$

where

$$c_j = \int_{\alpha_{2j-1}} f_0' d\zeta \quad (j=1, \ldots, \kappa). \tag{13}$$

In the case of R of finite genus, (12) can be verified by the well known method. In the case of R of infinite genus, it can be verified by the use of the following lemma by Virtanen (cf. the lemma 2 of [9]).

LEMMA 2. *Let $\alpha_1, \alpha_2, \ldots$ be an arbitrary canonical homology basis of $R \in O_G$ modulo the ideal boundary $\mathfrak{J}$, and let $w = u + iv$ and $w^* = u^* + iv^*$ be two abelian integrals of the first kind. If u or v^* has only a finite number of non-zero α_j-periods, then there holds the relation*

$$D_R(w, w^*) = \sum_j \left(\int_{\alpha_{2j-1}} du \int_{\alpha_{2j}} dv^* - \int_{\alpha_{2j-1}} dv^* \int_{\alpha_{2j}} du \right),$$

where $D_R(w, w^*)$ is the Dirichlet integral of w and w^* over R.

We put

$$dw \equiv f_0' d\zeta = du + idv, \; dw^* \equiv \sum_{j=1}^{\kappa} c_j dw_j = du^* + idv^*,$$

where du, dv, du^* and dv^* are real differentials. Then by the lemma 2 and (13), we have

$$D_R(w - w^*) = \sum_{j=1}^{\kappa} \left(\int_{\alpha_{2j-1}} (du - du^*) \int_{\alpha_{2j}} (dv - dv^*) - \int_{\alpha_{2j-1}} (dv - dv^*) \int_{\alpha_{2j}} (du - du^*) \right) = 0,$$

for

$$\int_{\alpha_j} du = \int_{\alpha_j} dv = 0 \qquad (j = 2\kappa + 1, \ldots).$$

Hence we have $dw \equiv dw^*$ and (12).

On calculating the periodicity moduli of (12) along each α_j $(j=1, 2, \ldots)$, we have

$$\begin{cases} \displaystyle\int_{\alpha_{2k-1}} f_0' d\zeta = \sum_{j=1}^{\kappa} c_j \int_{\alpha_{2k-1}} dw_j = \sum_{j=1}^{\kappa} c_j \delta_{j,k} = \begin{cases} c_k & (k=1, \ldots, \kappa), \\ 0 & (k=\kappa+1, \ldots); \end{cases} \\ \displaystyle\int_{\alpha_{2k}} f_0' d\zeta = \sum_{j=1}^{\kappa} c_j \int_{\alpha_{2k}} dw_j = \sum_{j=1}^{\kappa} c_j \tau_{j,k} \quad (k=1, 2, \ldots). \end{cases} \tag{14}$$

On the other hand, by the theorem 2 and (9), we have

$$\int_{\alpha_k} f_0' d\zeta = \begin{cases} m_{1,k} + m_{2,k}\lambda & (k=1, \dots, 2\kappa), \\ 0 & (k=2\kappa+1, \dots). \end{cases} \tag{15}$$

By (14) and (15), we have the following system of equations

$$\begin{cases} c_k = m_{1,2k-1} + m_{2,2k-1}\lambda & (k=1, \dots, \kappa), \\ \sum_{j=1}^{\kappa} c_j \tau_{j,k} = m_{1,2k} + m_{2,2k}\lambda & (k=1, \dots, \kappa), \\ \sum_{j=1}^{\kappa} c_j \tau_{j,k} = 0 & (k=\kappa+1, \dots), \end{cases} \tag{16}$$

and thus we have a system of algebraic equations:

$$\frac{\sum_{j=1}^{\kappa} m_{2,2j-1}\tau_{j,1} - m_{22}}{\sum_{j=1}^{\kappa} m_{1,2j-1}\tau_{j,1} - m_{12}} = \frac{\sum_{j=1}^{\kappa} m_{2,2j-1}\tau_{j,2} - m_{24}}{\sum_{j=1}^{\kappa} m_{1,2j-1}\tau_{j,2} - m_{14}} = \cdots \tag{17}$$

$$(m_{1,2k} = m_{2,2k} = 0 \quad \text{for} \quad k > \kappa).$$

Here it is understood that numerator and denominator vanish whenever one does.

11. LEMMA 3. *Let $\tilde{R}$ be a Riemann surface of the class O_G which admits a conformal transformation group $\mathfrak{G} = \{T_1, T_2\}$ of the type* II. *of* **2**. *If $\tilde{R}$ has finite spherical area, there holds the equation*

$$d_0 = \operatorname{sign}(\Im\lambda) \sum_{j=1}^{\kappa} \begin{vmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j} \end{vmatrix}, \tag{18}$$

where d_0 is the maximum valence of the function f_0 of the theorem 2; $d_0 = \max_w \mathfrak{v}_{f_0}(w)$.

Proof. Let F_0 be the fundamental region of $\mathfrak{G}$ defined in 8. The area of the image of F_0 under f_0 and that of the parallelogram

$$l_1 \leqq \Im w \leqq l_1 + |\Im\lambda|, \quad l_2 \leqq \Im(e^{-i\alpha}w) < l_2 + |\sin\alpha|$$

are equal to $D_{F_0}(f_0)$ and $|\Im\lambda|$, respectively. Then it is immediately seen that

$$d_0 = \frac{D_{F_0}(f_0)}{|\Im\lambda|}. \tag{19}$$

Since we can regard the differential df_0 as the abelian differential of the first kind on R, then we have

$$D_{F_0}(f_0)=D_R(f_0). \tag{20}$$

On the other hand, by (12) and (16) we have

$$df_0=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)dw_j. \tag{21}$$

If we put

$$df_0=du+idv;\quad du,\ dv \text{ being real differentials,}$$

then by the lemma 2 there holds the relation

$$D_R(f_0)=\sum_{j=1}^{\kappa}\left(\int_{\alpha_{2j-1}}du\int_{\alpha_{2j}}dv-\int_{\alpha_{2j-1}}dv\int_{\alpha_{2j}}du\right). \tag{22}$$

Since by (16)

$$\int_{\alpha_{2k-1}}df_0=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)\int_{\alpha_{2k-1}}dw_j=m_{1,2k-1}+m_{2,2k-1}\lambda,$$

$$\int_{\alpha_{2k}}df_0=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)\int_{\alpha_{2k}}dw_j$$

$$=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)\tau_{j,k}=m_{1,2k}+m_{2,2k}\lambda,$$

then we have

$$\left\{\begin{aligned}&\int_{\alpha_{2k-1}}du=m_{1,2k-1}+m_{2,2k-1}\Re\lambda, &&\int_{\alpha_{2k-1}}dv=m_{2,2k-1}\Im\lambda,\\ &\int_{\alpha_{2k}}du=m_{1,2k}+m_{2,2k}\Re\lambda, &&\int_{\alpha_{2k}}dv=m_{2,2k}\Im\lambda.\end{aligned}\right. \tag{23}$$

On substituting (23) in (22), we obtain

$$D_R(f_0)=\Im\lambda\sum_{j=1}^{\kappa}(m_{1,2j-1}m_{2,2j}-m_{2,2j-1}m_{1,2j}). \tag{24}$$

(19), (20) and (24) imply (18).

By the lemma 3, we obtain immediately

Corollary 1. *The maximum valence d_0 of the function f_0 of the theorem 2 is uniquely determined by the structure of the system of generators $\alpha_1, \ldots, \alpha_{2\kappa}$ as elements of the covering transformation group $\mathfrak{G}$.*

Remark. The relation (23)

$$\sum_{k=1}^{\kappa}(m_{2k-1}m_{2k}^{*}-m_{2k}m_{2k-1}^{*})=d_0 \tag{25}$$

of the previous paper [3] for the case of the type I. (H) of **2**, can also be verified by the method similar to the lemma 3 though it has been done by another method in [3]. The maximum valence d_0 in (25) is not uniquely determined by the structure of the system of generators $\alpha_1, \ldots, \alpha_{2\kappa}$ as elements of $\mathfrak{G}=\{T\}$, for the possibility of the choice of m_{2k-1}^{*} and m_{2k}^{*} $(k=1, \ldots, \kappa)$ remains. In **26** later, we shall concern ourselves with the problem of finding the Riemann surface $\tilde{R}$ minimizing d_0 in the class of the Riemann surfaces which have a given structure of the system of generators $\alpha_1, \ldots, \alpha_{2\kappa}$ as elements of $\mathfrak{G}=\{T\}$.

12. Now we shall proceed to the converse problem of **10**. We assume that there holds (17) for the period matrix $(\tau_{j,k})_{j,k=1,2,\ldots}$ and the system of integers $\{m_{1,j};\ m_{2,j}\}_{j=1}^{2\kappa}$ $(\kappa\geqq 1)$ which satisfies the condition (11) for a system of integers $n_{1,j}, n_{2,j}$ $(j=1, \ldots, 2\kappa)$. Then, when we put

$$\lambda=-\frac{\sum_{j=1}^{\kappa}m_{1,2j-1}\tau_{j,1}-m_{12}}{\sum_{j=1}^{\kappa}m_{2,2j-1}\tau_{j,1}-m_{22}}=-\frac{\sum_{j=1}^{\kappa}m_{1,2j-1}\tau_{j,2}-m_{14}}{\sum_{j=1}^{\kappa}m_{2,2j-1}\tau_{j,2}-m_{24}}=\cdots$$

$$(m_{1,2k}=m_{2,2k}=0 \quad \text{for} \quad k>\kappa),$$

we see immediately that the differential

$$df=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)dw_j$$

satisfies the period relations

$$\begin{cases} \displaystyle\int_{\alpha_{2k-1}}df=\begin{cases} m_{1,2k-1}+m_{2,2k-1}\lambda & (k=1, \ldots, \kappa), \\ 0 & (k=\kappa+1, \ldots), \end{cases} \\ \displaystyle\int_{\alpha_{2k}}df=\sum_{j=1}^{\kappa}(m_{1,2j-1}+m_{2,2j-1}\lambda)\tau_{j,k}=\begin{cases} m_{1,2k}+m_{2,2k}\lambda & (k=1, \ldots, \kappa), \\ 0 & (k=\kappa+1, \ldots). \end{cases} \end{cases} \tag{26}$$

Here the imaginary part of λ is not reduced to zero, because, if λ were real then by (26) and the lemma 2 we would have

$$D_R(f)=\sum_{j=1}^{\kappa}\left(\int_{\alpha_{2j-1}}du\int_{\alpha_{2j}}dv-\int_{\alpha_{2j-1}}dv\int_{\alpha_{2j}}du\right)=0$$

and thus $df\equiv 0$, where

$$df=du+idv;\quad du,\ dv \text{ being real differentials.}$$

Let $\tilde{R}$ be the abelian covering surface of R with the covering transformation group

$$\mathfrak{G}=\{T_1,\ T_2;\ \alpha_j=T_1^{m_{1,j}}\circ T_2^{m_{2,j}}\ (j=1,\ \ldots,\ 2\kappa),\ \alpha_j=I\ (j=2\kappa+1,\ \ldots)\}$$

of the type II. of **2**. Of course, here we assume that no dividing cycle on R is a non-trivial generator of $\mathfrak{G}$. Let $\tilde{p}_0$ be an arbitrary fixed point on $\tilde{R}$, $\tilde{C}(\tilde{p})$ be a path from $\tilde{p}_0$ to any point $\tilde{p}$ on $\tilde{R}$, and $C(\tilde{p})$ be the projection of $\tilde{C}(\tilde{p})$ on R. Then the analytic function

$$f(\tilde{p})=\int_{C(\tilde{p})} df$$

is one-valued and regular on $\tilde{R}$, for the value $f(\tilde{p})$ of f at $\tilde{p}$ is independent of the choice of the path $\tilde{C}(\tilde{p})$ by (26) and the structure of the system of generators $\alpha_1,\ \ldots,\ \alpha_{2\kappa}$ of the group $\mathfrak{G}$. Further, f satisfies the functional relation

$$f\circ T_1(\tilde{p})=f(\tilde{p})+1,\quad f\circ T_2(\tilde{p})=f(\tilde{p})+\lambda \qquad \text{for any} \quad \tilde{p}\in\tilde{R}, \tag{27}$$

for, by (10), (11) and (26)

$$f\circ T_1(\tilde{p})-f(\tilde{p})=\sum_{j=1}^{2\kappa} n_{1,j}\int_{\alpha_j} df=\sum_{j=1}^{2\kappa} n_{1,j}\ (m_{1,j}+m_{2,j}\lambda)=1,$$

$$f\circ T_2(\tilde{p})-f(\tilde{p})=\sum_{j=1}^{2\kappa} n_{2,j}\int_{\alpha_j} df=\sum_{j=1}^{2\kappa} n_{2,j}\ (m_{1,j}+m_{2,j}\lambda)=\lambda.$$

Then, if we note that $D_R(f)<\infty$, by the similar method to the lemma 3 we have

$$d=\operatorname{sign}(\mathfrak{J}\lambda)\sum_{j=1}^{\kappa}\begin{vmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j}\end{vmatrix}<\infty, \tag{28}$$

where d is the maximum valence of f. Thus, $f\in\mathfrak{B}(\tilde{R})$ and $\tilde{R}\notin O_{MD}$. Further, by (27), (28) and the lemma 3, we see that f provides the property of the function f_0 of the theorem 2.

13. By the argument throughout **10** and **12**, we obtain the following result.

THEOREM 3. *Let R be a Riemann surface of the class O_G, $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$, and $(\tau_{j,k})_{j,k=1,2,\ldots}$ be the period matrix corresponding to the canonical homology basis $\alpha_1, \alpha_2, \ldots$. Let $\tilde{R}$ be an abelian covering surface of R which is of the class O_G and which admits the covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type* II. *of* **2**, *and let $\mathfrak{G}$ have a system of generators $\alpha_1, \ldots, \alpha_{2\kappa}$ ($\kappa\geqq 1$) with*

$$\alpha_j=T_1^{m_{1,j}}\circ T_2^{m_{2,j}} \qquad (j=1,\ \ldots,\ 2\kappa).$$

Further we assume that $\alpha_j=I$ ($j=2\kappa+1, \ldots$) and no dividing cycle on R is a non-trivial generator of $\mathfrak{G}$, where I is the identical transformation of $\tilde{R}$ onto itself.

