INFINITESIMAL TRANSFORMATIONS OF A MANIFOLD WITH f-STRUCTURE

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Professor Yano [3] introduced the concept of f-structure on an n-dimensional differentiable manifold and investigated it from the global viewpoint. The f-structure may be regarded as a generalization of the almost complex structure and the almost contact structure. The main purpose of this paper is to study such an infinitesimal transformation v^h of a differentiable manifold with f-structure as leaves the structure tensor f_i^h invariant, that is, $\underset{v}{\mathcal{L}} f_i^h=0$.

§1. Preliminaries.

We consider an n-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{x^h\}$, and a tensor field f_i^h of type (1, 1) and of class C^∞ satisfying

$$f_i^t f_t^s f_s^h + f_i^h = 0, \tag{1. 1}$$

where the Latin indices run over 1, 2, ⋯, n.

In a manifold with (1. 1), the operations

$$l_i^h = -f_i^t f_t^h \qquad \text{and} \qquad m_i^h = f_i^t f_t^h + \delta_i^h \tag{1. 2}$$

applied to the tangent space at a point of the manifold are complementary projection operators. Thus there exist complementary distributions L and M corresponding to the projection operators l_i^h and m_i^h, respectively.

If the rank of f is r, then we call such a structure an *f-structure of rank r* ($r \leqq n$). If the rank of f is n, then $l_i^h = -\delta_i^h$ and $m_i^h=0$, so that we find that the f-structure of rank n is an almost complex structure. And if the rank of f is $n-1$, then the distribution L is $(n-1)$-dimensional and the distribution M is one dimensional, consequently m_i^h should have the form $m_i^h = p^h q_i$, where p^h and q_i are contravariant and covariant vector fields respectively. Therefore, we find that the f-structure of rank $(n-1)$ is an almost contact structure defined by Sasaki [1]. (Yano [3].)

Making use of (1. 1) and (1. 2), we find

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$$l_i^t f_t^h = f_i^t l_t^h = f_i^h, \qquad l_i^t l_t^h = l_i^h, \tag{1.3}$$

$$m_i^t m_t^h = m_i^h, \tag{1.4}$$

$$f_i^t m_t^h = l_i^t m_t^h = l_t^h m_i^t = 0. \tag{1.5}$$

In a manifold with f-structure of rank r, we put

$$*O_{ji}^{ts} = \frac{1}{2}(l_j^t l_i^s + 2m_j^t m_i^s + f_j^t f_i^s). \tag{1.6}$$

This operation is a formal generalization of purity of an almost complex manifold to the manifold. For example, we have

$$*O_{js}^{ti} f_t^s = 0, \qquad *O_{js}^{ti} l_t^s = 0, \tag{1.7}$$

$$*O_{js}^{ti} m_t^s = m_j^i. \tag{1.8}$$

It is known [3] that a manifold with f-structure of rank r always admits a positive definite Riemannian metric tensor g_{ih} such that

$$f_i^t f_h^s g_{ts} = g_{ih} - m_{ih}, \qquad \text{where} \qquad m_{ih} \equiv m_i^t g_{th}. \tag{1.9}$$

From which we see that the tensor m_{ih} is a symmetric one.

If an f-structure of rank r admits a positive definite Riemannian metric defined by (1. 9), then we shall call the structure an *(f_r, g)-structure.*

In a manifold with (f_r, g)-structure, using (1. 6) the equation (1. 9) can be written as

$$*O_{ih}^{ts} g_{ts} = g_{ih}. \tag{1.10}$$

Transvecting this equation with f_j^i, it follows that

$$*O_{jh}^{ts} f_{ts} = f_{jh}, \qquad \text{where} \qquad f_{jh} \equiv f_j^t g_{th}. \tag{1.11}$$

From this equation and (1. 9), we have

$$f_{ji} = f_j^t f_i^s f_{ts} = f_i^s (-g_{js} + m_{js}) = -f_{ij}.$$

Thus, in a manifold with (f_r, g)-structure the tensor f_{ji} is a skew-symmetric one [3].