Then, $\tilde{R}$ has finite spherical area if and only if there holds

$$(17)\qquad \frac{\sum_{j=1}^{\kappa} m_{2,2j-1}\tau_{j,1}-m_{22}}{\sum_{j=1}^{\kappa} m_{1,2j-1}\tau_{j,1}-m_{12}}=\frac{\sum_{j=1}^{\kappa} m_{2,2j-1}\tau_{j,2}-m_{24}}{\sum_{j=1}^{\kappa} m_{1,2j-1}\tau_{j,2}-m_{14}}=\cdots$$

$$(m_{1,2k}=m_{2,2k}=0 \quad \textit{for} \quad k>\kappa)$$

for the period matrix $(\tau_{j,k})_{j,k=1,2,\ldots}$ and the system of integers $\{m_{1,j};\ m_{2,j}\}_{j=1}^{2\kappa}$ which satisfies the condition

$$(11)\qquad \begin{pmatrix} m_{11} & \cdots & m_{1,2\kappa} \\ m_{21} & \cdots & m_{2,2\kappa} \end{pmatrix}\begin{pmatrix} n_{11} & n_{21} \\ \vdots & \vdots \\ n_{1,2\kappa} & n_{2,2\kappa} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for a system of integers $n_{1,j}$, $n_{2,j}$ $(j=1, \ldots, 2\kappa)$.

REMARK. In the theorem 4 of the previous paper [3], we assumed that R is a Riemann surface of the class O'' and $\alpha_1, \alpha_2, \ldots$ are a canonical homology basis belonging to the exhaustion $E=\{R_n\}$ satisfying $\lambda\{\mathfrak{L}_E\}=0$ (cf. [3] for the notations and words). However we can now verify that there holds the theorem 4 of [3] under the weaker assumption that R is a Riemann surface of the class O_G and $\alpha_1, \alpha_2, \ldots$ is a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$. In fact, if we note that, by the theorem 1 of [3] and the lemma 14 of [3], the differential $(f_0'/f_0)d\zeta$ of **24** in [3] has only a finite number of non-zero α_j-periods, then we can easily see that the argument of **10** and **12** in the present paper remains valid for the case.

§ 3. Topological properties of abelian covering surfaces with finite spherical area.

14. In the present chapter we shall investigate topological properties of abelian covering surfaces. Especially we shall investigate in detail the topological structure of an abelian covering surface with finite spherical area.

Let R be an arbitrary Riemann surface and $\tilde{R}$ be an abelian covering surface of the type II. of **2**. Let $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$, and $\alpha_1, \alpha_2, \ldots$ have the expression

$$(29)\qquad \alpha_j=T_1^{m_{1,j}}\circ T_2^{m_{2,j}} \qquad (j=1, 2, \ldots)$$

as elements of the covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ of $\tilde{R}$.[7]

Then we have the following lemma.

7) Here it may arise that either $\{\alpha_j\}$ is vacuous or all the α_j are trivial generators of $\mathfrak{G}$. Then the lemma 4 is trivial.

LEMMA 4. *From a given canonical homology basis* $\alpha_1, \alpha_2, \ldots$, *we can always select the canonical homology basis* $\bar{\alpha}_1, \bar{\alpha}_2, \ldots$ *of* R *modulo* $\mathfrak{F}$, *called regular for* $\tilde{R}$, *which satisfies the conditions*:

(i) *In the case*

$$\begin{vmatrix} m_{1,2j-1} & m_{1,2j} \\ m_{2,2j-1} & m_{2,2j} \end{vmatrix} = 0 \tag{30}$$

in (29),

$$\bar{\alpha}_{2j-1} = I, \quad \bar{\alpha}_{2j} = T_1^{\bar{m}_{1,2j}} \circ T_2^{\bar{m}_{2,2j}}, \tag{31}$$

where

$$\bar{m}_{1,2j} = (m_{1,2j-1}, m_{1,2j}), \quad \bar{m}_{2,2j} = \varepsilon(m_{2,2j-1}, m_{2,2j}),^{8)} \tag{32}$$

ε *being defined by*

$$\varepsilon = \mathrm{sign}(m_{1,2j-1}m_{2,2j-1} + m_{1,2j}m_{2,2j}) \qquad (\mathrm{sign}\ 0 = 1);$$

(ii) *In the case*

$$\begin{vmatrix} m_{1,2j-1} & m_{1,2j} \\ m_{2,2j-1} & m_{2,2j} \end{vmatrix} \neq 0 \tag{33}$$

in (29),

$$\bar{\alpha}_{2j-1} = T_1^{\bar{m}_{1,2j-1}}, \quad \bar{\alpha}_{2j} = T_1^{\bar{m}_{1,2j}} \circ T_2^{\bar{m}_{2,2j}}, \tag{34}$$

where $\bar{m}_{1,2j-1}$, $\bar{m}_{1,2j}$ *and* $\bar{m}_{2,2j}$ *satisfy the conditions*

$$\left\{ \begin{array}{l} \begin{pmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j} \end{pmatrix} = \begin{pmatrix} a_{1,2j-1} & a_{1,2j} \\ a_{2,2j-1} & a_{2,2j} \end{pmatrix} \begin{pmatrix} \bar{m}_{1,2j-1} & 0 \\ \bar{m}_{1,2j} & \bar{m}_{2,2j} \end{pmatrix}, \\ 0 \leq \bar{m}_{1,2j} < \bar{m}_{1,2j-1} \end{array} \right. \tag{35}$$

for a system of integers $a_{1,2j-1}$, $a_{1,2j}$, $a_{2,2j-1}$, $a_{2,2j}$ *such that*

$$\begin{vmatrix} a_{1,2j-1} & a_{1,2j} \\ a_{2,2j-1} & a_{2,2j} \end{vmatrix} = 1. \tag{36}$$

Conversely, from a given regular canonical homology basis (31) *and* (34) *for* $\tilde{R}$, *we can always select a canonical homology basis* $\alpha_1, \alpha_2, \ldots$ *of* R *which satisfies the condition*

$$\left\{ \begin{array}{l} \alpha_{2j-1} = T_1^{m_{1,2j-1}} \circ T_2^{m_{2,2j-1}} \\ \alpha_{2j} = T_1^{m_{1,2j}} \circ T_2^{m_{2,2j}} \end{array} \right. \qquad (j = 1, 2, \ldots)$$

8) (m, n) means the non-negative greatest common measure of integers m and n. For convenience, we take $(0, 0)$ as 0.

for an arbitrarily given system of integers $m_{1,2j-1}$, $m_{2,2j-1}$, $m_{1,2j}$ *and* $m_{2,2j}$ $(j=1, 2, \dots)$ *that satisfies* (32) *in the case* (31) *and does* (35) *for a system of integers* $a_{1,2j-1}$, $a_{1,2j}$, $a_{2,2j-1}$ *and* $a_{2,2j}$ *with* (36) *in the case* (34).

Proof. The case (i): If $m_{1,2j-1}=m_{1,2j}=m_{2,2j-1}=m_{2,2j}=0$, we put $\bar{\alpha}_{2j-1}=\alpha_{2j-1}$, $\bar{\alpha}_{2j}=\alpha_{2j}$. Then

$$\bar{\alpha}_{2j-1}=I,\ \bar{\alpha}_{2j}=I.$$

If $m_{1,2j-1}\neq 0$ or $m_{1,2j}\neq 0$, we put

$$\bar{\alpha}_{2j-1}=\alpha_{2j-1}{}^{m_{1,2j}/(m_{1,2j-1},\, m_{1,2j})}\alpha_{2j}{}^{-m_{1,2j-1}/(m_{1,2j-1},\, m_{1,2j})}. \tag{37}$$

Then, by (29) and (30), we have

$$\bar{\alpha}_{2j-1}=T_2{}^{(m_{2,2j-1}m_{1,2j}-m_{2,2j}m_{1,2j-1})/(m_{1,2j-1},\, m_{1,2j})}=I. \tag{38}$$

Now there always exists a pair of integers $x_{1,2j-1}$ and $x_{1,2j}$ such that

$$x_{1,2j-1}m_{1,2j-1}+x_{1,2j}m_{1,2j}=(m_{1,2j-1},\, m_{1,2j}). \tag{39}$$

For such $x_{1,2j-1}$ and $x_{1,2j}$, we put

$$\bar{\alpha}_{2j}=\alpha_{2j-1}{}^{x_{1,2j-1}}\alpha_{2j}{}^{x_{1,2j}}. \tag{40}$$

Then, by (29) and (39), we have

$$\bar{\alpha}_{2j}=T_1{}^{(m_{1,2j-1},\, m_{1,2j})}\circ T_2{}^{x_{1,2j-1}m_{2,2j-1}+x_{1,2j}m_{2,2j}}. \tag{41}$$

By (37), (39) and (40), we have

$$\begin{cases} \alpha_{2j-1}=\bar{\alpha}_{2j-1}{}^{x_{1,2j}}\bar{\alpha}_{2j}{}^{m_{1,2j-1}/(m_{1,2j-1},\, m_{1,2j})}, \\ \alpha_{2j}=\bar{\alpha}_{2j-1}{}^{-x_{1,2j-1}}\bar{\alpha}_{2j}{}^{m_{1,2j}/(m_{1,2j-1},\, m_{1,2j})}. \end{cases} \tag{42}$$

We note that if $m_{1,2j-1}\neq 0$ or $m_{1,2j}\neq 0$, and $m_{2,2j-1}\neq 0$ or $m_{2,2j}\neq 0$, by (30) either $m_{1,2j-1}/m_{2,2j-1}$ or $m_{1,2j}/m_{2,2j}$ is finite and

$$\frac{m_{1,2j-1}}{m_{2,2j-1}}=\frac{m_{1,2j}}{m_{2,2j}}=\varepsilon\,\frac{(m_{1,2j-1},\, m_{1,2j})}{(m_{2,2j-1},\, m_{2,2j})},$$

where $\varepsilon=\operatorname{sign}(m_{1,2j-1}m_{2,2j-1}+m_{1,2j}m_{2,2j})$ (sign $0=1$). Thus, by (30) and (39), we have

$$\begin{aligned} x_{1,2j-1}m_{2,2j-1}+x_{1,2j}m_{2,2j}&=(m_{1,2j-1},\, m_{1,2j})\,\frac{m_{2,2j-1}}{m_{1,2j-1}}\left(\text{or } (m_{1,2j-1},\, m_{1,2j})\,\frac{m_{2,2j}}{m_{1,2j}}\right) \\ &=\varepsilon(m_{2,2j-1},\, m_{2,2j}), \end{aligned}$$

provided that $m_{1,2j-1}\neq 0$ or $m_{1,2j}\neq 0$. Thus (41) takes the form

$$\bar{\alpha}_{2j}=T_1{}^{(m_{1,2j-1},\, m_{1,2j})}\circ T_2{}^{\varepsilon(m_{2,2j-1},\, m_{2,2j})}. \tag{43}$$

If $m_{1,2j-1}=m_{1,2j}=0$, and $m_{2,2j-1}\neq 0$ or $m_{2,2j}\neq 0$, we put

$$(44)\qquad \begin{cases} \bar{\alpha}_{2j-1}=\alpha_{2j-1}{}^{m_{2,2j}/(m_{2,2j-1},\, m_{2,2j})}\alpha_{2j}{}^{-m_{2,2j-1}/(m_{2,2j-1},\, m_{2,2j})}, \\ \bar{\alpha}_{2j}=\alpha_{2j-1}{}^{x_{2,2j-1}}\alpha_{2j}{}^{x_{2,2j}}, \end{cases}$$

where $x_{2,2j-1}$ and $x_{2,2j}$ are a pair of integers determined by the condition

$$(45)\qquad x_{2,2j-1}m_{2,2j-1}+x_{2,2j}m_{2,2j}=(m_{2,2j-1},\, m_{2,2j}).$$

Then we have

$$\bar{\alpha}_{2j-1}=I,\ \bar{\alpha}_{2j}=T_1{}^{(m_{2,2j-1},\, m_{2,2j})},$$

and by (44) and (45)

$$(46)\qquad \begin{cases} \alpha_{2j-1}=\bar{\alpha}_{2j-1}{}^{x_{2,2j}}\bar{\alpha}_{2j}{}^{m_{2,2j-1}/(m_{2,2j-1},\, m_{2,2j})}, \\ \alpha_{2j}=\bar{\alpha}_{2j-1}{}^{-x_{2,2j-1}}\bar{\alpha}_{2j}{}^{m_{2,2j}/(m_{2,2j-1},\, m_{2,2j})}. \end{cases}$$

The case (ii): In the case, $m_{2,2j-1}=m_{2,2j}=0$ does not take place. Then we put

$$(47)\qquad \begin{cases} \alpha_{2j-1}'=\alpha_{2j-1}{}^{\varepsilon' m_{2,2j}/(m_{2,2j-1},\, m_{2,2j})}\alpha_{2j}{}^{-\varepsilon' m_{2,2j-1}/(m_{2,2j-1},\, m_{2,2j})}, \\ \alpha_{2j}'=\alpha_{2j-1}{}^{\varepsilon' x_{2,2j-1}}\alpha_{2j}{}^{\varepsilon' x_{2,2j}}, \end{cases}$$

where $x_{2,2j-1}$ and $x_{2,2j}$ are a pair of integers determined by the condition

$$(48)\qquad x_{2,2j-1}m_{2,2j-1}+x_{2,2j}m_{2,2j}=(m_{2,2j-1},\, m_{2,2j}),$$

and

$$\varepsilon' = \operatorname{sign}\begin{vmatrix} m_{1,2j-1} & m_{1,2j} \\ m_{2,2j-1} & m_{2,2j} \end{vmatrix}.$$

Then we have

$$(49)\qquad \begin{cases} \alpha_{2j-1}'=T_1{}^{|m_{1,2j-1}m_{2,2j}-m_{1,2j}m_{2,2j-1}|/(m_{2,2j-1},\, m_{2,2j})}, \\ \alpha_{2j}'=T_1{}^{\varepsilon'(x_{2,2j-1}m_{1,2j-1}+x_{2,2j}m_{1,2j})}\circ T_2{}^{\varepsilon'(m_{2,2j-1},\, m_{2,2j})}, \end{cases}$$

and by (47) and (48)

$$(50)\qquad \begin{cases} \alpha_{2j-1}=\alpha_{2j-1}'{}^{\varepsilon' x_{2,2j}}\alpha_{2j}'{}^{\varepsilon' m_{2,2j-1}/(m_{2,2j-1},\, m_{2,2j})}, \\ \alpha_{2j}=\alpha_{2j-1}'{}^{-\varepsilon' x_{2,2j-1}}\alpha_{2j}'{}^{\varepsilon' m_{2,2j}/(m_{2,2j-1},\, m_{2,2j})}. \end{cases}$$

If we put

$$(51)\qquad \begin{cases} \overline{m}_{1,2j-1}=\dfrac{|m_{1,2j-1}m_{2,2j}-m_{1,2j}m_{2,2j-1}|}{(m_{2,2j-1},\, m_{2,2j})}\ (>0), \\ m_{1,2j}'=\varepsilon'(x_{2,2j-1}m_{1,2j-1}+x_{2,2j}m_{1,2j}),\quad \overline{m}_{2,2j}=\varepsilon'(m_{2,2j-1},\, m_{2,2j}), \end{cases}$$

then (49) takes the form

$$\alpha_{2j-1}'=T_1^{\bar{m}_{1,2j-1}},\ \alpha_{2j}'=T_1^{m'_{1,2j}}\circ T_2^{\bar{m}_{2,2j}}. \tag{52}$$

There exists a pair of integers $\bar{m}_{1,2j}$ and n_j such that

$$\bar{m}_{1,2j}=m_{1,2j}'-n_j\bar{m}_{1,2j-1} \tag{53}$$

and

$$0\leqq\bar{m}_{1,2j}<\bar{m}_{1,2j-1}. \tag{54}$$

Then, if we put

$$\bar{\alpha}_{2j-1}=\alpha_{2j-1}',\qquad \bar{\alpha}_{2j}=\alpha_{2j-1}'^{-n_j}\alpha_{2j}', \tag{55}$$

then we have

$$\alpha_{2j-1}=\bar{\alpha}_{2j-1}',\qquad \alpha_{2j}'=\bar{\alpha}_{2j-1}^{n_j}\bar{\alpha}_{2j}, \tag{56}$$

and

$$\bar{\alpha}_{2j-1}=T_1^{\bar{m}_{1,2j-1}},\qquad \bar{\alpha}_{2j}=T_1^{\bar{m}_{1,2j}}\circ T_2^{\bar{m}_{2,2j}}. \tag{57}$$

It is immediately verified that the combined system of the conditions (51), (53) and (54) are equivalent with (35).

We must show that the system of the cycles $\bar{\alpha}_1$, $\bar{\alpha}_2$, ... constructed by the above procedure satisfies the conditions (a) or (a′), and (b) or (b′) of **3** in [3]. It is obvious by (42), (46), (50) and (56) that the condition (a) or (a′) is satisfied. Further it is immediately verified by simple calculations that the condition (b) or (b′) for the intersection number is satisfied.

The converse statement is also obvious by the above procedure of the proof.

15. Let R be an arbitrary Riemann surface, $\tilde{R}$ be an abelian covering surface of R and $\mathfrak{G}$ be its covering transformation group. Let α_1, α_2, ... be a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$. Then the cycles α_1, α_2, ... can be regarded as elements of the group $\mathfrak{G}$. Let $\bar{\alpha}_{2j-1}$, $\bar{\alpha}_{2j}$ be a system of conjugate cycles obtained by a linear combination of the conjugate cycles α_{2j-1}, α_{2j} for each j ($j=1, 2, \ldots$) such that the system of the cycles $\bar{\alpha}_1$, $\bar{\alpha}_2$, ... forms a canonical homology basis modulo $\mathfrak{J}$. For convenience, we shall define the following terminology. If we can take from the system of the conjugate cycles α_{2j-1}, α_{2j} a system of conjugate cycles $\bar{\alpha}_{2j-1}$, $\bar{\alpha}_{2j}$ with $\bar{\alpha}_{2j-1}=I$, then we shall call that the system of the conjugate cycles α_{2j-1}, α_{2j} is of the *tube type on* $\tilde{R}$, and if we can never take from α_{2j-1}, α_{2j} a system of conjugate cycles $\bar{\alpha}_{2j-1}$, $\bar{\alpha}_{2j}$ with $\bar{\alpha}_{2j-1}=I$ (or $\bar{\alpha}_{2j}=I$), then we shall call that the system of the conjugate cycles α_{2j-1}, α_{2j} is of the *card type on* $\tilde{R}$.

By the lemma 4, we can state the result.

COROLLARY 2. *Let $\tilde{R}$ be an abelian covering surface of R of which the covering transformation group $\mathfrak{G}$ is of the type* II. *of* **2**. *Then, the system of the conjugate cycles α_{2j-1}, α_{2j} is of the tube type on $\tilde{R}$ if and only if* (30) *holds, and it is of the card type on $\tilde{R}$ if and only if* (33) *holds.*

Further, by the lemma 10 of [3], we obtain the result.

COROLLARY 3. *Let $\tilde{R}$ be an abelian covering surface of R of which the covering transformation group $\mathfrak{G}$ is of the type* I. (H) *of* **2**. *Then, each system of the conjugate cycles α_{2j-1}, α_{2j} is of the tube type on $\tilde{R}$.*

Next, let $\tilde{R}$ be an abelian covering surface of R of which the covering transformation group $\mathfrak{G}$ is of the type I. (P) of **2**. Then for such an $\tilde{R}$ we can obtain a lemma of the same type as the lemma 10 of [3] and it can be verified by the argument similar to it. Therefore we have the result.

COROLLARY 4. *Let $\tilde{R}$ be an abelian covering surface of R of which the covering transformation group $\mathfrak{G}$ is of the type* I. (P) *of* **2**. *Then, each system of the conjugate cycles α_{2j-1}, α_{2j} is of the tube type on $\tilde{R}$.*

16. Throughout **16—25**, we shall assume that $\tilde{R}$ is a Riemann surface which is of the class O_G and admits a covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ of the type II. of **2**. Let $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis of $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) modulo $\mathfrak{I}$, and $\beta_{j_1\cdots j_n}$, $j_n>1$ be a canonical homology basis of dividing cycles of R (cf. **4** of [3] for the notation).

LEMMA 5. *If $\tilde{R}$ has finite spherical area, then no dividing cycle on R can be a non-trivial generator of $\mathfrak{G}$.*

Proof. (i) We would assume that an infinite number of elements of a canonical homology basis of dividing cycles were non-trivial generators of $\mathfrak{G}$, and let $\mathfrak{B}=\{\beta_1, \beta_2, \ldots\}$ be the system of such ones.

Let p_j be a point on β_j $(j=1, 2, \ldots)$ and $\tilde{p}_j$ be the point lying over p_j on the fundamental region F_0 of the group $\mathfrak{G}$ defined in **8**. We can select a subsequence $\{\tilde{p}_{j_\nu}\}_{\nu=1}^{\infty}$ of the point sequence $\{\tilde{p}_j\}_{j=1}^{\infty}$ such that both point sequences $\{\tilde{p}_{j_\nu}\}_{\nu=1}^{\infty}$ and $\{\beta_{j_\nu}(\tilde{p}_{j_\nu})\}_{\nu=1}^{\infty}$ simultaneously tend to ideal boundary components γ and γ', respectively. Then we see that the limit point $\lim_{\nu\to\infty} f_0(\tilde{p}_{j_\nu})$ must lie on the parallelogram

$$Z_0=\{z|l_1\leqq\Im z\leqq l_1+|\Im\lambda|,\ l_2\leqq\Im(e^{-i\alpha}z)\leqq l_2+|\sin\alpha|\}\qquad(\alpha=\arg\lambda),$$

where f_0 is the function in the theorem 2. Since β_j $(j=1, 2, \ldots)$ are non-trivial generators of $\mathfrak{G}$, by the theorem 2 we have

$$|f_0(\beta_{j_\nu}(\tilde{p}_{j_\nu}))-f_0(\tilde{p}_{j_\nu})|\geqq\min(1, \lambda).$$

Thus, by the lemma 12 of [3], we know that $\lim_{\nu\to\infty} f_0(\beta_{j_\nu}(\tilde{p}_{j_\nu}))\neq\lim_{\nu\to\infty} f_0(\tilde{p}_{j_\nu})$ or $\gamma\neq\gamma'$. On the other hand, since for any compact region $K\subset\tilde{R}$ there exists a number ν_0 such that two points $\tilde{p}_{j_\nu}$ and $\beta_{j_\nu}(\tilde{p}_{j_\nu})$ can be connected by one $\tilde{\beta}_{j_\nu}$ of the curves on $\tilde{R}-K$ lying over β_{j_ν} for all $\nu\geqq\nu_0$, we know that it must be $\gamma=\gamma'$. Contradiction.

(ii) We would assume that only a (non-zero) finite number of elements of a canonical homology basis of dividing cycles are non-trivial generators of $\mathfrak{G}$, and let $\mathfrak{B}$ be a system of such ones. Let $\beta_{j_1^0\cdots j_N^0}$ $(j_N^0>1)$ be one of the elements of $\mathfrak{B}$ such that

(58) $$N=\max n \qquad \text{for} \quad \beta_{j_1\cdots j_n}\in\mathfrak{B}.$$

Then we see that

$$\beta_{j_1^0\cdots j_N^0 1}\sim\beta_{j_1^0\cdots j_N^0}\prod_{j_{N+1}=2}^{s_{j_1^0\cdots j_N^0}}(\beta_{j_1^0\cdots j_N^0 j_{N+1}})^{-1}$$

and

$$\beta_{j_1^0\cdots j_N^0 j_{N+1}}=I \qquad (j_{N+1}=2,\ \ldots,\ s_{j_1^0\cdots j_N^0}).$$

For, if $\beta_{j_1^0\cdots j_N^0 j_{N+1}}\neq I$, $\beta_{j_1^0\cdots j_N^0 j_{N+1}}$ must belong to $\mathfrak{B}$ which contradicts (58). Thus $\beta_{j_1^0\cdots j_N^0 1}=\beta_{j_1^0\cdots j_N^0}$ as generators of $\mathfrak{G}$. By the similar procedure, we have that

$$\beta_{j_1^0\cdots j_N^0}=\beta_{j_1^0\cdots j_N^0 1}=\beta_{j_1^0\cdots j_N^0 11}=\cdots$$

and then $\beta_{j_1^0\cdots j_N^0 1}$, $\beta_{j_1^0\cdots j_N^0 11}$, ... are non-trivial generators. Thus we may apply the argument of (i) for a system of such ones and deduce a contradiction.

LEMMA 6. *If $\tilde{R}$ has finite spherical area, then only a finite number of α_j can be non-trivial generators of $\mathfrak{G}$.*

We can prove the lemma by the argument similar to the proof of the case (i) of the lemma 5. We omit its proof.

17. By the lemmas 5 and 6, if $\tilde{R}$ has finite spherical area, a suitably chosen system of a finite number of $\alpha_1, \alpha_2, \ldots$ forms a system of generators of $\mathfrak{G}$ and thus we may assume that there exists a number κ such that

$$\alpha_{2j-1}\neq I \quad \text{or} \quad \alpha_{2j}\neq I \quad \text{for each} \quad j=1,\ \ldots,\ \kappa,$$

and

$$\alpha_j=I \qquad \text{for all} \quad j\geqq 2\kappa+1,$$

if necessary, by a suitable change of indices of α_j. Then, $\alpha_1, \alpha_2, \ldots$ have the expressions

(59) $$\alpha_j=T_1^{m_{1,j}}\circ T_2^{m_{2,j}} \quad (j=1,\ 2,\ \ldots;\ m_{1,j}=m_{2,j}=0 \quad \text{for} \quad j\geqq 2\kappa+1),$$

as the generators of $\mathfrak{G}$, where the system of integers $m_{1,j}$, $m_{2,j}$ $(j=1, \ldots, 2\kappa)$ must satisfy the condition (11) for a system of integers $n_{1,j}$, $n_{2,j}$ $(j=1, \ldots, 2\kappa)$.

Then, we have the following lemma.

LEMMA 7. *If $\tilde{R}$ has finite spherical area, then there holds*

$$\mathrm{sign}(\Im\lambda)\begin{vmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j} \end{vmatrix} \geqq 0 \qquad (j=1, 2, \ldots),$$

λ being the constant in the theorem 2, *and thus $m_{1,j}$, $m_{2,j}$ $(j=1, 2, \ldots)$ satisfies the condition*:

(a) *The determinants*

$$\begin{vmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j} \end{vmatrix} \qquad (j=1, 2, \ldots)$$

are all non-negative or all non-positive.

Proof. It is sufficient to prove the lemma only for the case $\Im\lambda>0$. Because, if $\Im\lambda<0$, we take T_1, T_2^{-1} in place of T_1, T_2 as the basis of the group $\mathfrak{G}$, respectively, and take $-\lambda$, $m_{1,j}$, $-m_{2,j}$ $(j=1, 2, \ldots)$ in place of λ, $m_{1,j}$, $m_{2,j}$, respectively, then it is reduced to the case $\Im\lambda>0$. Further, by the lemma 4, it is sufficient to prove that $m_{2,2j}>0$ if the pair α_{2j-1}, α_{2j} is of the type:

$$\text{(60)} \qquad \alpha_{2j-1}=T_1^{m_{1,2j-1}}, \quad \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}} \quad (0\leqq m_{1,2j}<m_{1,2j-1},\ m_{2,2j}\neq 0),$$

in the case where α_1, α_2, ... are the regular canonical homology basis.