Next, applying ∇_j to (1. 1), we get

$$f_s^t f_t^h \nabla_j f_i^s + f_i^t f_s^h \nabla_j f_t^s + f_i^t f_t^s \nabla_j f_s^h + \nabla_j f_i^h = 0,$$

where ∇_j denotes the operator of covariant derivative with respect to the Riemannian connection formed with g_{ih}. The last equation can be written as

(1. 12) $$^*O_{ih}^{ts}\nabla_j f_{ts}=0, \qquad \text{or} \qquad f_i^{\ t}\, l_h^{\ s}\,\nabla_j f_{ts}=l_i^{\ t} f_h^{\ s}\,\nabla_j f_{ts}.$$

If we proceed in similar manner with equation (1. 9), we get

$$\nabla_j m_{ih}+f_h^{\ t}\nabla_j f_{it}+f_i^{\ t}\nabla_j f_{th}=0.$$

Operating $^*O_{ba}^{ih}$ to this equation, we find by virtue of (1. 12)

$$^*O_{ba}^{ji}\nabla_h m_{ji}={}^*O_{ba}^{ji}\ (f_j^{\ t}\nabla_h f_{ti}-f_i^{\ t}\nabla_h f_{jt})=0.$$

Hence we have

(1. 13) $$^*O_{ji}^{ts}\nabla_h m_{ts}=0.$$

Operating ∇_h to (1. 4), we have

(1. 14) $$m_j^{\ t}\nabla_h m_t^{\ i}=\nabla_h m_j^{\ i}-m_t^{\ i}\nabla_h m_j^{\ t}$$

from which by using (1. 4)

$$m_j^{\ t}\nabla_h m_{ti}=m_j^{\ t}\nabla_h m_{ti}-m_j^{\ t}m_i^{\ s}\nabla_h m_{ts},$$

and hence

(1. 15) $$m_j^{\ t}m_i^{\ s}\nabla_h m_{ts}=0.$$

Now, we shall prove the following

THEOREM 1. 1. *If, in a manifold with (f_r, g)-structure, the skew-symmetric tensor f_{ji} is closed, that is,*

(1. 16) $$f_{jih}\equiv\partial_j f_{ih}+\partial_i f_{hj}+\partial_h f_{ji}=0,$$

then the distribution M is integrable, where $\partial_j\equiv\partial/\partial x^j$.

Proof. Transvecting (1. 16) with $m_t^{\ j}m_s^{\ i}f_r^{\ h}$, we get

$$m_t^{\ j}m_s^{\ i}f_r^{\ h}(\nabla_j f_{ih}+\nabla_i f_{hj}+\nabla_h f_{ji})=0,$$

or using (1. 5)

$$-m_t^{\ j}f_{ih}f_r^{\ h}\nabla_j m_s^{\ i}-m_s^{\ i}f_r^{\ h}f_{hj}\nabla_i m_t^{\ j}=0.$$

Substituting (1. 2) in the last equation

$$-m_t^{\ j}\nabla_j m_{sr}+m_t^{\ j}m_{ri}\nabla_j m_s^{\ i}+m_s^{\ i}\nabla_i m_{tr}-m_s^{\ i}m_{rj}\nabla_i m_t^{\ j}=0,$$

and transvecting this with $m_h^{\ t}m_k^{\ s}$, we have in consequence of (1. 15)

(1. 17) $$m_h{}^j m_k{}^i (\nabla_j m_{ir} - \nabla_i m_{jr}) = 0.$$

This equation shows that the distribution M is integrable. q.e.d.

§2. Symmetric Killing tensors.

Now, in a Riemannian manifold V_n if a symmetric tensor $T_{i_q \cdots i_1}$ of type $(0, q)$ satisfies

(2. 1) $$\nabla_{(j} T_{i_q \cdots i_1)} = 0,$$

and

(2. 2) $$g^{ji} \nabla_j T_{i_q \cdots i_2 i} = 0,$$

then we shall call it a *symmetric Killing tensor*. We owe this definition to Professor S. Tachibana.