Let the two shores of each of the curves α_{2j-1}, α_{2j} $(j=1, 2, \ldots)$ be denoted by $\alpha_{2j-1}{}^{+}$, $\alpha_{2j-1}{}^{-}$, $\alpha_{2j}{}^{+}$, $\alpha_{2j}{}^{-}$, respectively, in such a manner that oriented curve α_{2j} intersects α_{2j-1} from $\alpha_{2j-1}{}^{+}$ to the other shore $\alpha_{2j-1}{}^{-}$ and that α_{2j-1} intersects α_{2j} from $\alpha_{2j}{}^{+}$ to $\alpha_{2j}{}^{-}$. Let F be the surface obtained from R by scissoring along the curves α_{2j-1}, α_{2j} $(j=1, \ldots, \kappa)$. Then, by **6** of [3] and (59), $\tilde{R}$ is constructed from an infinite number of replicas of F by a suitable identification process along the curves $\alpha_{2j-1}{}^{+}$, $\alpha_{2j-1}{}^{-}$, $\alpha_{2j}{}^{+}$ and $\alpha_{2j}{}^{-}$ $(j=1, \ldots, \kappa)$ of the replicas, and thus we may regard F as a fundamental region of $\mathfrak{G}$ on $\tilde{R}$. Further we may regard $\tilde{R}$ as the covering surface of the w-plane S with the projection map f_0 of $\tilde{R}$ onto S.

Now let α_{2j-1}, α_{2j} be of the card type (60). Then, by the theorem 2, we have

$$f_0\circ\alpha_{2j-1}(\tilde{p})=f_0(\tilde{p})+m_{1,2j-1} \quad \text{and} \quad \alpha_{2j-1}(\tilde{p})\in\alpha_{2j}{}^{+} \quad \text{for any} \quad \tilde{p}\in\alpha_{2j}{}^{-},$$

$$f_0\circ\alpha_{2j}(\tilde{p})=f_0(\tilde{p})+m_{1,2j}+m_{2,2j}\lambda \quad \text{and} \quad \alpha_{2j}(\tilde{p})\in\alpha_{2j-1}{}^{+} \quad \text{for any} \quad \tilde{p}\in\alpha_{2j-1}{}^{-}.$$

Let C_j be the Jordan curve $\alpha_{2j-1}{}^{-}\alpha_{2j}{}^{+}(\alpha_{2j-1}{}^{+})^{-1}(\alpha_{2j}{}^{-})^{-1}$ on $\tilde{R}$ which has the four vertices

$$\tilde{p}_0,\ T_1^{m_{1,2j-1}}(\tilde{p}_0),\ T_1^{m_{1,2j-1}+m_{1,2j}}\circ T_2^{m_{2,2j}}(\tilde{p}_0) \quad \text{and} \quad T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}}(\tilde{p}_0),$$

where $\tilde{p}_0$ is the common point of $\alpha_{2j-1}{}^{-}$, $\alpha_{2j}{}^{-}$.[9] Let G_j be a doubly-connected subregion of F contained in a neighborhood of C_j, one of the boundary components

9) We must take simple closed curves α_{2j-1} and α_{2j} in the homology classes (α_{2j-1}) and (α_{2j}), respectively, which always exist.

of which is C_j. Further, let $\tilde{G}_j$ be the connected component containing G_j of the subset of $\tilde{R}$ consisting of all the regions equivalent with G_j modulo $\mathfrak{G}$. Then, $\tilde{G}_j$ is an infinitely-connected covering surface of genus zero on S and admits the covering transformation group $\mathfrak{G}'$ the basis of which is

$$\alpha_{2j-1}=T_1^{m_{1,2j-1}},\ \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}}.$$

Hence, there exists a region $\tilde{G}_j^*$ and a homeomorphic map of $\tilde{G}_j$ onto $\tilde{G}_j^*$ which satisfy the conditions:

(i) the map is homotopically deformable to the identity map on fixing all points equivalent with $\tilde{p}_0$ modulo $\mathfrak{G}'$;

(ii) $\tilde{G}_j^*$ is univalent over the w-plane S;

(iii) $\tilde{G}_j^*$ admits the covering transformation group $\mathfrak{g}'$ the basis of which is

$$t_1=m_{1,2j-1},\ t_2=m_{1,2j}+\lambda m_{2,2j}.$$

Here we note that $\alpha_{2j-1}\times\alpha_{2j}=1$ ((b) or (b′) of **3** in [3]). Then we know from the structure of $\tilde{G}_j$ that there should hold that $m_{2,2j}>0$.

REMARK. We note that the condition (a) of the lemma 7 does not necessarily hold unless there is the assumption that $\tilde{R}$ has finite spherical area. It is evident by the following simple example.

Let

$$\mathfrak{g}=\{t_1,\ t_2;\ t_1=z+1,\ t_2=z+i\},$$

$$l=\left\{z\,\middle|\,\Re z=\frac{1}{2},\ \frac{1}{3}\leqq\Im z\leqq\frac{2}{3}\right\},$$

$$\tilde{F}_1=\tilde{F}_2=\{|z|<\infty\}-\bigcup_{\mu,\nu=-\infty}^{+\infty}t_1^{\mu}\circ t_2^{\nu}(l),$$

$$\tilde{\alpha}_1=\tilde{\alpha}_3=\{z|0\leqq\Re z\leqq1,\ \Im z=0\},$$

$$\tilde{\alpha}_2=\tilde{\alpha}_4=\{z|0\leqq\Im z\leqq1,\ \Re z=0\}.$$

We draw the curves $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ on $\tilde{F}_1$, and $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ on the reverse side of $\tilde{F}_2$, respectively. Let $\tilde{R}$ be the Riemann surface obtained from $\tilde{F}_1$ and $\tilde{F}_2$ by connecting along the *common* shores of each pair of the slits with the common projection on the z-plane. Let $\mathfrak{G}$ be the covering transformation group of $\tilde{R}$ generated by the basis T_1 and T_2 which transform an arbitrary point $\tilde{p}$ on $\tilde{F}_j$ ($j=1, 2$) to the points $\tilde{p}_1$ and $\tilde{p}_2$ on the *same* $\tilde{F}_j$ such that

$$f_0(\tilde{p}_1)=t_1\circ f_0(\tilde{p})\quad\text{and}\quad f_0(\tilde{p}_2)=t_2\circ f_0(\tilde{p}),$$

respectively, where f_0 is the projection map of $\tilde{R}$ onto the z-plane.[10] Let F be a subregion of $\tilde{R}$ lying over $\{z|0\leqq\Re z\leqq1,\ 0\leqq\Im z\leqq1\}$, and R be a Riemann surface

10) We should note that the map f_0 is not analytic.

obtained from F by identifying the points of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ with those of $T_2(\tilde{\alpha}_1)$, $T_1(\tilde{\alpha}_2)$, $T_2(\tilde{\alpha}_3)$ and $T_1(\tilde{\alpha}_4)$ equivalent modulo $\mathfrak{G}$. Then $\tilde{R}$ is an abelian covering surface of R with the covering transformation group $\mathfrak{G}$ of the type II. of **2**, and the images α_1, α_2, α_3 and α_4 on R of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$, respectively, form a canonical homology basis of R. We can take the orientations of α_1, α_2, α_3 and α_4 in such a way that

$$\alpha_1 = T_1,\ \alpha_2 = T_2;\ \alpha_3 = T_1,\ \alpha_4 = T_2^{-1}.$$

This gives an example desired.

The canonical homology basis $\alpha_1, \alpha_2, \ldots$ of R satisfying the condition (a) of the lemma 7 shall be called to have the *uni-orientation* with respect to $\tilde{R}$ or $\mathfrak{G}$.

18. Now we shall classify the abelian covering surfaces $\tilde{R}$. Let $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ $(q \geqq \kappa)$ be the class of the abelian covering surfaces $\tilde{R}$ such that $R \equiv \tilde{R}$ (mod $\mathfrak{G}$) are open or closed Riemann surfaces of genus q $(1 \leqq q \leqq \infty)$, and which satisfy the following conditions:

(i) No dividing cycle on R is a non-trivial generator of $\mathfrak{G}$;

(ii) There exists a canonical homology basis $\alpha_1, \alpha_2, \ldots$ modulo the ideal boundary $\mathfrak{J}$ of R which has the forms

$$\begin{cases} \alpha_j = T_1^{m_{1j}} \circ T_2^{m_{2j}} & (j=1, \ldots, 2\kappa), \\ \alpha_j = I & (j=2\kappa+1, \ldots) \end{cases}$$

as a system of generators of $\mathfrak{G}$, where it does not occur that $m_{1,2j-1} = m_{1,2j} = m_{2,2j-1} = m_{2,2j} = 0$ for any j $(j=1, \ldots, 2\kappa)$, and $m_{1,j}$, $m_{2,j}$ $(j=1, \ldots, 2\kappa)$ satisfy the condition (11) for a system of integers $n_{1,j}$, $n_{2,j}$ $(j=1, \ldots, 2\kappa)$;

(iii) The canonical homology basis $\alpha_1, \alpha_2, \ldots$ has the uni-orientation with respect to $\tilde{R}$, i.e. $m_{1,j}$, $m_{2,j}$ satisfy the condition (a) of the lemma 7.

Further we shall divide each class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ into the following three families:

(A) *The family of* $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ *such that, for* $\tilde{R} \in \mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$, *each system of the conjugate cycles* α_{2j-1}, α_{2j} $(j=1, \ldots, \kappa)$ *is of the tube type on* $\tilde{R}$;

(B) *The family of* $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ *such that, for* $\tilde{R} \in \mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$, *each system of the conjugate cycles* α_{2j-1}, α_{2j} $(j=1, \ldots, \kappa)$ *is of the card type on* $\tilde{R}$;

(C) *The family of* $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ *such that, for* $\tilde{R} \in \mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$, α_{2j-1}, α_{2j} $(j=1, \ldots, \iota;\ 1 \leqq \iota < \kappa)$ *are of the card types on* $\tilde{R}$ *and* α_{2j-1}, α_{2j} $(j=\iota+1, \ldots, \kappa)$ *are of the tube types on* $\tilde{R}$.[11]

19. In the present section, we shall concern ourselves with the family (A). We have the lemma.

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11) We should note that each property of (A), (B), (C) does not depend on the choice of $\tilde{R} \in \mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$.

LEMMA 8. *If* $\mathfrak{A}(q;\ m_{11},\ \ldots,\ m_{1,2\kappa};\ m_{21},\ \ldots,\ m_{2,2\kappa})$ *belongs to the family* (A), *then* $\mathfrak{A}(q;\ m_{11},\ \ldots,\ m_{1,2\kappa};\ m_{21},\ \ldots,\ m_{2,2\kappa})\subset O_{MD}$.

Proof. If there existed $\tilde{R}$ such that $\tilde{R}\in\mathfrak{A}(q;\ m_{11},\ \ldots,\ m_{1,2\kappa};\ m_{21},\ \ldots,\ m_{2,2\kappa})\in(\mathrm{A})$ and $\tilde{R}\notin O_{MD}$, then by the lemma 3 we would have

$$d_0=\operatorname{sign}(\mathfrak{J}\lambda)\sum_{j=1}^{\kappa}\begin{vmatrix} m_{1,2j-1} & m_{2,2j-1} \\ m_{1,2j} & m_{2,2j}\end{vmatrix}. \tag{61}$$

By the corollary 2, the right hand side of (61) must be zero and thus $d_0=0$ which is evidently impossible.

20. In the present section, we shall concern ourselves with the family (B). First we have a lemma.

LEMMA 9. *If* $\mathfrak{A}(q;\ m_{11},\ m_{12};\ m_{21},\ m_{22})$ *belongs to the family* (B) *and* $2\leqq q\leqq\infty$, *then* $\mathfrak{A}(q;\ m_{11},\ m_{22};\ m_{21},\ m_{22})\subset O_{MD}$.

Proof. By the assumption, $\mathfrak{G}$ has the system of generators

$$\begin{cases} \alpha_1=T_1^{m_{11}}\circ T_2^{m_{21}}, & \alpha_2=T_1^{m_{12}}\circ T_2^{m_{22}}, \\ \alpha_j=I & (3\leqq j\leqq 2q;\ 2\leqq q\leqq\infty), \end{cases} \tag{62}$$

and

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22}\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \tag{63}$$

for a system of integers n_{11}, n_{12}, n_{21} and n_{22}. If $\tilde{R}\notin O_{MD}$, then by the lemma 3, (62) and (63) we would have

$$d_0=\operatorname{sign}(\mathfrak{J}\lambda)\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{vmatrix}=1.$$

On the other hand, each system of the branches $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$ with a common point which are the images on $\tilde{R}$ of the conjugate cycles α_{2j-1}, α_{2j} $(2\leqq j\leqq q)$, forms a system of conjugate cycles on $\tilde{R}$ (cf. **6** of [3]). Then, $\tilde{R}$ has infinite genus and thus it is evidently impossible that $\tilde{R}$ is univalently mapped onto the w-plane.

Next, we shall show that, except for the case of the lemma 9, there always exists an abelian covering surface $\tilde{R}$ with finite spherical area in each class $\mathfrak{A}(q;\ m_{11},\ \ldots,\ m_{1,2\kappa};\ m_{21},\ \ldots,\ m_{2,2\kappa})$ of the family (B). In fact, we have the lemmas 10 and 11.

LEMMA 10. *There exists an abelian covering surface* $\tilde{R}$ *with finite spherical area in each class* $\mathfrak{A}(1;\ m_{11},\ m_{12};\ m_{21},\ m_{22})$.

Proof. Each class $\mathfrak{A}(1;\ m_{11},\ m_{12};\ m_{21},\ m_{22})$ necessarily belongs to the family (B). Then, by the lemma 4 and the condition (11), it is sufficient to prove the lemma only for the case

$$\alpha_1=T_1,\ \alpha_2=T_2^{\pm 1}.$$

Further we may assume that $\alpha_1=T_1$, $\alpha_2=T_2$, otherwise we may take T_1, T_2^{-1} in place of T_1, T_2 as the basis of $\mathfrak{G}$.

Let $\tilde{R}=\{z\,||z|<\infty\}$, $\mathfrak{G}=\{T_1,\ T_2;\ T_1=z+1,\ T_2=z+i\}$, $\tilde{\alpha}_1=\{\Im z=0,\ 0\leqq\Re z\leqq 1\}$, $\tilde{\alpha}_2=\{\Re z=0,\ 0\leqq\Im z\leqq 1\}$ and $F_0=\{0\leqq\Re z\leqq 1,\ 0\leqq\Im z\leqq 1\}$. Let R be a Riemann surface (torus) constructed from F_0 by identifying the points of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ with those of $T_2(\tilde{\alpha}_1)$ and $T_1(\tilde{\alpha}_2)$ equivalent modulo $\mathfrak{G}$, and α_1, α_2 be the images on R of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ respectively. Then we see immediately that $\alpha_1=T_1$, $\alpha_2=T_2$ by a suitable selection of the orientation of α_1, α_2 and thus $\tilde{R}$ satisfies the condition of the present lemma.

LEMMA 11. *If $\kappa\geqq 2$, then there exists an abelian covering surface $\tilde{R}$ with finite spherical area in each class $\mathfrak{A}(q;\ m_{11},\ \dots,\ m_{1,2\kappa};\ m_{21},\ \dots,\ m_{2,2\kappa})$ of the family* (B).

Proof. By the lemma 4, it is sufficient to prove the lemma only for the case

$$\text{(64)}\qquad \begin{cases} \alpha_{2j-1}=T_1^{m_{1,2j-1}},\ \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}} & \\ \qquad\qquad (j=1,\ \dots,\ \kappa;\ 0\leqq m_{1,2j}<m_{1,2j-1},\ m_{2,2j}\neq 0), & \\ \alpha_j=I \qquad (j=2\kappa+1,\ \dots). & \end{cases}$$

Further, by the condition (iii) of **18** we may assume that $m_{2,2j}>0$ $(j=1,\ \dots,\ \kappa)$, otherwise we may take T_1, T_2^{-1} in place of T_1, T_2 as the basis of $\mathfrak{G}$.

Let

$$t_1(z)=z+1,\ t_2(z)=z+i,$$

$$\text{(65)}\qquad \begin{cases} \tilde{\alpha}_{2j-1}=\{\Im z=0,\ 0\leqq\Re z\leqq m_{1,2j-1}\}, \\ \tilde{\alpha}_{2j}=\left\{\arg z=\tan^{-1}\dfrac{m_{2,2j}}{m_{1,2j}},\ 0\leqq|z|\leqq\sqrt{m_{1,2j}^2+m_{2,2j}^2}\right\} \end{cases}\qquad (j=1,\ \dots,\ \kappa),$$

let Φ_j $(j=1,\ \dots,\ \kappa)$ be the closed parallelogram surrounded by $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$, $t_1^{m_{1,2j}}\circ t_2^{m_{2,2j}}(\tilde{\alpha}_{2j-1})$ and $t_1^{m_{1,2j-1}}(\tilde{\alpha}_{2j})$ and let $\Phi_0=\{0\leqq\Re z\leqq 1,\ 0\leqq\Im z\leqq 1\}$. Then we can take a disk

$$\text{(66)}\qquad D=\{z\,||z-z_0|<r\}$$

such that

$$\text{(67)}\qquad D\subset\bigcap_{j=0}^{\kappa}(\Phi_j)^{\circ}.$$

Let

$$\text{(68)}\qquad l_j=\left\{\Im(z-z_0)=0,\quad \frac{2j-3}{2\kappa-1}\,r\leqq\Re(z-z_0)\leqq\frac{2(j-1)}{2\kappa-1}\,r\right\}\qquad (j=2,\ \dots,\ \kappa),$$

$$l_k' = \left\{ \Im(z-z_0)=0, \quad -\frac{r}{2k} \leqq \Re(z-z_0) \leqq -\frac{r}{2k+1} \right\} \tag{69}$$

$$(k=1, \ldots, q-\kappa, \text{ if } q<\infty; \ k=1, 2, \ldots, \text{ if } q=\infty).$$

Let

$$\tilde{F}_1(0, 0)=\{|z|<\infty\}-\bigcup_{n_1, n_2=-\infty}^{+\infty} t_1^{n_1 m_{11}+n_2 m_{12}} \circ t_2^{n_2 m_{22}}\left(l_2 \smile \bigcup_k l_k'\right),$$

$$\tilde{F}_j(0, 0)=\{|z|<\infty\}-\bigcup_{n_1, n_2=-\infty}^{+\infty} t_1^{n_1 m_{1,2j-1}+n_2 m_{1,2j}} \circ t_2^{n_2 m_{2,2j}}(l_j \smile l_{j+1})$$

$$(j=2, \ldots, \kappa-1),$$

$$\tilde{F}_\kappa(0, 0)=\{|z|<\infty\}-\bigcup_{n_1, n_2=-\infty}^{+\infty} t_1^{n_1 m_{1,2k-1}+n_2 m_{1,2k}} \circ t_2^{n_2 m_{2,2k}}\left(l_\kappa \smile \bigcup_k l_k'\right),$$

$$\tilde{F}_j(\mu, \nu)=t_1^\mu \circ t_2^\nu(\tilde{F}_j(0, 0)) \qquad (j=1, \ldots, \kappa; \ \mu, \nu\text{: integers}),$$

where $\bigcup_k l_k'=\phi$ if $q=\kappa$. We shall agree that

$$\tilde{F}_j(\mu', \nu')=\tilde{F}_j(\mu, \nu) \qquad (j=1, \ldots, \kappa),$$

if the system of integers $\mu, \nu; \mu', \nu'$ satisfies the relation

$$\begin{pmatrix} \mu'-\mu \\ \nu'-\nu \end{pmatrix} = \begin{pmatrix} m_{1,2j-1} & m_{1,2j} \\ 0 & m_{2,2j} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

for some integers n_1, n_2. The system of all mutually distinct $\tilde{F}_j(\mu, \nu)$ is given by

$$\tilde{F}_j(\mu, \nu) \qquad (\mu=0, \ldots, m_{1,2j-1}-1; \ \nu=0, \ldots, m_{2,2j}-1; \ j=1, \ldots, \kappa).$$

We draw the curves $\tilde{\alpha}_{2j-1}, \tilde{\alpha}_{2j}$ on $\tilde{F}_j(0, 0)$ $(j=1, \ldots, \kappa)$, and let $F_j(0, 0)$ be the subregion of $\tilde{F}_j(0, 0)$ surrounded by $\tilde{\alpha}_{2j-1}, \tilde{\alpha}_{2j}, t_1^{m_{1,2j}} \circ t_2^{m_{2,2j}}(\tilde{\alpha}_{2j-1})$ and $t_1^{m_{1,2j-1}}(\tilde{\alpha}_{2j})$, i.e. $F_1(0, 0) =\Phi_1-(l_2 \smile \bigcup_k l_k')$, $F_j(0, 0)=\Phi_j-(l_j \smile l_{j+1})$ $(j=2, \ldots, \kappa-1)$, $F_\kappa(0, 0)=\Phi_\kappa-(l_\kappa \smile \bigcup_k l_k')$. Let $\tilde{R}$ be the surface obtained from

$$\tilde{F}_j(\mu, \nu) \qquad (\mu=0, \ldots, m_{1,2j-1}-1; \ \nu=0, \ldots, m_{2,2j}-1; \ j=1, \ldots, \kappa)$$

by connecting crosswise along each pair of the slits with the common projection on the z-plane, where each slit corresponds obviously to one and only one slit.

First we can see as follows that $\tilde{R}$ is connected. It is sufficient to be proved that a point on $\tilde{F}_1(0, 0)$ can be connected to a point on $\tilde{F}_j(\mu, \nu)$ by a curve on $\tilde{R}$ for any j $(1 \leqq j \leqq \kappa)$, μ and ν, denoted by

$$\tilde{F}_1(0, 0) \sim \tilde{F}_j(\mu, \nu).$$

It is immediately seen that $\tilde{F}_j(0, 0) \sim \tilde{F}_j(\mu, \nu)$ if and only if $\tilde{F}_j(\mu', \nu') \sim \tilde{F}_j(\mu+\mu', \nu+\nu')$ $(j=1, \ldots, \kappa)$ for any μ', ν', and that

$$\tilde{F}_1(\mu, \nu) \sim \dots \sim \tilde{F}_\kappa(\mu, \nu) \qquad \text{for any} \quad \mu, \nu.$$

Hence, it is sufficient to be verified that

$$\tilde{F}_1(0, 0) \sim \tilde{F}_\kappa(1, 0) \qquad \text{and} \qquad \tilde{F}_1(0, 0) \sim \tilde{F}_\kappa(0, 1). \tag{70}$$

It is immediately seen that if

$$\begin{pmatrix} \mu_j - \mu_{j-1} \\ \nu_j - \nu_{j-1} \end{pmatrix} = \begin{pmatrix} m_{1, 2j-1} & m_{1, 2j} \\ 0 & m_{2, 2j} \end{pmatrix} \begin{pmatrix} n_{1, 2j-1} \\ n_{1, 2j} \end{pmatrix}$$

then

$$\tilde{F}_j(\mu_{j-1}, \nu_{j-1}) = \tilde{F}_j(\mu_j, \nu_j) \qquad (j=1, \dots, \kappa),$$

and that

$$\tilde{F}_j(\mu_j, \nu_j) \sim \tilde{F}_{j+1}(\mu_j, \nu_j) \qquad (j=1, \dots, \kappa-1),$$

where $n_{1, j}$ $(j=1, \dots, 2\kappa)$ are the system of integers satisfying the condition (11). Then we have that

$$\tilde{F}_1(\mu_0, \nu_0) \sim \tilde{F}_\kappa(\mu_\kappa, \nu_\kappa).$$

If we take $\mu_0=0$, $\nu_0=0$, then by (11) we see that $\mu_\kappa=1$, $\nu_\kappa=0$. Thus we obtain the first relation of (70). The second relation of (70) is also obtained similarly.

Next there exist the conformal transformations T_1 and T_2 of $\tilde{R}$ onto itself which transform an arbitrary point $\tilde{p}$ on $\tilde{F}_j(\mu, \nu)$ for each j $(j=1, \dots, \kappa)$ to the points $\tilde{p}_1$ on $\tilde{F}_j(\mu+1, \nu)$ and $\tilde{p}_2$ on $\tilde{F}_j(\mu, \nu+1)$ such that

$$f_0(\tilde{p}_1) = t_1 \circ f_0(\tilde{p}) \qquad \text{and} \qquad f_0(\tilde{p}_2) = t_2 \circ f_0(\tilde{p}),$$

respectively, where f_0 is the projection map of $\tilde{R}$ onto the z-plane. Thus $\tilde{R}$ admits the covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ generated by T_1 and T_2. Let F be a subregion of $\tilde{R}$ surrounded by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $T_1^{m_{12}} \circ T_2^{m_{22}}(\tilde{\alpha}_1)$, $T_1^{m_{11}}(\tilde{\alpha}_2)$, $\dots$, $\tilde{\alpha}_{2\kappa-1}$, $\tilde{\alpha}_{2\kappa}$, $T_1^{m_{1, 2\kappa}} \circ T_2^{m_{2, 2\kappa}}(\tilde{\alpha}_{2\kappa-1})$ and $T_1^{m_{1, 2\kappa-1}}(\tilde{\alpha}_{2\kappa})$, and thus F consists of the portions $F_1(0, 0)$, $\dots$, $F_\kappa(0, 0)$ connected along the corresponding slits. Let R be a Riemann surface obtained from F by identifying the points of $\tilde{\alpha}_{2j-1}$ and $\tilde{\alpha}_{2j}$ $(j=1, \dots, \kappa)$ with those of $T_1^{m_{1, 2j}} \circ T_2^{m_{2, 2j}}(\tilde{\alpha}_{2j-1})$ and $T_1^{m_{1, 2j-1}}(\tilde{\alpha}_{2j})$ equivalent modulo $\mathfrak{G}$, respectively, and α_j be the images on R of $\tilde{\alpha}_j$ $(j=1, \dots, 2\kappa)$, respectively. Then, $R \equiv \tilde{R}$ (mod $\mathfrak{G}$), for F is a fundamental region of the group $\mathfrak{G}$. Further, on selecting suitably the orientation of $\alpha_1, \dots, \alpha_{2\kappa}$ and the remaining basis $\alpha_{2\kappa-1}, \dots$ on R, we see that the condition (64) is satisfied. Hence, $\tilde{R}$ is the one which satisfies the condition of the present lemma.

21. In **21**$\sim$**24**, we shall concern ourselves with the family (C). First we have a lemma.

LEMMA 12. *If* $\mathfrak{A}(q;\ m_{11}, \dots, m_{1, 2\kappa};\ m_{21}, \dots, m_{2, 2\kappa})$ *belongs to the family* (C), $\iota=1$ *and*

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = \pm 1,$$

then $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa}) \subset O_{MD}$, *where* ι *is the integer in* (C) *of* **18**.

Proof. If there existed R in the class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ of the lemma such that $\tilde{R} \notin O_{MD}$, then by the lemma 3, we would have

$$d_0 = \operatorname{sign}(\Im \lambda) \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = 1. \tag{71}$$

Now we may assume that the system $\alpha_1, \alpha_2, \ldots$ is regular for $\tilde{R}$. Then, by **6** of [3], any branch $\tilde{\alpha}_{2j-1}$ of the images on $\tilde{R}$ of α_{2j-1} $(j=2, \ldots, \kappa)$ forms a cycle on $\tilde{R}$, the surface $\tilde{F}$ obtained from $\tilde{R}$ by scissoring along all branches $\tilde{\alpha}_{2j-1}$ for each j $(j=2, \ldots, \kappa)$ is connected, and thus there exists a cycle on $\tilde{R}$ conjugate to each $\tilde{\alpha}_{2j-1}$. Hence, the genus of $\tilde{R}$ is infinite, which contradicts (71).

22. In **23** and **24**, we shall show that, except for the case of the lemma 12, there always exists an abelian covering surface $\tilde{R}$ with finite spherical area in each class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ of the family (C). For the purpose we shall prepare a lemma in the present section.