It is easily seen that when $q=1$ the symmetric Killing tensor coincides with the notion of a Killing vector. For a symmetric Killing tensor, we shall prove the following

THEOREM 2. 1. *In a compact orientable Riemannian manifold, a necessary and sufficient condition that a symmetric tensor field $T_{i_q \cdots i_1}$ be symmetric Killing is that it satisfies*

(2. 3) $$g^{ts} \nabla_t \nabla_s T_{i_q \cdots i_1} + \sum_{s=1}^{q} K_{i_s}{}^r T_{i_q \cdots r \cdots i_1} - (q-1) \sum_{t<s}^{q} K^j{}_{i_t i_s}{}^h T_{i_q \cdots j \cdots h \cdots i_1} = 0,$$

and

$$g^{ji} \nabla_j T_{i_q \cdots i_2 i} = 0,$$

where $K_{kji}{}^h$ is the Riemannian curvature tensor and $K^k{}_{ji}{}^h \equiv K_{sji}{}^h g^{sk}$, $K_{ji} \equiv K_{sji}{}^s$.

Proof. We first establish that the condition (2. 3) is necessary. Operating ∇^j to (2. 1), we get

$$\nabla^j \nabla_j T_{i_q \cdots i_1} + \nabla^j \nabla_{i_1} T_{i_q \cdots i_2 j} + \cdots + \nabla^j \nabla_{i_q} T_{j i_{q-1} \cdots i_1} = 0.$$

On the other hand, using the Ricci identities and (2. 2), we get

$$\nabla^j \nabla_{i_s} T_{i_q \cdots j \cdots i_1} = K_{i_s}{}^t T_{i_q \cdots t \cdots i_1} - K^j{}_{i_s i_1}{}^t T_{i_q \cdots j \cdots i_2 t} - \cdots - K^j{}_{i_s i_q}{}^t T_{t i_{q-1} \cdots j \cdots i_1}.$$

Consequently, for a symmetric Killing tensor $T_{i_q \cdots i_1}$, we have

$$\Delta T_{i_q \cdots i_1} \equiv g^{ts} \nabla_t \nabla_s T_{i_q \cdots i_1} + \sum_{s=1}^{q} K_{i_s}{}^t T_{i_q \cdots t \cdots i_1}$$
$$- (q-1) \sum_{t<s}^{q} K^j{}_{i_t i_s}{}^h T_{i_q \cdots j \cdots h \cdots i_1} = 0.$$

To prove the sufficiency of this theorem, we put

$$(q+1)U_{ji_q\cdots i_1} \equiv \nabla_{(j} T_{i_q\cdots i_1)},$$

then taking account of the Ricci identities and (2. 2), it follows that

$$(q+1)\nabla_j[T_{i_q\cdots i_1}(\nabla^j T^{i_q\cdots i_1}+q\nabla^{i_1}T^{i_q\cdots i_2 j})]$$
$$=U^2+T^{i_q\cdots i_1}\cdot(\Delta T_{i_q\cdots i_1})+qT_{i_q\cdots i_1}\cdot\nabla^{i_1}\nabla_j T^{i_q\cdots i_2 j}$$

and

$$\nabla^{i_1}(T_{i_q\cdots i_1}\nabla_j T^{i_q\cdots i_2 j})=V^2+T_{i_q\cdots i_1}\nabla^{i_1}\nabla_j T^{i_p\cdots i_2 j},$$

where $T^{i_q\cdots i_1}\equiv T_{j_q\cdots j_1}g^{j_q i_q}\cdots g^{j_1 i_1}$ and $V_{i_q\cdots i_2}\equiv\nabla_j T_{i_q\cdots i_2}{}^j$. Since the manifold is compact orientable, applying the Green's theorem we have

$$\int_{V_n}[U^2+\Delta T_{i_q\cdots i_1}T^{i_q\cdots i_1}-qV^2]d\sigma=0.$$

Hence, if $\Delta T_{i_q\cdots i_1}=0$ and $V_{i_q\cdots i_2}=0$, then we obtain $U_{i_q\cdots i_1}=0$. q.e.d.

Now, in a Riemannian manifold, let us consider a point x^h and a direction v^h at x^h which is contained in a distribution M. Then the geodesic is uniquely determined by the initial point x^h and the initial direction v^h.

If the tangent to the geodesic thus determined is always contained in M, then we say that the distribution is *geodesic*. [2], [4, p. 243].