Let $\mathfrak{g}$ be the transformation group the basis of which is a system of transformations

$$t_1(z) = z+1,\ t_2(z) = z+i$$

of the finite z-plane $Z = \{|z| < \infty\}$. One of the simplest fundamental regions of $\mathfrak{g}$ is given by the square

$$\Phi_0 = \{0 \leqq \Re z \leqq 1,\ 0 \leqq \Im z \leqq 1\}.$$

We denote

$$\tilde{\alpha}_1^0 = \{\Im z = 0,\ 0 \leqq \Re z \leqq 1\},\ \tilde{\alpha}_2^0 = \{\Re z = 0,\ 0 \leqq \Im z \leqq 1\}.$$

Let K be a bounded set arbitrarily given on Z consisting of a finite number of continua or isolated points $K_1, \ldots, K_n$ which satisfies the conditions:

(i) The complementary set of K is a domain;

(ii) Two distinct points z, z' equivalent each other modulo $\mathfrak{g}$ do not simultaneously belong to K;

(iii) Any lattice point (the point the real part and the imaginary part of which are integers) does not belong to K.

Then we have the following lemma proved in [4].

Lemma 13. *There exist a fundamental region Φ of $\mathfrak{g}$ and a homeomorphic map h of Φ_0 onto Φ which have the properties*:

(a) *The four points* $0, 1, 1+i, i$ *are fixed points of* h;

(b) $h\circ t_1(z)=t_1\circ h(z)$ *for any* $z\in\tilde{\alpha}_2^0$,

$h\circ t_2(z)=t_2\circ h(z)$ *for any* $z\in\tilde{\alpha}_1^0$;

(c) $K\subset(\Phi)^\circ$.

23.[12] LEMMA 14. *If* $\iota\geqq 2$, *then there exists an abelian covering surface* $\tilde{R}$ *with finite spherical area in each class* $\mathfrak{A}(q;\ m_{11},\ \dots,\ m_{1,2\kappa};\ m_{21},\ \dots,\ m_{2,2\kappa})$ *of the family* (C), *where* ι *is the integer in* (C) *of* **18**.

Proof. By the lemma 4, it is sufficient to prove the lemma only for the case

$$(72)\qquad \begin{cases} \alpha_{2j-1}=T_1^{m_{1,2j-1}},\ \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}} & \\ \qquad\qquad (j=1,\ \dots\ \iota;\ 0\leqq m_{1,2j}<m_{1,2j-1},\ m_{2,2j}\neq 0), & \\ \alpha_{2j-1}=I,\ \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}} & (j=\iota+1,\ \dots,\ \kappa), \\ \alpha_j=I & (j=2\kappa+1,\ \dots). \end{cases}$$

Further, by the condition (iii) of **18** we may assume that $m_{2,2j}>0$ $(j=1,\ \dots,\ \iota)$.

Let $\tilde{\alpha}_{2j-1}, \tilde{\alpha}_{2j}$ $(j=1,\ \dots,\ \iota)$ be those defined by (65) for $j=1,\ \dots,\ \iota$ in place of $j=1,\ \dots,\ \kappa$, and let l_j $(j=2,\ \dots,\ \kappa)$ and l_k' $(k=1,\ \dots,\ q-\kappa$, if $q<\infty$; $k=1, 2, \dots$, if $q=\infty)$ be those defined by (68) and (69), respectively, for a disk (66) such that $D\subset\bigcap_{j=0}^{\iota}(\Phi_j)^\circ$ in place of (67).

Let $\mathfrak{g}'$ be the transformation group the basis of which is a system of transformations

$$t_1'(z)=z+m_{11},\quad t_2'(z)=z+m_{12}+im_{22}$$

of the finite z-plane Z. Then, the closed paralellogram Φ_0' surrounded by $\tilde{\alpha}_1$ $\tilde{\alpha}_2$, $t_2'(\tilde{\alpha}_1)$ and $t_1'(\tilde{\alpha}_2)$ is a fundamental region of $\mathfrak{g}'$. Then, there exist a fundamental region Φ' of $\mathfrak{g}'$ and a homeomorphic map h' of Φ_0' onto Φ' which have the properties:

(a') The four points $0,\ m_{11},\ m_{12}+im_{22},\ m_{11}+m_{12}+im_{22}$ are fixed points of h';

(b') $h'\circ t_1'(z)=t_1'\circ h'(z)$ for any $z\in\tilde{\alpha}_2$,

$h'\circ t_2'(z)=t_2'\circ h'(z)$ for any $z\in\tilde{\alpha}_1$;

(c') $$K'\equiv\bigcup_{j=2}^{\iota} l_j\cup\bigcup_{j=\iota+1}^{\kappa} l_j^*\cup\bigcup_k l_k'\subset(\Phi')^\circ,$$

where

$$l_j^*=t_1^{m_{1,2j}}\circ t_2^{m_{2,2j}}(l_j)\qquad (j=\iota+1,\ \dots,\ \kappa).$$

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12) The signatures in the previous section will be taken for the same meaning as those in **23** and **24**.

In fact, we define a homeomorphic (affine) map g of Z onto itself by

$$g(z)=u(x, y)+iv(x, y) \qquad (z=x+iy),$$

$$\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then, g maps homeomorphically Φ_0 onto Φ_0', satisfies

$$g(0)=0,\ g(1)=m_{11},\ g(1+i)=m_{11}+m_{12}+im_{22},$$

$$g(i)=m_{12}+im_{22};$$

$$g(\tilde{\alpha}_1^0)=\tilde{\alpha}_1, \qquad g(\tilde{\alpha}_2^0)=\tilde{\alpha}_2,$$

and further has the property

$$g\circ t_1(z)=t_1'\circ g(z), \qquad g\circ t_2(z)=t_2'\circ g(z).$$

Let K be the homeomorphic image of K' under the inverse map g^{-1}: $K=g^{-1}(K')$. It is easily seen that the present K satisfies the conditions (i), (ii) and (iii) of **22.** Here we should note that the possibility that a number of components of $\cup_k l_k'$ is infinite does not interrupt the validity of the lemma 13. If we apply the lemma 13 to the present K, then it is immediately verified that $\Phi'=g(\Phi)$ and $h'=g\circ h\circ g^{-1}$ have the properties (a′), (b′) and (c′).

Let $\tilde{\alpha}_1'=h'(\tilde{\alpha}_1)$, $\tilde{\alpha}_2'=h'(\tilde{\alpha}_2)$. Then, by (a′) and (b′), $\tilde{\alpha}_1'$ and $\tilde{\alpha}_2'$ are the curves which run from 0 to m_{11} and $m_{12}+im_{22}$, respectively, and $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$, $t_2'(\tilde{\alpha}_1')$ and $t_1'(\tilde{\alpha}_2')$ form the boundary of the fundamental region Φ' of $\mathfrak{g}'$. Since any confusion might not occur we shall denote $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$, for the sake of simplicity, by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ in the following.

Let

$$\tilde{F}_1(0, 0)=\{|z|<\infty\}-\bigcup_{n_1,n_2=-\infty}^{+\infty} t_1^{n_1m_{11}+n_2m_{12}}\circ t_2^{m_2m_{22}}\Big(l_2\smile\bigcup_{j=\iota+1}^{\kappa} l_j^*\smile\bigcup_k l_k'\Big),$$

$$\tilde{F}_j(0, 0)=\{|z|<\infty\}-\bigcup_{n_1,n_2=-\infty}^{+\infty} t_1^{n_1m_{1,2j-1}+n_2m_{1,2j}}\circ t_2^{n_2m_{2,2j}}(l_j\smile l_{j+1}) \qquad (j=2, \dots, \iota-1),$$

$$\tilde{F}_\iota(0, 0)=\{|z|<\infty\}-\bigcup_{n_1,n_2=-\infty}^{+\infty} t_1^{n_1m_{1,2\iota-1}+n_2m_{1,2\iota}}\circ t_2^{n_2m_{2,2\iota}}\Big(\bigcup_{j=\iota}^{\kappa} l_j\smile\bigcup_k l_k'\Big),$$

$$\tilde{F}_j(\mu, \nu)=t_1^{\mu}\circ t_2^{\nu}(\tilde{F}_j(0, 0)) \qquad (j=1, \dots, \iota;\ \mu, \nu:\ \text{integers}),$$

where $\cup_k l_k'=\phi$ if $q=\kappa$. We shall agree that

$$\tilde{F}_j(\mu', \nu')=\tilde{F}_j(\mu, \nu) \qquad (j=1, \dots, \iota),$$

if the system of integers μ, ν; μ', ν' satisfies the relation

$$\begin{pmatrix} \mu'-\mu \\ \nu'-\nu \end{pmatrix}=\begin{pmatrix} m_{1,2j-1} & m_{1,2j} \\ 0 & m_{2,2j} \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

for some integers n_1, n_2. Then, the system of all mutually distinct $F_j(\mu, \nu)$ is given by

$$\tilde{F}_j(\mu, \nu) \qquad (\mu=0, \ldots, m_{1,2j-1}-1;\ \nu=0, \ldots, m_{2,2j}-1;\ j=1, \ldots, \iota).$$

We draw the curves $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$ on $\tilde{F}_j(0, 0)$ $(j=1, \ldots, \iota)$, respectively, and let $F_j(0, 0)$ $(j=1, \ldots, \iota)$ be the subregion of $\tilde{F}_j(0, 0)$ surrounded by $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$, $t_1^{m_{1,2j}}\circ t_2^{m_{2,2j}}(\tilde{\alpha}_{2j-1})$ and $t_1^{m_{1,2j-1}}(\tilde{\alpha}_{2j})$. Then, by (c′), we have

$$F_1(0, 0)=\Phi'-\left(l_2\cup\bigcup_{j=\iota+1}^{\kappa} l_j{}^*\cup\bigcup_{k} l_k'\right).$$

Further, we have obviously

$$F_j(0, 0)=\Phi_j-(l_j\cup l_{j+1}) \qquad (j=2, \ldots, \iota-1),$$

$$F_\iota(0, 0)=\Phi_\iota-\left(\bigcup_{j=\iota}^{\kappa} l_j\cup\bigcup_{k} l_k'\right),$$

where Φ_j $(j=2, \ldots, \iota)$ are the closed parallelograms defined in the lemma 11. Let $\tilde{R}$ be the surface obtained from

$$\tilde{F}_j(\mu, \nu) \qquad (\mu=0, \ldots, m_{1,2j-1}-1;\ \nu=0, \ldots, m_{2,2j}-1;\ j=1, \ldots, \iota)$$

by connecting crosswise along each pair of the slits with the common projection on the z-plane, where each slit corresponds obviously to one and only one slit.

First we can see as follows that $\tilde{R}$ is connected. By the reasoning similar to the lemma 11, it is sufficient to be verified that

(73) $$\tilde{F}_1(0, 0)\sim\tilde{F}_\iota(1, 0) \quad\text{and}\quad \tilde{F}_1(0, 0)\sim\tilde{F}_{\iota-1}(0, 1).$$

By the reasoning similar to the lemma 11, if

(74) $$\begin{pmatrix} \mu_\iota-\mu_0 \\ \nu_\iota-\nu_0 \end{pmatrix}=\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1,2\iota-1} & m_{1,2\iota} \\ 0 & m_{22} & \cdots & 0 & m_{2,2\iota} \end{pmatrix}\begin{pmatrix} n_{11} \\ \vdots \\ n_{1,2\iota} \end{pmatrix}$$

then

(75) $$\tilde{F}_1(\mu_0, \nu_0)\sim\tilde{F}_\iota(\mu_\iota, \nu_\iota).$$

Further, we note that

$$\tilde{F}_1(0, 0)\sim\cdots\sim\tilde{F}_\iota(0, 0)\sim\tilde{F}_\iota(m_{1,2j}, m_{2,2j}) \qquad (j=\iota+1, \ldots, \kappa).$$

Then, we have that

$$\tilde{F}_\iota(\mu_{j-1}, \nu_{j-1}) \sim \tilde{F}_\iota(\mu_j, \nu_j) \qquad (j=\iota+1, \dots, \kappa), \tag{76}$$

provided

$$\begin{pmatrix} \mu_j-\mu_{j-1} \\ \nu_j-\nu_{j-1} \end{pmatrix} = \begin{pmatrix} 0 & m_{1,2j} \\ 0 & m_{2,2j} \end{pmatrix} \begin{pmatrix} n_{1,2j-1} \\ n_{1,2j} \end{pmatrix} \qquad (j=\iota+1, \dots, \kappa). \tag{77}$$

By (75) and (76)

$$\tilde{F}_1(\mu_0, \nu_0) \sim \tilde{F}_\iota(\mu_\kappa, \nu_\kappa).$$

Further, if we take $\mu_0=0$, $\nu_0=0$, then, by (74), (77) and the condition (11), we know that $\mu_k=1$, $\nu_k=0$. Thus we obtain the first relation of (73). The second relation of (73) is also obtained similarly.