It is known [2] that the condition for M to be geodesic distribution is

$$m_j{}^t m_i{}^s(\nabla_t m_s{}^h+\nabla_s m_t{}^h)=0. \tag{2. 4}$$

Next, let us consider a vector field. If the vector is parallel when we displace in any direction contained in a distribution M, we say that the vector is parallel along M. We can use the same terminology also for the distribution M, that is, if a distribution is parallel when we displace in any direction contained in M, we say that the distribution is parallel along M. When we displace a vector contained in a distribution M parallelly along M, if the displaced vector is always contained in the distribution M, we say that the distribution M is *flat*. [2], [4, p. 242].

It is known [2] that the condition for M to be a flat distribution is

$$m_j{}^t\nabla_t m_i{}^h=0. \tag{2. 5}$$

Now, we shall return to our manifold with (f_r, g)-structure. In this manifold, if the tensor m_{ji} defined by (1. 9) satisfies

$$m_{jih}\equiv\nabla_j m_{ih}+\nabla_i m_{hj}+\nabla_h m_{ji}=0, \tag{2. 6}$$

then transvecting this equation with g^{ji}, we find

$$\nabla_s m_h{}^s=0, \tag{2. 7}$$

by virtue of (1. 13). Hence, the tensor m_{ji} is a symmetric Killing tensor. Transvecting (2. 6) with $m_t{}^j m_s{}^i$ and using (1. 15), we get (2. 4). Therefore we have the following

THEOREM 2. 2. *In a manifold with (f_r, g)-structure, if m_{ji} is a symmetric Killing tensor, then the distribution M is geodesic.*

§ 3. Normal (f_r, g)-structures.

Now, we shall call an (f_r, g)-structure a *normal (f_r, g)-structure* if the following conditions are satisfied:

$$f_{jih} \equiv \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji} = 0, \tag{3. 1}$$

$$m_{jih} \equiv \nabla_j m_{ih} + \nabla_i m_{hj} + \nabla_h m_{ji} = 0. \tag{3. 2}$$

The condition (3. 1) shows that the skew-symmetric tensor f_{ji} is closed and hence by virtue of Theorem 1. 1, the distribution M is integrable. The condition (3. 2) means that the distribution M is geodesic by virtue of Theorem 2. 2.

If the rank of f is $(n-1)$, then since the distribution M is one-dimensional the tensor $m_j{}^i$ should have the form $m_j{}^i = p^i q_j$. Therefore, (3. 2) reduces

$$\nabla_j p_i + \nabla_i p_j = 0, \tag{3. 3}$$

that is, the vector p^i is a Killing vector. Hence a manifold with normal (f_{n-1}, g)-structure is similar with a normal contact manifold defined by S. Sasaki.

In a manifold with normal (f_r, g)-structure, from (1. 17) and (2. 4) we get $m_j{}^s m_i{}^t \nabla_s m_t{}^h = 0$. Hence by virtue of (1. 14), we get (2. 5), that is, $m_j{}^t \nabla_t m_i{}^h = 0$. Thus we have the following

THEOREM 3. 1. *In a manifold with normal (f_r, g)-structure, the distribution M is flat.*

Next, we shall prove the following two theorems which are useful in later sections.

THEOREM 3. 2. *In a manifold with normal (f_r, g)-structure, we have*

$$*O_{ji}^{ts} f_r{}^h \nabla_t f_s{}^r = 0. \tag{3. 4}$$

Proof. If we put $T_{ji} \equiv *O_{ji}^{ts} f_h{}^r \nabla_t f_{sr}$, then by virtue of (1. 12) and (3. 1), we find

$$T_{jih} = *O_{ji}^{ts} f_h{}^r (-\nabla_s f_{rt} - \nabla_r f_{ts})$$

$$= *O_{ji}^{ts} f_h{}^r \nabla_s f_{tr} = T_{ijh}.$$

Consequently we have

(3. 5) $$T_{jih}=T_{ijh}.$$

On the other hand, taking account of (1. 6), we find

$$2T_{jih}=f_h{}^r\,(l_j{}^t l_i{}^s+2m_j{}^t m_i{}^s+f_j{}^t f_i{}^s)\nabla_t f_{sr}$$

$$=f_h{}^r\,(l_j{}^t l_i{}^s+f_j{}^t f_i{}^s)\nabla_t f_{sr}, \qquad \text{by virtue of (2. 5)},$$

$$=(l_j{}^t l_h{}^r f_i{}^s+f_j{}^t f_h{}^r f_i{}^s)\nabla_t f_{sr}, \qquad \text{from (1. 12)},$$

$$=-2f_i{}^s {}^*O_{jh}^{tr}\nabla_t f_{rs}=-2T_{jhi}.$$

Hence we have

(3. 6) $$T_{jhi}=-T_{jih}.$$

From (3. 5) and (3. 6), we get $-T_{ihj}=T_{hij}$. Using (3. 5) again, we have $2T_{hij}=0$. q.e.d.

Transvecting (3. 4) with g^{ji}. we find by virtue of (1. 10)

(3. 7) $$f_h{}^r f_r=0, \qquad \text{or} \qquad f_h=f_r{}^s\nabla_s m_h{}^r,$$

where we put $f_h\equiv\nabla_s f_h{}^s$. From which we have

THEOREM 3. 3. *In a manifold with normal (f_r, g)-structure, if a vector field v^h admits $\underset{v}{\mathcal{L}} f_j{}^i=0$, then we have $f_h{}^t\underset{v}{\mathcal{L}} f_t=0$.*

§4. Infinitesimal transformations.

In this section, we shall consider in a manifold with normal (f_r, g)-structure a vector field v^h satisfying

(4. 1) $$\underset{v}{\mathcal{L}} f_j{}^i=0 \qquad \text{and} \qquad m_s{}^i v^s=0.$$

In this case, from (1. 2) we easily get

(4. 2) $$\underset{v}{\mathcal{L}} m_j{}^i=0,$$

and consequently

(4. 3) $$\underset{v}{\mathcal{L}} {}^*O_{ji}^{ts}=0.$$

Now, we shall prove the following

THEOREM 4. 1. *In a manifold with normal (f_r, g)-structure, if a contravariant vector v^h which is orthogonal to the distribution M admits $\underset{v}{\mathcal{L}} f_j{}^i=0$, then we have*

$$m_s{}^h g^{ji}\underset{v}{\mathcal{L}}\left\{{s \atop ji}\right\}=0.$$

Proof. Multiplying (2. 5) by v^h and contracting, we find

$$0=v^h m_j{}^t \nabla_t m_{ih}=-m_j{}^t m_i{}^s \nabla_t v_s=-\frac{1}{2} m_j{}^t m_i{}^s \underset{v}{\mathcal{L}} g_{ts},$$

by virtue of Theorem 3. 1 and $m_s{}^i v^s=0$. Hence taking account of (1. 4) and (4. 2), we have

$$\underset{v}{\mathcal{L}} m_{ji}=0. \tag{4. 4}$$

Next, from (1. 13) and (4. 3), we find

$$*O_{ji}^{ts} \underset{v}{\mathcal{L}} \nabla_h m_{ts}=0.$$

Taking account of (4. 4), this implies

$$*O_{ji}^{ts}[\underset{v}{\mathcal{L}} \{{}_{hs}^{r}\} m_{rt}+\underset{v}{\mathcal{L}} \{{}_{ht}^{r}\} m_{sr}]=0.$$

Transvecting this equation with g^{ji}, we obtain

$$m_s{}^t \underset{v}{\mathcal{L}} \{{}_{ht}^{s}\}=0, \tag{4. 5}$$

by virtue of (1. 10).

Lastly, by making use of the identity

$$\underset{v}{\mathcal{L}} \nabla_j m_{ih}=\nabla_j \underset{v}{\mathcal{L}} m_{ih}-\underset{v}{\mathcal{L}} \{{}_{ji}^{s}\} m_{sh}-\underset{v}{\mathcal{L}} \{{}_{jh}^{s}\} m_{is},$$

and taking account of (3. 2) and (4. 4), we find

$$\underset{v}{\mathcal{L}} \{{}_{ji}^{s}\} m_{sh}+\underset{v}{\mathcal{L}} \{{}_{jh}^{s}\} m_{si}+\underset{v}{\mathcal{L}} \{{}_{ih}^{s}\} m_{sj}=0.$$

Transvecting this equation with g^{ji} and using (4. 5), we have

$$g^{ji} \underset{v}{\mathcal{L}} \{{}_{ji}^{s}\} m_s{}^h=0.$$ q.e.d.