Next there exist the conformal transformations T_1 and T_2 of $\tilde{R}$ onto itself which transform an arbitrary point $\tilde{p}$ on $\tilde{F}_j(\mu, \nu)$ for each j ($j=1, \dots, \iota$) to the points $\tilde{p}_1$ on $\tilde{F}_j(\mu+1, \nu)$ and $\tilde{p}_2$ on $\tilde{F}_j(\mu, \nu+1)$ such that

$$f_0(\tilde{p}_1)=t_1 \circ f_0(\tilde{p}) \qquad \text{and} \qquad f_0(\tilde{p}_2)=t_2 \circ f_0(\tilde{p}),$$

respectively, where f_0 is the projection map of $\tilde{R}$ onto the z-plane. Thus $\tilde{R}$ admits the covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ generated by the basis T_1 and T_2. Let F be a bounded subregion of $\tilde{R}$ obtained by scissoring along $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$, $T_1^{m_{1,2j}} \circ T_2^{m_{2,2j}}(\tilde{\alpha}_{2j-1})$, $T_1^{m_{1,2j-1}}(\tilde{\alpha}_{2j})$ ($j=1, \dots, \iota$) and along both shores of slits l_j, l_j^* ($j=\iota+1, \dots, \kappa$). Then F consists of the portions $F_1(0, 0), \dots, F_\iota(0, 0)$ connected crosswise along each pair of the slits l_j ($j=2, \dots, \iota$) and l_k' ($k=1, \dots, q-\kappa$, if $q<\infty$; $k=1, 2, \dots$, if $q=\infty$) with the common projection on the w-plane, where the slits l_j, l_j^* ($j=\iota+1, \dots, \kappa$) remain free from the connecting process. Let F' be the subregion of F obtained by removing from F a domain G surrounded by two Jordan curves C_1 and C_ι surrounding $\cup_k l_k'$ in $(F_1(0, 0))^\circ$ and $(F_\iota(0, 0))^\circ$, respectively, but not surrounding l_2, l_j ($j=\iota, \dots, \kappa$) and l_j^* ($j=\iota+1, \dots, \kappa$). Then, F' is connected and thus we can draw a simple curve $\tilde{\alpha}_{2j}$ on $(F')^\circ$ which runs from the mid-point of the upper shore of l_j to the mid-point of the lower shore of l_j^* for each j ($j=\iota+1, \dots, \kappa$) such that $\tilde{\alpha}_{2j}$ ($j=\iota+1, \dots, \kappa$) mutually have no common points. Further we can draw a Jordan curve $\tilde{\alpha}_{2j-1}$ on $(F')^\circ$ surrounding l_j for each j ($j=\iota+1, \dots, \kappa$) which intersects $\tilde{\alpha}_{2j}$ only once but not $\tilde{\alpha}_{2k-1}$ and $\tilde{\alpha}_{2k}$ for $k \neq j$. Let R be a Riemann surface obtained from F by identifying the points of $\tilde{\alpha}_{2j-1}$, $\tilde{\alpha}_{2j}$ ($j=1, \dots, \iota$) and l_j ($j=\iota+1, \dots, \kappa$) with those of $T_1^{m_{1,2j}} \circ T_2^{m_{2,2j}}(\tilde{\alpha}_{2j-1})$, $T_1^{m_{1,2j-1}}(\tilde{\alpha}_{2j})$ and l_j^* equivalent modulo $\mathfrak{G}$, respectively, where l_j ($j=\iota+1, \dots, \kappa$) are connected crosswise with l_j^*, respectively. Then $R \equiv \tilde{R}$ (mod $\mathfrak{G}$), for F is a fundamental region of the group $\mathfrak{G}$. Let α_j be the images on R of $\tilde{\alpha}_j$ ($j=1, \dots, 2\kappa$), respectively. Then each pair α_{2j-1} and α_{2j} ($j=1, \dots, \kappa$) forms a system of conjugate cycles on R. We can select the remaining elements $\alpha_{2\kappa+1}$, … of the canonical homology basis on the subdomain G of R. Then, on taking a suitable orientation of $\alpha_1, \alpha_2, \dots$, we see that the condition (72) is satisfied. Hence, $\tilde{R}$ is the one which satisfies the condition of the present lemma.

24. LEMMA 15. *If $\iota=1$ and*

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \neq \pm 1, \tag{78}$$

then there exists an abelian covering surface $\tilde{R}$ with finite spherical area in each class $\mathfrak{A}(q;\ m_{11},\ \ldots,\ m_{1,2\kappa};\ m_{21},\ \ldots,\ m_{2,2\kappa})$ of the family (C), *where ι is the integer in* (C) *of* **18**.

Proof. By the lemma 4, it is sufficient to prove the lemma only for the case

$$\begin{cases} \alpha_1=T_1^{m_{11}}, & \alpha_2=T_1^{m_{12}}\circ T_2^{m_{22}} \qquad (0\leqq m_{12}<m_{11},\ m_{22}\neq 0), \\ \alpha_{2j-1}=I, & \alpha_{2j}=T_1^{m_{1,2j}}\circ T_2^{m_{2,2j}} \qquad (j=2,\ \ldots,\ \kappa), \\ \alpha_j=I & \qquad\qquad\qquad\qquad (j=2\kappa+1,\ \ldots). \end{cases} \tag{79}$$

Further, we may assume that $m_{22}>0$. By (78) and the condition (11), we can see that there exists at least one number k $(2\leqq k\leqq\kappa)$ such that there does not hold

$$\begin{pmatrix} m_{1,2k} \\ m_{2,2k} \end{pmatrix}=\begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix}\begin{pmatrix} \mu_{1,2k} \\ \mu_{2,2k} \end{pmatrix}$$

for any system of integers $\mu_{1,2k}$, $\mu_{2,2k}$. Without loss of generality, we may assume that $j=2,\ \ldots,\ \iota'$ $(2\leqq\iota'\leqq\kappa)$ are all the numbers for which

$$\begin{pmatrix} m_{1,2j} \\ m_{2,2j} \end{pmatrix}=\begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix}\begin{pmatrix} \mu_{1,2j} \\ \mu_{2,2j} \end{pmatrix}$$

does not hold for any system of integers $\mu_{1,2j}$, $\mu_{2,2j}$. Let $\mathfrak{g}'$ be the transformation group the basis of which is a system of transformations

$$t_1'(z)=z+m_{11},\quad t_2'(z)=z+m_{12}+im_{22}$$

of the finite z-plane Z. Then we can easily see that, for any point $z\in Z$, z and $t_1^{m_{1,2j}}\circ t_2^{m_{2,2j}}(z)$ $(j=2,\ \ldots,\ \iota')$ are not mutually equivalent modulo $\mathfrak{g}'$, but z and $t_1^{m_{1,2j}}\circ t_2^{m_{2,2j}}(z)$ $(j=\iota'+1,\ \ldots,\ \kappa)$ are mutually equivalent modulo $\mathfrak{g}'$.

Let

$$\begin{cases} \tilde{\alpha}_1=\{\Im z=0,\ 0\leqq\Re z\leqq m_{11}\}, \\ \tilde{\alpha}_2=\left\{\arg z=\tan^{-1}\dfrac{m_{22}}{m_{12}},\ 0\leqq|z|\leqq\sqrt{m_{12}^2+m_{22}^2}\right\}, \end{cases}$$

and let l_j $(j=2,\ \ldots,\ \kappa)$ and l_k' $(k=1,\ \ldots,\ q-\kappa$, if $q<\infty$; $k=1,\ 2,\ \ldots$, if $q=\infty)$ be those defined by (68) and (69), respectively, for a disk (66) such that $D\subset(\Phi_0\frown\Phi_1)^\circ$ in place of (67). Further let

$$l_j^* = \begin{cases} T_1^{m_{1,2j}} \circ T_2^{m_{2,2j}}(l_j) & (j=2, \ldots, \iota') \\ T_1^{m_{14}+m_{1,2j}} \circ T_2^{m_{24}+m_{2,2j}}(l_j) & (j=\iota'+1, \ldots, \kappa), \end{cases}$$

$$l_k'^* = T_1^{m_{14}} \circ T_2^{m_{24}}(l_k') \qquad (k=1, \ldots, q-\kappa, \text{ if } q<\infty;\ k=1, 2, \ldots, \text{ if } q=\infty).$$

Then, we can easily see that the set

$$K' \equiv \bigcup_{j=2}^{\kappa} (l_j \cup l_j^*) \cup \bigcup_k (l_k' \cup l_k'^*)$$

does not simultaneously contain two distinct points z, z' equivalent each other modulo $\mathfrak{g}'$. Then, by the method similar to **23**, we know that there exists a fundamental region Φ' of $\mathfrak{g}'$ and a homeomorphic map h' of Φ_0' onto Φ' which satisfy the conditions (a′), (b′) of **23**, and

$$\text{(c}''\text{)} \qquad K' \equiv \bigcup_{j=2}^{\kappa} (l_j \cup l_j^*) \cup \bigcup_k (l_k' \cup l_k'^*) \subset (\Phi')^\circ.$$

Let $\tilde{\alpha}_1' = h'(\tilde{\alpha}_1)$, $\tilde{\alpha}_2' = h'(\tilde{\alpha}_2)$. Then, by (a′) and (b′), $\tilde{\alpha}_1'$ and $\tilde{\alpha}_2'$ are the curves which run from 0 to m_{11} and $m_{12}+im_{22}$, respectively, and $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$, $t_2'(\tilde{\alpha}_1')$ and $t_1'(\tilde{\alpha}_2')$ form the boundary of the fundamental region Φ' of $\mathfrak{g}'$. Since any confusion might not occur we shall denote $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$, for the sake of simplicity, by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ in the following.

Let

$$\tilde{F}(0, 0) = \{|z|<\infty\} - \bigcup_{n_1, n_2=-\infty}^{+\infty} t_1^{n_1 m_{11}+n_2 m_{12}} \circ t_2^{n_2 m_{22}} \left(\bigcup_{j=2}^{\kappa} (l_j \cup l_j^*) \cup \bigcup_k (l_k' \cup l_k'^*) \right),$$

$$\tilde{F}(\mu, \nu) = t_1^{\mu} \circ t_2^{\nu}(\tilde{F}(0, 0)) \qquad (\mu, \nu\text{: integers}),$$

where $\cup_k l_k' = \phi$ if $q=\kappa$. We shall agree that

$$\tilde{F}(\mu', \nu') = \tilde{F}(\mu, \nu),$$

if the system of integers μ, ν, μ', ν' satisfies the relation

$$\begin{pmatrix} \mu'-\mu \\ \nu'-\nu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

for some integers n_1, n_2. Then, the system of all mutually distinct $\tilde{F}(\mu, \nu)$ is given by

$$\tilde{F}(\mu, \nu) \qquad (\mu=0, \ldots, m_{11}-1;\ \nu=0, \ldots, m_{22}-1).$$

We draw the curves $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ on $\tilde{F}(0, 0)$, and let $F(0, 0)$ be the subregion of $\tilde{F}(0, 0)$ surrounded by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $t_1^{m_{12}} \circ t_2^{m_{22}}(\tilde{\alpha}_1)$ and $t_1^{m_{11}}(\tilde{\alpha}_2)$. Then, by (c″) we have

$$F(0, 0)=\Phi'-\left(\bigcup_{j=2}^{\kappa}(l_j\cup l_j^*)\cup\bigcup_{k}(l_k'\cup l_k'^*)\right).$$

Let $\tilde{R}$ be the Riemann surface obtained from

$$\tilde{F}(\mu, \nu) \qquad (\mu=0, \ldots, m_{11}-1;\ \nu=0, \cdots, m_{22}-1)$$

by connecting crosswise along each pair of the slits with the common projection on the w-plane, where each slit corresponds obviously to one and only one slit. We can see by the method similar to **23** that $\tilde{R}$ is connected. Further there exist the conformal transformations T_1 and T_2 of $\tilde{R}$ onto itself which transform arbitrary point $\tilde{p}$ on $\tilde{F}(\mu, \nu)$ to the points $\tilde{p}_1$ and $\tilde{p}_2$ on $\tilde{F}(\mu+1, \nu)$ and $\tilde{F}(\mu, \nu+1)$ such that

$$f_0(\tilde{p}_1)=t_1\circ f_0(\tilde{p}) \qquad \text{and} \qquad f_0(\tilde{p}_2)=t_2\circ f_0(\tilde{p}),$$

respectively, where f_0 is the projection map of $\tilde{R}$ onto the z-plane. Thus $\tilde{R}$ admits the covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ generated by T_1 and T_2. We can easily see that $F(0, 0)$ is a fundamental region of the group $\mathfrak{G}$.

Let p_2 and p_j $(j=\iota'+1, \ldots, \kappa)$ be mutually distinct points on the upper shore of l_2, and p_2^* and p_j^* $(j=\iota'+1, \ldots, \kappa)$ be the points on the lower shore of l_2^* equivalent with p_2 and p_j modulo $\mathfrak{g}=\{t_1, t_2\}$, respectively.[13] Let $\tilde{\alpha}_3$ be a Jordan curve surrounding $l_2\cup\bigcup_{j=\iota'+1}^{\kappa} l_j\cup\bigcup_k l_k'$ in $(F(0, 0))^\circ$ but not surrounding l_j $(j=3, \ldots, \iota')$, l_j^* $(j=2, \ldots, \kappa)$ and all $l_k'^*$, and let G_3 be the Jordan domain surrounded by $\tilde{\alpha}_3$. Let C and C^* be Jordan curves surrounding $\bigcup_k l_k'$ and $\bigcup_k l_k'^*$ in $(F(0, 0))^\circ$, respectively, but not surrounding l_j and l_j^* $(j=2, \ldots, \kappa)$ such that $C\subset G_3$, and let G and G^* be subdomains of $F(0, 0)$ surrounded by C and C^*, respectively. Let $\tilde{\alpha}_4$ be a simple curve on $(F(0, 0))^\circ-\bar{G}-\bar{G}^*$ which runs from p_2 to p_2^* and intersects $\tilde{\alpha}_3$ only once. Further, let $\tilde{\alpha}_{2j}'$ be a simple curve on $G_3-\bar{G}$ which runs from the mid-point of the upper shore of l_j to p_j for each j $(j=\iota'+1, \ldots, \kappa)$ and such that $\tilde{\alpha}_4$ and $\tilde{\alpha}_{2j}'$ $(j=\iota'+1, \ldots, \kappa)$ mutually have no common points, and let $\tilde{\alpha}_{2j-1}$ $(j=\iota'+1, \ldots, \kappa)$ be a Jordan curve on $G_3-\bar{G}$ surrounding l_j but not surrounding l_k for $k\neq j$ which intersects $\tilde{\alpha}_{2j}'$ only once but not $\tilde{\alpha}_4$, $\tilde{\alpha}_{2k-1}$ and $\tilde{\alpha}_{2k}'$ for $k\neq j$. Let $\tilde{\alpha}_{2j}''$ be a simple curve on $(F(0, 0))^\circ-\bar{G}_3-\bar{G}^*$ which runs from p_j^* to the mid-point of the lower shore of l_j^* for each j $(j=\iota'+1, \ldots, \kappa)$ and such that $\tilde{\alpha}_{2j}''$ $(j=\iota'+1, \ldots, \kappa)$ mutually have no common points, and let $\tilde{\alpha}_{2j}=\tilde{\alpha}_{2j}'\cup\tilde{\alpha}_{2j}''$ $(j=\iota'+1, \ldots, \kappa)$. Let $\tilde{\alpha}_{2j}$ $(j=3, \ldots, \iota')$ be simple curves on $(F(0, 0))^\circ-\bar{G}_3-\bar{G}^*-\bigcup_{j=\iota'+1}^{\kappa}\tilde{\alpha}_{2j}''$ which run from the mid-points of the upper shores of l_j $(j=3, \ldots, \iota')$ to the mid-points of the lower shores of l_j^* $(j=3, \ldots, \iota')$, respectively, and such that $\tilde{\alpha}_{2j}$ $(j=3, \ldots, \iota')$ mutually have no common points. Let $\tilde{\alpha}_{2j-1}$ $(j=3, \ldots, \iota')$ be a Jordan curve on $(F(0, 0))^\circ-\bar{G}_3-\bar{G}^*-\bigcup_{j=\iota'+1}^{\kappa}\tilde{\alpha}_{2j}''$ surrounding l_j but not surrounding l_k for $k\neq j$ and any l_j^* which intersects $\tilde{\alpha}_{2j}$ $(j=3, \ldots, \iota')$ only once but not $\tilde{\alpha}_{2k-1}$ and $\tilde{\alpha}_{2k}$ for $k\neq j$.