Now, operating $\underset{v}{\mathcal{L}}$ to (3. 4), we get by means of (4. 3),

$$*O_{ji}^{ts} f_r{}^h \underset{v}{\mathcal{L}} \nabla_t f_s{}^r=0.$$

Transvecting this equation with g^{ji} and using (1. 10), we find

$$0=g^{ts} f_r{}^h \underset{v}{\mathcal{L}} \nabla_t f_s{}^r=g^{ts} f_r{}^h [f_s{}^i \underset{v}{\mathcal{L}} \{{}_{ti}^{r}\}-\underset{v}{\mathcal{L}} \{{}_{ts}^{i}\} f_i{}^r],$$

or

$$l_r{}^h g^{ts} \underset{v}{\mathcal{L}} \{{}_{ts}^{r}\}=0.$$

Hence from Theorem 4. 1, we have the following

THEOREM 4. 2. *In a manifold with normal* (f_r, g)*-structure, if a contravariant vector field* v^h *which is orthogonal to the distribution* M *admits* $\underset{v}{\mathcal{L}} f_j^i=0$, *then it is a geodesic vector, that is,*

$$g^{ts}\underset{v}{\mathcal{L}}\left\{ {h \atop ts} \right\}=0.$$

From this theorem, we easily get the following

THEOREM 4. 3. *In a compact orientable manifold with normal* (f_r, g)*-structure, if an infinitesimal transformation* v^h *satisfying* $m_s{}^i v^s=0$ *and* $\underset{v}{\mathcal{L}} f_j^i=0$ *is volume preserving, then it is an infinitesimal isometry.*

THEOREM 4. 4. *In a compact orientable manifold with normal* (f_r, g)*-structure, if a conformal (projective) Killing vector* v^h *admits* $m_s{}^i v^s=0$ *and* $\underset{v}{\mathcal{L}} f_j^i=0$, *then it is a Killing vector.*

§ 5. An integral formula.

In this section, we shall consider a compact orientable manifold with normal (f_r, g)-structure and by using the Green's theorem we shall obtain conditions that a contravariant vector field v^h which is orthogonal to the distribution M, leaves f_j^i invariant.

If we put

$$T_j{}^i \equiv \underset{v}{\mathcal{L}} f_j{}^i = v^t\nabla_t f_j{}^i - f_j{}^t\nabla_t v^i + f_t{}^i\nabla_j v^t, \tag{5. 1}$$

then in a manifold with normal (f_r, g)-structure, we have

$$\begin{aligned}\frac{1}{2}T^2 = &\frac{1}{2}[v^s\nabla_s f_{ji} - f_j{}^s\nabla_s v_i - f_i{}^s\nabla_j v_s]\\ &\times[v^t\nabla_t f^{ji} - f^{jt}\nabla_t v^i - f^{it}\nabla^j v_t]\\ = &\frac{1}{2}[v^t v^s\nabla_t f_{ji}(\nabla_s f^{ji}) - m^{ts}\nabla_t v_i(\nabla_s v^i) - m^{ts}\nabla_i v_t(\nabla^i v_s)]\\ &+\nabla_j v_i(\nabla^j v^i) - v^s f^{tj}\nabla_s f_{ti}(\nabla_j v^i) - v^s f^{ti}\nabla_s f_{jt}(\nabla^j v_i) - f_j{}^t f_i{}^s\nabla_t v^i(\nabla^j v_s).\end{aligned}$$

On the other hand, we find that $\nabla^j[f_t{}^i v_i T_j{}^t]$ is sum of the following three terms:

$$\begin{aligned}(\nabla^j f_t{}^i)v_i T_j{}^t &= v_i\nabla^j f_t{}^i[v^s\nabla_s f_j{}^t - f_j{}^s\nabla_s v^t + f_s{}^t\nabla_j v^s]\\ &= -\frac{1}{2}v^t v^s\nabla_t f_{ji}(\nabla_s f^{ji}) + v^s\nabla^j v^i(f_j{}^t\nabla_t f_{is} + f_i{}^t\nabla_j f_{ts}),\end{aligned}$$

$$f_t{}^i(\nabla^j v_i)T_j{}^t = v^s\nabla^j v_i(f_t{}^i\nabla_s f_j{}^t) - f_j{}^t f_i{}^s\nabla^j v^i(\nabla_t v_s) - \nabla_j v_i(\nabla^j v^i) + m_s{}^i\nabla^j v_i(\nabla_j v^s),$$

and

$$f_t{}^i v_i \nabla^j T_j{}^t = f_t{}^i v_i g^{sj}(\nabla_s \underset{v}{\mathcal{L}} f_j{}^t)$$

$$= [-f_r{}^j g^{ts} \underset{v}{\mathcal{L}} \{{}^{\,r}_{ts}\} - \underset{v}{\mathcal{L}} f^j - \nabla^t f^{sj}(\underset{v}{\mathcal{L}} g_{ts})] f_j{}^i v_i$$

$$= [-f_r{}^j g^{ts} \underset{v}{\mathcal{L}} \{{}^{\,r}_{ts}\} - \underset{v}{\mathcal{L}} f^j] f_j{}^i v_i + v^s \nabla^j v^i (f_i{}^t \nabla_j f_{st} + f_j{}^t \nabla_i f_{st} + \nabla_j m_{is} + \nabla_i m_{js}).$$

Gathering above formulas, we obtain

$$\frac{1}{2} T^2 + \nabla^j [f_t{}^i v_i T_j{}^t]$$

$$= [f_r{}^j g^{ts} \underset{v}{\mathcal{L}} \{{}^{\,r}_{ts}\} - \underset{v}{\mathcal{L}} f^j] f_j{}^i v_i + f_j{}^t v^s \nabla^j v^i [\nabla_s f_{ti} + \nabla_t f_{is} + \nabla_i f_{st}]$$

$$- \frac{1}{2} [m^{ts} \nabla_t v_i (\nabla_s v^i) + m^{ts} \nabla_i v_t (\nabla^i v_s)] + v^s \nabla^j v^i (\nabla_j m_{is} + \nabla_i m_{js}).$$

In this case, if $m_i{}^s v_s = 0$, then in a manifold with normal (f_r, g)-structure we have

$$\frac{1}{2} T^2 + \nabla^j [f_t{}^i v_i T_j{}^t]$$

$$= [f_r{}^j g^{ts} \underset{v}{\mathcal{L}} \{{}^{\,r}_{ts}\} - \underset{v}{\mathcal{L}} f^j] f_j{}^i v_i - \frac{1}{2} U^2,$$

where $U_{ji} \equiv m_j{}^t \underset{v}{\mathcal{L}} g_{ti}$. Thus, we have the following

THEOREM 5.1. *In a compact orientable manifold with normal (f_r, g)-structure, the integral formula*

$$\int_{V_n} \left[\frac{1}{2} T^2 + \frac{1}{2} U^2 - l_r{}^i v_i g^{ts} \underset{v}{\mathcal{L}} \{{}^{\,r}_{ts}\} + f_j{}^i v_i \underset{v}{\mathcal{L}} f^j \right] d\sigma = 0.$$

is valid for a contravariant vecror field v^h satisfying $m_s{}^h v^s = 0$ where $T_j{}^i \equiv \underset{v}{\mathcal{L}} f_j{}^i$ and $U_{ji} \equiv m_j{}^t \underset{v}{\mathcal{L}} g_{ti}$.

From Theorems 3.1, 4.2 and 5.1, we have the following

THEOREM 5.2. *A necessary and sufficient condition that in a compact orientable manifold with normal (f_r, g)-structure a contravariant vector field v^h which is orthogonal to the distribution M, leave $f_j{}^i$ invariant is that it satisfies*

$$g^{ts} \nabla_t \nabla_s v^h + K_s{}^h v^s = 0 \qquad \text{and} \qquad f_h{}^r \underset{v}{\mathcal{L}} f_r = 0.$$

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