Let R be the Riemann surface obtained from $F(0, 0)$ by identifying the points

13) We shall take that $\{p_j, p_j^*\}_{j=\iota'+1}^{\kappa}$ is vacuous in the case $\iota'=\kappa$. The similar note should be taken in the following.

of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, l_j $(j=2, \ldots, \kappa)$ and l_k' for each k with those of $T_1^{m_{12}} \circ T_2^{m_{22}}(\tilde{\alpha}_1)$, $T_1^{m_{11}}(\tilde{\alpha}_2)$, l_j^* and $l_k'^*$ equivalent modulo $\mathfrak{G}$, respectively, where l_j $(j=2, \ldots, \kappa)$ and l_k' are connected crosswise with l_j^* and $l_k'^*$, respectively. Then $R \equiv \tilde{R}$ (mod $\mathfrak{G}$). Let α_j be the images on R of $\tilde{\alpha}_j$ $(j=1, \ldots, 2\kappa)$, respectively. Then each pair α_{2j-1} and α_{2j} $(j=1, \ldots, \kappa)$ forms a system of conjugate cycles on R. We can select the remaining elements $\alpha_{2\kappa+1}, \ldots$ of the canonical homology basis on the subdomain of R corresponding to G and G^*. Then, on taking a suitable orientation of $\alpha_1, \alpha_2, \ldots$, we see that the condition (79) is satisfied. Hence, $\tilde{R}$ is the one which satisfies the condition of the present lemma.

25. On summing up the results of **16**$\sim$**24**, we obtain the following theorem.

THEOREM 4. *Let R be a Riemann surface of the class O_G, $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$. Let $\tilde{R}$ be an abelian covering surface of R which is of the class O_G and whose covering transformation group $\mathfrak{G}=\{T_1, T_2\}$ is of the type* II. *of* **2**. *If $\tilde{R}$ has finite spherical area, then $\tilde{R}$ and $\mathfrak{G}$ satisfies the conditions* (i)$\sim$(vi):

(i) *No dividing cycle on R can be a non-trivial generator of $\mathfrak{G}$* (LEMMA 5);

(ii) *Only a finite number of α_j can be non-trivial generators of $\mathfrak{G}$* (LEMMA 6);

(iii) *$\{\alpha_j\}$ has the uni-orientation with respect to $\mathfrak{G}$* (LEMMA 7);

(iv) *$\tilde{R}$ does not belong to any class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ of the family* (A) (LEMMA 8);

(v) *$\tilde{R}$ does not belong to a class $\mathfrak{A}(q;\ m_{11}, m_{12};\ m_{21}, m_{22})$ $(2 \leqq q \leqq \infty)$ of the family* (B) (LEMMA 9);

(vi) *$\tilde{R}$ does not belong to a class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ of the family* (C) *such that $\iota=1$ and*

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = \pm 1,$$

where ι is the integer in (C) *of* **18** (LEMMA 12).

On the other hand, there always exists an abelian covering surface $\tilde{R}$ with finite spherical area in each class $\mathfrak{A}(q;\ m_{11}, \ldots, m_{1,2\kappa};\ m_{21}, \ldots, m_{2,2\kappa})$ except for the classes in (iv), (v) *and* (vi) (LEMMAS 10, 11, 14, 15).

REMARK 1. It is admitted in the theorem 4 that R is a closed Riemann surface, where the conditions (i), (ii) are trivial.

REMARK 2. On the class of the abelian covering surfaces of the class O_G with finite spherical area, there is a notable difference on the topological property between the cases I. (H) and II. of **2**. The theorem 4 should be compared with the theorem 3 of [3].

26. Let R be a Riemann surface of the class O_G, $\alpha_1, \alpha_2, \ldots$ be a canonical homology basis of R modulo the ideal boundary $\mathfrak{J}$. Throughout the present section, we assume that $\tilde{R}$ is an abelian covering surface of R which is of the class O_G and whose covering transformation group $\mathfrak{G}=\{T\}$ is of the type I. (H) of **2**. By the theorem 1 of [3], no dividing cycle on R can be a non-trivial generator of $\mathfrak{G}$ and only a finite number of $\alpha_1, \alpha_2, \ldots$ are non-trivial generators of $\mathfrak{G}$. Then we may assume that

$$\begin{cases} \alpha_j = T^{m_j} & (j=1, \ldots, 2\kappa),\ (m_1, \ldots, m_{2\kappa})=1, \\ \alpha_j = I & (j=2\kappa+1, \ldots), \end{cases}$$

for a $\kappa \geqq 1$, for T is a basis of $\mathfrak{G}$, where m_j $(j=1, \ldots, 2\kappa)$ are integers being not $m_{2j-1}=m_{2j}=0$ for any j $(j=1, \ldots, \kappa)$ and $(m_1, \ldots, m_{2\kappa})$ denotes the greatest common measure of the integers $m_1, \ldots, m_{2\kappa}$.

Let $\{m_j\}_{j=1}^{2\kappa}$ be a system of integers with $(m_1, \ldots, m_{2\kappa})=1$ being not $m_{2j-1}=m_{2j}=0$ for any j $(j=1, \ldots, \kappa)$. Let $\mathfrak{A}(q;\ m_1, \ldots, m_{2\kappa})$ be the class of the abelian covering surface $\tilde{R}$ with finite spherical area such that $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) are open or closed Riemann surface of finite or infinite genus q $(\kappa \leqq q \leqq \infty)$, and such that there exists a basis $\alpha_1, \alpha_2, \ldots$ on R which has the forms

$$\begin{cases} \alpha_j = T^{m_j} & (j=1, \ldots, 2\kappa), \\ \alpha_j = I & (j=2\kappa+1, \ldots) \end{cases}$$

as generators of $\mathfrak{G}$. Then, by the theorem 3 of [3], $\mathfrak{A}(q;\ m_1, \ldots, m_{2\kappa}) \neq \phi$.

In the present section, we shall verify the theorem.

THEOREM 5. *Let*

$$d_0 = \min_{f\in\mathfrak{V}(\tilde{R})} \max_{w} \mathfrak{v}_f(w). \tag{80}$$

Then

$$\min_{\tilde{R}\in\mathfrak{A}(q;\ m_1,\cdots,m_{2\kappa})} d_0 = \begin{cases} 1 & (m_0=1,\ q=1), \\ 2 & (m_0=1,\ q\geqq 2), \\ m_0 & (m_0 \geqq 2), \end{cases} \tag{81}$$

where $m_0 = \min_{1\leqq j\leqq\kappa} (m_{2j-1}, m_{2j})$.

Proof. We have already known that the function attaining d_0 of (80) is given by the function f_0 of the theorem 2 of [3] (cf. **21** of [3]) and further between d_0 and $m_1, \ldots, m_{2\kappa}$ there holds the relation of (23) of [3]:

$$\sum_{j=1}^{\kappa} (m_{2j-1}m_{2j}^* - m_{2j}m_{2j-1}^*) = d_0 \tag{82}$$

for a system of integers m_j^* $(j=1, \ldots, 2\kappa)$.

Let $\bar{\alpha}_1, \bar{\alpha}_2, \ldots$ be the canonical homology basis regular for $\tilde{R}$ constructed from $\alpha_1, \alpha_2, \ldots$ such that

$$\begin{cases} \bar{\alpha}_{2j-1}=I, \\ \bar{\alpha}_{2j}=T^{\bar{m}_j}, \end{cases} \qquad (j=1, \dots, \kappa),$$

where $\bar{m}_j=(m_{2j-1}, m_{2j})>0$ (cf. the lemma 10 of [3]). By the lemma 10 of [3], it is sufficient to prove the theorem for such a regular canonical homology basis for $\tilde{R}$. Then (82) takes the form

$$\sum_{j=1}^{\kappa} \bar{m}_j{}^* \bar{m}_j = d_0, \tag{83}$$

and by the method similar to the lemma 7 it can be shown that

$$\bar{m}_j{}^* \geqq 0 \qquad (j=1, \dots, \kappa).$$

First it is obvious by (83) that the minimum d_0 is not smaller than the value of the right hand side of (81).

Next we shall construct the abelian covering surface $\tilde{R}$ attaining the value of the right hand side of (81) as d_0.

(i) The case $m_0=1$, $q=1$:
It is easily constructed.

(ii) The case $m_0=1$, $q\geqq 2$:
Without loss of generality, we may assume that $\bar{m}_1=m_0$. By the method of the proof of the lemma 15, we construct the abelian covering surface $\tilde{R}_2$ of the type II. of 2 which has the covering transformation group $\mathfrak{G}$ with the system of generators such that

$$\begin{cases} \alpha_1=T_1{}^2, \ \alpha_2=T_2, \\ \alpha_{2j-1}=I, \quad \alpha_{2j}=T_1\circ T_2{}^{\bar{m}_j} \qquad (j=2, \dots, \kappa), \\ \alpha_j=I \qquad\qquad (j=2\kappa+1, \dots), \end{cases}$$

and which has finite spherical area. Let F be the subregion of $\tilde{R}_2$ lying over the parallel strip $\{0\leqq\Re z\leqq 1\}$, and $\tilde{R}$ be the Riemann surface constructed from F on identifying points of ∂F equivalent modulo T_1. Then it is easily seen that $\tilde{R}$ is the desired one.

(iii) The case $m_0\geqq 2$:
We may assume that $\bar{m}_1=m_0$. By the method of the proof of the lemma 15, we construct the abelian covering surface $\tilde{R}_2$ with the system of generators:

$$\begin{cases} \alpha_1=T_1, \ \alpha_2=T_2{}^{\bar{m}_1}, \\ \alpha_{2j-1}=I, \quad \alpha_{2j}=T_2{}^{\bar{m}_j} \qquad (j=2, \dots, \kappa), \\ \alpha_j=I \qquad\qquad (j=2\kappa+1, \dots). \end{cases}$$

Then, by the method similar to (ii), we can construct the desired $\tilde{R}$ from $\tilde{R}_2$.

27. Let $\tilde{R}$ be an abelian covering surface the covering transformation group $\mathfrak{G}$ of which is of the type I. (P) of **2**, throughout the present section. By the theorem 1 of the previous paper [3], we have that $\tilde{R}$ of the type I. of **2** is of the type I. (P) if and only if either

(i) There exists a dividing cycle on $R\equiv\tilde{R}(\mathfrak{G})$ being a non-trivial generator of $\mathfrak{G}$; or

(ii) There exists an infinite number of the elements of a canonical homology basis modulo $\mathfrak{Z}$ being non-trivial generators of $\mathfrak{G}$.

Thus we see that there holds (i) or (ii) for $\tilde{R}$ of the type I. (P) even if $\tilde{R}$ has finite spherical area, which shows a notable difference from the cases of the types I. (H) and II. (cf. the lemmas 7 and 8 of [3] and the lemmas 5 and 6). In fact, we can easily construct examples of $\tilde{R}$ with finite spherical area which have the properties (i) or (ii). Further, by the theorem 1, the differential $f_0'(\tilde{p})d\zeta$ can be regarded as an abelian differential of the first kind on $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) and

$$D_R(f_0)=\infty,$$

which shows again a notable difference from the cases of types I. (H) and II. where there hold

$$D_R(\log f_0)<\infty \qquad \text{and} \qquad D_R(f_0)<\infty,$$

respectively.

By the above reasoning, it seems that it is difficult to obtain the results similar to the theorems 3 and 4 for the case of the type I. (P).

28. Let $\tilde{R}$ be an abelian covering surface the covering transformation group $\mathfrak{G}$ of which is of the type III. of **2**, throughout the present section. Then no more $\tilde{R}$ cannot belong to the class O_G even if $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) is closed (cf. [5]). Let $\mathfrak{G}_j$ $(j=1, \dots, N;\ N\geqq 3)$ be the subgroups of $\mathfrak{G}$ generated by the only one element T_j of the basis of $\mathfrak{G}$, respectively. Then, each $\mathfrak{G}_j$ $(j=1, \dots, N)$ is of the type I. (P). Here we shall assume that

(i) there holds the same conclusion as in the theorem 1 for the present $\tilde{R}$ and $\mathfrak{G}_j$ $(j=1, \dots, N)$ in place of $\mathfrak{G}$ of the theorem 1.[14)]

Then, we know by the method similar to the proof of the theorem 2 that if $\tilde{R}$ has finite spherical area there exists a function $f_0\in\mathfrak{B}(\tilde{R})$ which satisfies the conditions

$$f_0\circ T_j(\tilde{p})=f_0(\tilde{p})+\lambda_j \qquad (j=1, \dots, N). \tag{84}$$

On the other hand, it is obviously impossible that there exists such a function f_0 for the case $N\geqq 3$. Thus we have that $\tilde{R}\in O_{MD}$. Unfortunitely, we have not yet known if (i) is true.

Conjecture. Let $\tilde{R}$ be an abelian covering surface the covering transformation group $\mathfrak{G}$ of which is of the type III. and such that $R\equiv\tilde{R}$ (mod $\mathfrak{G}$) belongs to the class O_G. Then $\tilde{R}$ would have in finite spherical area.

14) Here we should note that $\tilde{R}\notin O_G$. If $\tilde{R}\in O_G$, (i) would follow from the theorem 1.

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